Free vibrations of circular curved beams

Ajinkya Baxy[1,*] and *Abhijit* Sarkar[1,**]

[1]Department of Mechanical Engineering, Indian Institute of Technology Madras, Chennai, India.

Abstract. The study of free vibrations of curved beams has relevance in engineering applications like modeling turbo machinery blades, propellers, arch design, etc. Vibration characteristics of structures are generally evaluated using the Finite Element Method. The governing equations for the curved beam using the inextensional theory are available in the literature. These equations are solved analytically for two different boundary conditions, namely (a) simply-supported, (b) cantilever. The results obtained for all the cases are compared against the FEM simulation results. It is found that the present solutions are in agreement with the FEM solutions up to an opening angle of 40°.

1 Introduction

Dynamics of curved beams is an important aspect in various fields such as turbo machinery, design of springs and structures, arches, etc. The prevalent engineering practice is to use Finite Element (FE) based simulations to determine the modal parameters of such structures. In this work, we present a novel analytical approach to determine the natural frequencies and mode shapes of simply-supported and cantilever curved beam structures.

2 Formulation

Consider a curved beam with radius of curvature R and opening angle θ_c. Length of the beam is $L = R\,\theta_c$, $w(s)$ and $u(s)$ are the displacement functions in the transverse and longitudinal direction, and s is the arc-length. In the general theory, the transverse and circumferential deflection of the curved beam is coupled [1]. However, with the assumption of vanishing of the strain over mid-line of the beam these equations are decoupled [2]. This leads to the inextensional theory of curved beams. In the inextensional theory the transverse displacement and the circumferential displacement are related as $w = -\frac{\partial u}{\partial s}$ [2].

The governing equation of motion for transverse deflection is given as [3]

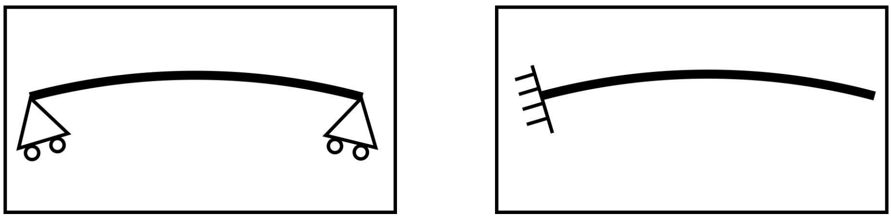
$$\frac{EI}{\rho A}\left(\frac{\partial^4 w}{\partial s^4} + \frac{2}{R^2}\frac{\partial^2 w}{\partial s^2} + \frac{w}{R^4}\right) + \frac{\partial^2 w}{\partial t^2} = 0. \tag{1}$$

Here, E represents the Young's modulus, ρ is the density, I represents the moment of inertia and A represents the cross-sectional area of the beam. In the limiting case of small curvature *viz.* $(1/R) \to 0$ the above equations reduce to that of the straight beam equation.

*e-mail: ajinkyaa3.14@gmail.com
**e-mail: asarkar.iitm.ac.in

2.1 Boundary conditions

The kinematic boundary conditions are given in terms of displacement w and slope $\frac{\partial w}{\partial s}$. The natural boundary conditions are given in terms of moment and shear force .



(a) **(b)**

Figure 1: Boundary conditions under consideration. (a) Simply-supported beam; (b) Cantilever beam

2.1.1 Simply-supported beam [2]

In a simply supported beam, the transverse displacements and moments at the ends are equal to zero at the boundary. Thus,

$$w(0) = 0 \text{ and } w(L) = 0 \tag{2a}$$

$$EI\left(\frac{\partial^2 w}{\partial s^2} + \frac{w}{R^2}\right)\Big|_{s=0,\,L} = 0 \implies EI\left(\frac{\partial^2 w}{\partial s^2}\right)\Big|_{s=0,\,L} = 0 \tag{2b}$$

2.1.2 Cantilever [4]

For curved cantilever, displacement and slope are zero at the fixed end ($s = 0$) whereas, moment and shear force are zero at the free end. Thus,

$$w(0) = 0 \text{ and } \frac{\partial w}{\partial s}\Big|_0 = 0 \tag{3a}$$

$$EI\left(\frac{\partial^2 w}{\partial s^2} + \frac{w}{R^2}\right)\Big|_L = 0 \text{ and } EI\left(\frac{\partial^3 w}{\partial s^3} + \frac{1}{R^2}\frac{\partial w}{\partial s}\right)\Big|_L = 0 \tag{3b}$$

Note, as $R \to \infty$, the natural boundary conditions of the curved cantilever beam revert back to their straight beam counterpart.

