

PAPER • OPEN ACCESS

## Free vibration of spring-mounted beams

To cite this article: S Vijay Kumar *et al* 2019 *J. Phys.: Conf. Ser.* **1264** 012026

View the [article online](#) for updates and enhancements.



**IOP | ebooks™**

Bringing together innovative digital publishing with leading authors from the global scientific community.

Start exploring the collection—download the first chapter of every title for free.

# Free vibration of spring-mounted beams

Vijay Kumar S, Abhijit Sarkar, Chandramouli Padmanabhan

Department of Mechanical Engineering, IIT Madras Chennai, India

E-mail: [asarkar@iitm.ac.in](mailto:asarkar@iitm.ac.in), [mouli@iitm.ac.in](mailto:mouli@iitm.ac.in)

**Abstract.** In the present work, free vibration characteristics of a spring-mounted beam is studied. The objective is to determine the natural frequencies and mode shapes of this beam. Towards this end, a novel semi-analytical method is developed. The methodology is based on Green's function of Euler-Bernoulli beam together with a Rayleigh-Ritz method. A minimal set of basis functions for the Rayleigh-Ritz method is found, by generating the beam mode shapes for extreme values (zero and infinity) of the connecting spring stiffnesses. Thus, for a beam with  $n$  intermediate spring connections,  $2^n$  beam modes are generated as basis functions. The Green's function approach is used to extract the beam mode shapes for the extreme values of the connecting spring stiffnesses. The results from the proposed formulation are compared with those from a finite element (FE) simulation. For most cases, the results obtained by the two methods are in excellent agreement. Guidelines for further improvement of the accuracy of the present results have been proposed. Detailed parametric studies on the effect of spring stiffness value and their location have been performed.

Keywords: Green's function, Rayleigh-Ritz, Natural frequencies, Mode shapes.

## 1. Introduction

Beams with springs support have been studied extensively. Most investigators have solved the problem by determining the roots of a frequency equation, which is usually transcendental in nature. Nicholson and Bergman [1] derived the governing differential equation for a beam with spring-mass attachment. The equation was solved using Green's function method and analysis was carried out for Simply supported and cantilever boundary conditions. Cha and Wong [2] proposed a novel method to determine the frequency equation of a beam mounted on  $s$  spring-mass system. The characteristic determinant is algebraically simplified such that the  $N \times N$  ( $N$  number of terms considered in assumed mode method) generalized eigenvalue problem is reduced to a  $s \times s$  ( $s$  number of spring locations) problem. Lau [3] derived closed form expressions for the fundamental frequency and mode shape of beam supported on  $n$  number of springs. The beam is discretized into  $n + 1$  beam elements and continuity is enforced at each beam-spring interface. Mohamad [4] extended the earlier research by deriving the governing differential equation for a beam attached with linear spring, rotational spring and mass at different locations. This is then solved using a Green's function. Wu and Lin [5] used a combined analytical-numerical method for finding the natural frequency of a beam with masses attached at multiple points. The frequency equation was obtained using an expansion theorem and solved numerically for frequencies and mode shapes.

Dowell [6] used a Lagrangian formulation for finding the natural frequency of a simply supported beam connected with a spring-mass system. He employed the component modes of the beam using the idea of a Lagrange multiplier at the interface. Gurgoze [7] derived a closed form



solution for a cantilever beam with mass-spring-mass attached at the tip using Lagrangian multiplier method. The frequency equation is solved numerically using regula-falsi method for various values of spring and mass. Later, he studied [8] the effect of oscillators attached to an Euler-Bernoulli beam at different locations using two methods: the first formulation uses assumed mode method in conjunction with Lagrangian multipliers. While the second formulation uses Euler-Lagrangian equation. Displacements of the beam at the point of spring interface is expressed in terms of generalized coordinates which leads to system of differential equations and are solved numerically.

Lin and Yang [9] developed a transfer matrix method for two beams connected by a spring-mass-spring device, by expressing the compatibility at the point of connection. The method is generalized to  $n$  spring-mass-spring devices at different locations for all classical boundary conditions. Rezaiee and Mojtaba [10] investigated the dynamics of two horizontal beams connected by a spring-mass system. The differential equations of the two beams are transformed to frequency domain, and solved analytically for the natural frequencies and mode shapes. Inceoglu and Gurgoze [11] employed the Green's function approach to study the transverse vibrations of a system consisting of two clamped-free beams carrying tip masses at their ends and  $n$  spring-mass-spring devices attached across the span.

It is evident from the literature that, both Green's functions and assumed mode methods are predominantly used for solving beam on spring-support. The use of Green's function involves solving transcendental equation which is tedious. The assumed mode method uses mode shapes of the beam without spring along with Lagrangian multiplier and requires inclusion of a large number of terms for desired accuracy.

In this paper a new approach is proposed. A combination of the Green's function approach with Rayleigh-Ritz formulation is used to estimate natural frequencies and mode shapes. A minimal set of basis functions are generated by assigning extreme values to the spring stiffness ( $K_i \rightarrow 0$  and  $K_i \rightarrow \infty$ ). Using these basis function in the Rayleigh-Ritz formulation, the natural frequency and the mode shapes of the spring-supported beam are obtained for any set of values for  $K_i$ .

## 2. Formulation

Consider a uniform beam connected by linear springs at multiple locations as shown in fig. 1. Let  $L, I, A$  be the length, area and moment of inertia of the beam respectively.  $\rho, E$  are the density and Young's modulus of beam material, respectively. The spring connected to the beam at  $j^{th}$  location has a spring stiffness  $k_j$ . The transverse displacement of the beam is denoted by  $y(x, t)$ . The governing differential equation of beam with springs attached is given by [1]

$$EI_{zz} \frac{\partial^4 y(x, t)}{\partial x^4} + \rho A \frac{\partial^2 y(x, t)}{\partial t^2} = \sum_{j=1}^n k_j y(\zeta_j, t) \delta(x - \zeta_j), \quad j = 1, 2, \dots, n, \quad i = 1, 2 \quad (1)$$

where  $n$  is the number of linear springs connected to beams. Assuming harmonic dependency in time, Eq. (1) is non-dimensionalized to a form as shown below:

$$\frac{d^4 \bar{\phi}(\bar{x})}{d\bar{x}^4} - \beta^4 \bar{\phi}(\bar{x}) = \sum_{j=1}^n K_j \bar{\phi}(\zeta_j) \delta(\bar{x} - \zeta_j) \quad (2)$$

$$\text{where } \bar{x} = \frac{x}{L}, \quad \bar{\phi}(\bar{x}) = \frac{y}{L}, \quad K_j = \frac{k_j L^3}{E I_{zz}}, \quad \beta^4 = \frac{\rho A \omega^2}{E I_{zz} L^4}$$

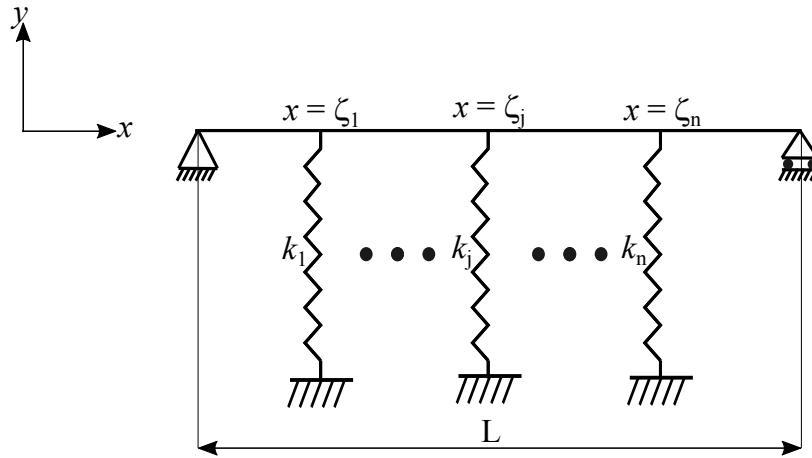


Figure 1: Simply supported beam with  $n$  linear springs attached in span with connecting stiffness  $k_j$ .