In the following, we solve equation (1) for natural frequencies and mode shapes for the above two boundary conditions.

3 Solution

For determination of the modal characteristics of the curved beam, a harmonic dependence of the displacement $w(\theta, t) = W(\theta)e^{i\omega t}$ is assumed. Substituting this simplification in equation (1) and noting $s = R\theta$, we get

$$\frac{d^4W}{d\theta^4} + 2\frac{d^2W}{d\theta^2} + \gamma^4 W = 0, \text{where } \Gamma = \sqrt{\frac{EI}{\rho A R^4}},\ \gamma^4 = \left(1 - \frac{\omega^2}{\Gamma^2}\right), \tag{4}$$

Assuming a solution of the form $w(\theta) = A\, e^{m\theta}$ we get the characteristic equation for m as $m^4 + 2m^2 + \gamma^4 = 0$. The four roots of the equation are given by,

$$m_1 = \sqrt{\frac{\omega}{\Gamma} - 1} = \psi\ ,\ m_2 = -\sqrt{\frac{\omega}{\Gamma} - 1} = -\psi\ ,\ m_3 = i\sqrt{\frac{\omega}{\Gamma} + 1} = i\delta\ ,\ m_4 = -i\sqrt{\frac{\omega}{\Gamma} + 1} = -i\delta \tag{5}$$

Using equation (5), $W(\theta)$ can be written as,

$$W(\theta) = A\cos(\delta\theta) + B\sin(\delta\theta) + C\cosh(\psi\theta) + D\sinh(\psi\theta) \tag{6}$$

$A,\ B,\ C,$ and D are the arbitrary constants which are determined by incorporating the boundary conditions of the problem.

3.1 Simply-supported beam

Incorporating the simply supported boundary conditions in equation (6) a system of equation is obtained which can be written in matrix form as,

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -\delta^2 & 0 & \psi^2 & 0 \\ \cos\left(\frac{\delta L}{R}\right) & \sin\left(\frac{\delta L}{R}\right) & \cosh\left(\frac{\psi L}{R}\right) & \sinh\left(\frac{\psi L}{R}\right) \\ -\cos\left(\frac{\delta L}{R}\right)\delta^2 & -\sin\left(\frac{\delta L}{R}\right)\delta^2 & \cosh\left(\frac{\psi L}{R}\right)\psi^2 & \sinh\left(\frac{\psi L}{R}\right)\psi^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \mathbf{0}$$

Setting the determinant to zero, we get,

$$-\sin\left(\frac{\delta L}{R}\right)\sinh\left(\frac{\psi L}{R}\right)\left(\delta^2 + \psi^2\right)^2 = 0$$

The physically feasible solution to the above equation is given by

$$\sin\left(\frac{\delta L}{R}\right) = 0 \implies \delta\frac{L}{R} = n\pi \implies \omega = \sqrt{\frac{EI}{\rho A}}\left(\frac{n^2\pi^2}{L^2} - \frac{1}{R^2}\right) \tag{7}$$

Equation (7) represents the natural frequency of a circularly curved simply supported beam. Using the above solution for δ, we find $\psi = \sqrt{\frac{\phi}{\Phi} - \frac{n^2\pi^2R^2}{L^2}}$. Substituting these solutions in the above matrix equation we find a non-trivial solution given by$A = C = D = 0 \neq B$. Thus mode shapes are given by,

$$w(s) = \sin(\delta\theta) = \sin\left(\frac{n\,\pi\,s}{L}\right) \tag{8}$$

3.1.1 Cantilever

Incorporating the cantilever boundary conditions in equation (6), a homogeneous matrix equation for the undetermined constants A, B, C, D is formulated as follows

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \delta & 0 & \psi \\ -\cos\left(\frac{\delta L}{R}\right)\frac{\omega}{\phi} & -\sin\left(\frac{\delta L}{R}\right)\frac{\omega}{\phi} & \cosh\left(\frac{\psi L}{R}\right)\frac{\omega}{\phi} & \sinh\left(\frac{\psi L}{R}\right)\frac{\omega}{\phi} \\ \sin\left(\frac{\delta L}{R}\right)\delta\frac{\omega}{\phi} & -\cos\left(\frac{\delta L}{R}\right)\delta\frac{\omega}{\phi} & \sinh\left(\frac{\psi L}{R}\right)\psi\frac{\omega}{\phi} & \cosh\left(\frac{\psi L}{R}\right)\psi\frac{\omega}{\phi} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \mathbf{0} \tag{9}$$

Equating the determinant of the above matrix equation to zero we get the following transcendental equation for the determination of the natural frequencies.

$$1+\cos\left(\frac{\delta L}{R}\right)\cosh\left(\frac{\psi L}{R}\right)+\frac{1}{\delta\psi}\sin\left(\frac{\delta L}{R}\right)\sinh\left(\frac{\psi L}{R}\right)=0 \tag{10a}$$

Finally, the mode shapes $W(s)$ are found as,

$$W=\cos\left(\frac{\delta s}{R}\right)-\cosh\left(\frac{\psi s}{R}\right)-C_c\left(\psi\sin\left(\frac{\delta s}{R}\right)-\delta\sinh\left(\frac{\psi s}{R}\right)\right) \tag{10b}$$

where,

$$C_c=\frac{\cos\left(\frac{\delta L}{R}\right)+\cosh\left(\frac{\psi L}{R}\right)}{\psi\sin\left(\frac{\delta L}{R}\right)+\delta\sinh\left(\frac{\psi L}{R}\right)}$$

Due to complexities of equation (10a) analytical solution is difficult. However, numerical solvers can be used to determine the solution of the above transcendental equation.

4 Results

The natural frequencies obtained in the derivation above are non-dimensionalized with respect to frequency parameter $\Gamma=\sqrt{\frac{EI}{\rho AL^4}}$. The results obtained from the present approach are compared with the results obtained through a Finite Element (FE) simulation in a commercial software package.

4.1 Simply-supported beam

Equation (7) is used to obtain the natural frequencies of a curved simply supported beam. Figures 2a, 2b, 2c compare the first three non-dimensional natural frequencies obtained using equation (7) and FE simulation. Figures 2d, 2e, 2f compare the first three mode shapes of curved simply-supported beam with opening angle 40°. The mode shapes are obtained using equation (8) and FE simulation.

From the results, it is noted that the formula correlates well with FE results for opening angles $< 50°$.

4.2 Cantilever

The natural frequencies of curved cantilever beam are obtained by solving the transcendental equation (10a). Figures 3a, 3b, 3c compare the first three non-dimensional natural frequencies obtained from equation (10a) and FE simulation. Figures 3d, 3e, 3f compare first three mode shapes of curved cantilever with opening angle 40°. The mode shapes are obtained using equation (10b) and FE simulation.

From the figures, it can be concluded that the results from the inextensional theory gives satisfactory results up to an opening angle of 40°.