The solution of the differential equation is given by

$$\bar{\phi}(\bar{x}) = \sum_{j=1}^n K_j \bar{\phi}(\zeta_j) G(\bar{x}, \zeta_j) \quad (3)$$

where the Green's function  $G(\bar{x}, \zeta_j)$  should satisfy the following differential equation

$$\frac{d^4 G}{d\bar{x}^4} - \beta^4 G = \delta(\bar{x} - \zeta_j) \quad (4)$$

From here on,  $\bar{x}$  and  $\bar{\phi}$  are treated as  $x$  and  $\phi$  respectively. The Green's function for simply supported boundary conditions is expressed as [1]

$$G(x, \zeta_j) = \begin{cases} g(x, \zeta_j) & x \leq \zeta_j \\ g(\zeta_j, x) & x > \zeta_j \end{cases} \quad (5)$$

$$\text{where } g(x, \zeta_j) = \frac{1}{2\beta^3} \left( \frac{\sin(\beta(L - \zeta_j)) \sin(\beta x)}{\sin(\beta L)} - \frac{\sinh(\beta(L - \zeta_j)) \sinh(\beta x)}{\sinh(\beta L)} \right)$$

Similar expressions are available for other boundary conditions [4].

### 2.1. Shape functions

In the following section, we demonstrate the methodology for obtaining the shape functions for a simply supported beam with single spring as shown in fig. 2a. In this case two shape functions are used, namely  $\phi_1(x)$  and  $\phi_2(x)$ . The former corresponds to highly compliant spring ( $K_1 \rightarrow 0$ ) whereas the later corresponds to stiff spring ( $K_1 \rightarrow \infty$ ). These  $\phi_1(x)$  and  $\phi_2(x)$  represents the two end points of a line spanning the range of  $K_1$  values see fig. 2b.

$$\phi_1(x) = K_{1,0} G(x, \zeta_1) \quad (6a)$$

$$\phi_2(x) = K_{1,\infty} G(x, \zeta_1), \quad (6b)$$

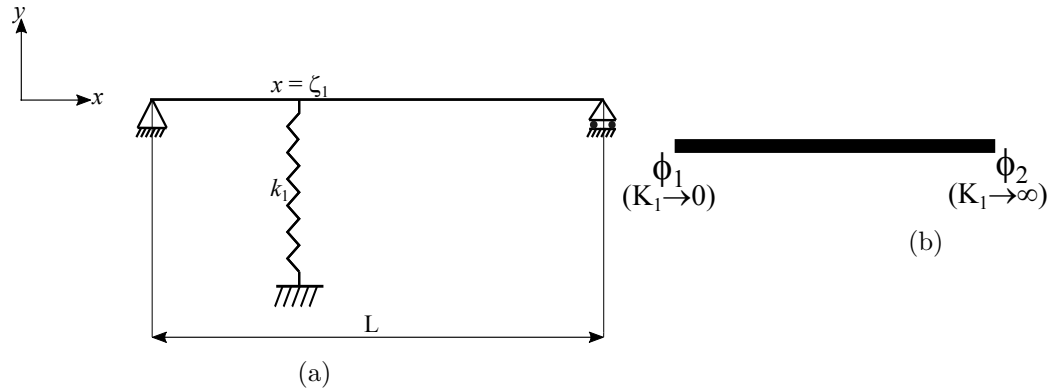


Figure 2: (a) Simply supported beams connected with single linear springs placed at  $x = \zeta_1$  with spring stiffness  $K_1$ , (b) Two asymptotic shape function.

In the above equation  $K_{1,0}$  is an extremely small value  $\epsilon$  ( $1e-6$ ) and  $K_{1,\infty}$  is an extremely large value of say  $\frac{1}{\epsilon}$ . The key idea proposed in this paper is to assume that for any intermediate value of spring stiffness  $K_1$ , the beam shape function  $y(x, t)$  can be expressed as a linear combination of the two extreme valued shape functions:

$$y(x, t) = \{B_1 \phi_1(x) + B_2 \phi_2(x)\}e^{i\omega t} \quad (7)$$

For '2' springs the asymptotic shape functions combinations represent the vertices of a square, as shown in fig. 3, and are given by

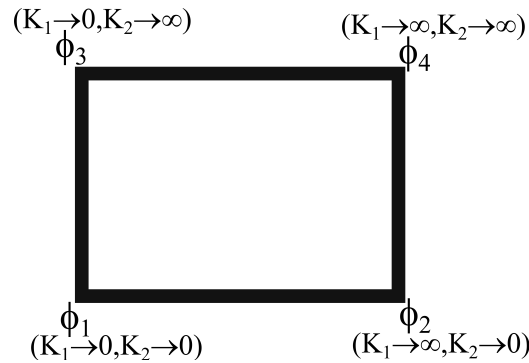


Figure 3: Extreme combinations of  $K_1$  and  $K_2$  leads to shape function which represents the vertices of the square.

$$y(x, t) = \{B_1 \phi_1(x) + B_2 \phi_2(x) + B_3 \phi_3(x) + B_4 \phi_4(x)\}e^{i\omega t} \quad (8)$$

$$\text{where } \phi_i(x) = \sum_{j=1}^2 K_j G(x, \zeta_j) \quad i = 1 \dots 4$$

Eq. (8) contains a linear combination of four extreme valued shape functions, which correspond to different combinations of extreme values of the two springs. For  $n$  springs the form of  $y(x, t)$  can be expressed as .

$$y(x, t) = \sum_{i=1}^{2^n} B_i \phi_i(x) e^{i\omega t} \quad (9)$$

The scaled kinetic energy and non-dimensional potential energy of the system are expressed as

$$\bar{\mathcal{V}} = \frac{1}{2} \int_0^1 \left( \frac{\partial^2 y(x, t)}{\partial x^2} \right)^2 dx + \sum_{j=1}^n \frac{1}{2} K_j (y(\zeta_j, t))^2 \quad (10a)$$

$$\bar{\mathcal{T}} = \frac{1}{2} \int_0^1 \left( \frac{\partial y(x, t)}{\partial t} \right)^2 dx \quad (10b)$$

$$\text{where } \bar{\mathcal{V}} = \frac{\mathcal{V} L}{E I_{zz}} \quad \bar{\mathcal{T}} = \frac{\mathcal{T}}{\rho A L^3}$$

The displacement of the combined system is expressed as a linear combination of basis functions generated from the extreme values of each spring ( $K_j \rightarrow 0$  and  $K_j \rightarrow \infty$ ). The Rayleigh-Ritz energy formulation is used to solve for the natural frequencies and mode shapes. The size of the algebraic equations is given by  $2^n$  where  $n$  is the number of springs. For example, if 4 linear springs exist then 16 shape functions are required. The shape functions are denoted by  $\phi_i(x)$ . The form of  $y(x, t)$  is constructed using the derived shape functions. This form is substituted in the kinetic energy and potential energy expressions. The Lagrangian is minimized with respect to the undetermined coefficients leading to a system of homogeneous equation.

$$[\mathbf{K} - \omega^2 \mathbf{M}] \{\mathbf{B}\} = \mathbf{0} \quad (11)$$

where,  $\mathbf{K}$  and  $\mathbf{M}$  are the stiffness and mass matrices respectively,  $\mathbf{B}$  is the unknown coefficient vector. Solving this eigenvalue problem yields the natural frequencies and mode shapes. While the results are presented only for a spring mounted simply supported beam, this procedure has been found to work for other boundary conditions. These results are however not shown here.