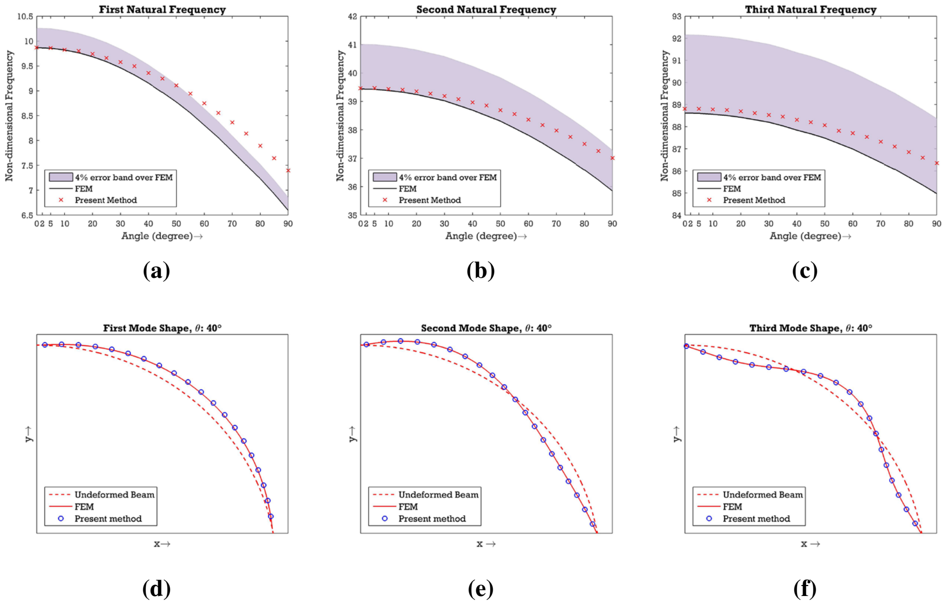


Figure 2: Comparison of natural frequencies and mode shapes: Curved Simply Supported Beam.

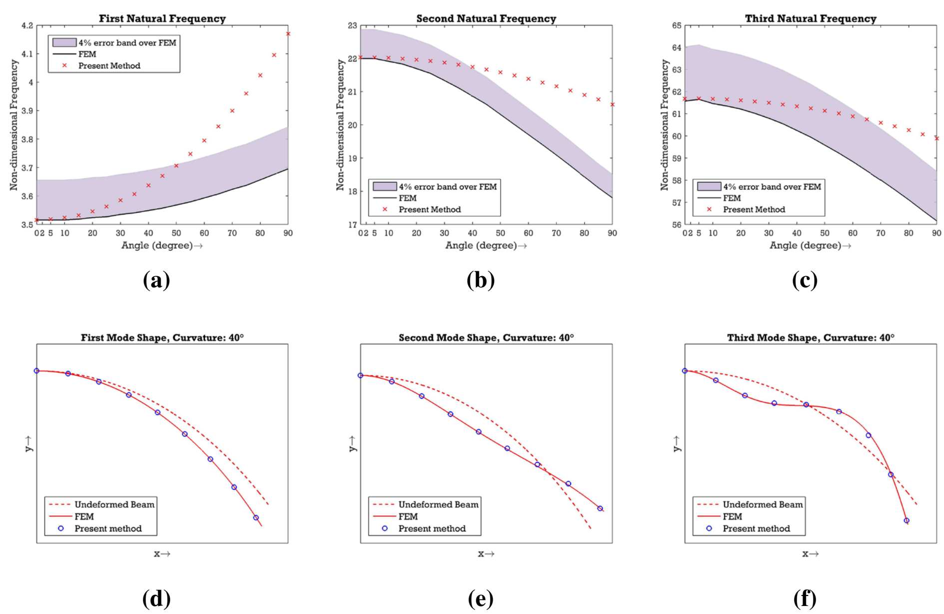


Figure 3: Comparison of Natural frequencies and mode shapes: Curved Cantilever.

5 Conclusion

In this work, the free vibration characteristics of two boundary conditions, namely simply supported and cantilever are studied in the inextensional theory framework. A simple formula has been derived in case of simply supported beam to calculate the natural frequencies. For curved cantilever beam, a transcendental equation has been derived to obtain the natural frequencies. Note, in the case of straight cantilever beam, an analytical formula to calculate the natural frequencies is not available. There too, a transcendental equation has to be solved. In the special case when $R \to \infty$(straight beam), it is observed that the formula(equation (7)) and the transcendental equation (10a) for curved beam, revert to their straight beam counterparts. This is also true for mode shapes. Thus, the present results can be specialized for straight beams. The results obtained using the present method are validated by comparing the solution with the results of a FE simulation. It is concluded that the present results are in agreement with the FE simulations. Thus, the utility of the present method lies in proposing an alternative analytical and efficient method in calculating the modal parameters of a curved beam. The present approach will be useful in design iterations. For larger opening angles, the present method based on inextensional theory is limited in its accuracy. This is due to the coupling of the transverse and circumferential displacement. This necessitates the use of extensional theory of curved beams[1].

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