### 3. Numerical Results

In this section several numerical results for a beam with single and two springs are presented to demonstrate the advantages of the present method. In the present analysis all the results are expressed in terms of a non dimensional frequency parameter  $\beta$ .

#### 3.1. Beam with single spring

First as an illustration, a simply supported beam carrying single spring ( $j = 1$ ) as shown in fig. 2a is considered. The admissible functions used in the Rayleigh-Ritz formulation are derived using Eq. (6) for two locations  $\zeta_1 = \frac{1}{2}$  and  $\zeta_1 = \frac{1}{3}$ . It is observed that the results obtained from the present method are in good agreement with those obtained from FEM; see Tables 1 and 2 for comparison. The lowest three mode shapes of the beam carrying a single spring at  $\zeta_1 = \frac{1}{3}$  with stiffness ranging from ( $K_1 = 0, \gamma, \infty$ ) are obtained from the present method. These mode shapes shows excellent match with those obtained from FEM.

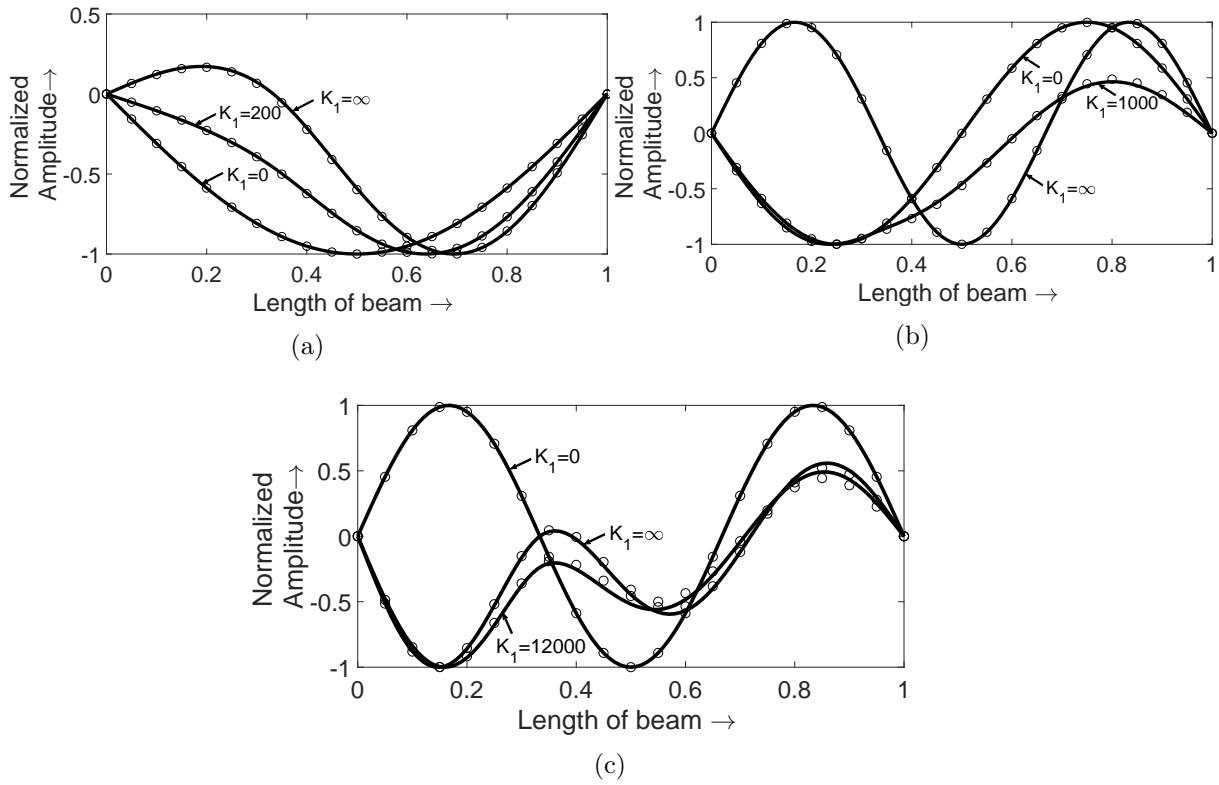


Figure 4: The first three mode shape of simply supported beam with spring attached at  $\zeta = \frac{1}{3}$ , for different values of  $K_1$ , key: present method —, FEM  $\circ$

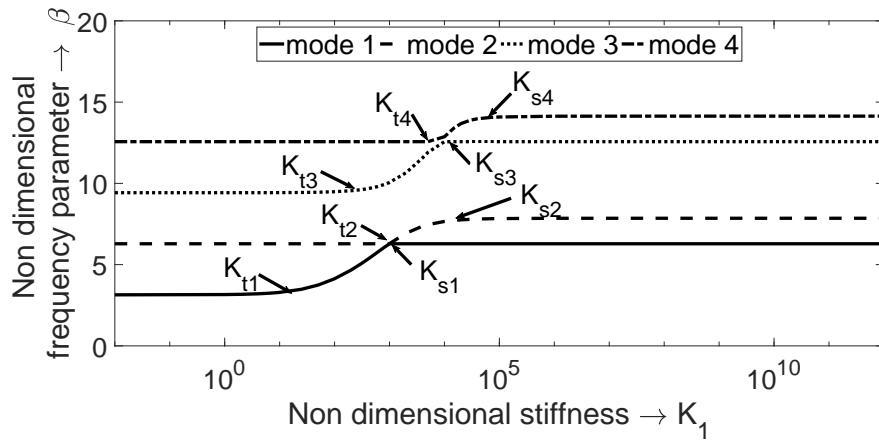


Figure 5: Transition curves for simply supported beam with spring mounted at  $\zeta_1 = \frac{1}{2}$ .

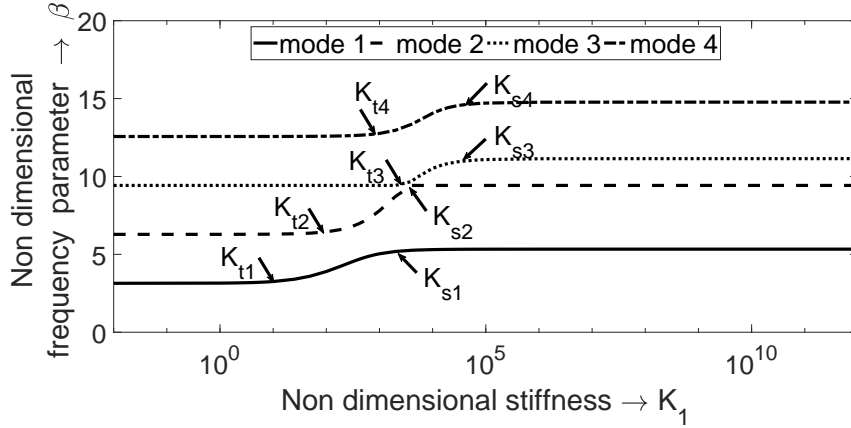


Figure 6: Transition curves for simply supported beam with spring mounted at  $\zeta_1 = \frac{1}{3}$ .

The variation of the non dimensional frequencies of the system as the non dimensional spring stiffness is increased is shown in figs. 5 and 6 for the springs at the two different locations mentioned earlier. It is observed from these curves, that the frequency parameter is insensitive to an increase in spring stiffness till it reaches a transition stiffness  $K_{t,i}$ ; the associated frequency is called as transition frequency  $\beta_{t,i}$ . Beyond  $K_{t,i}$  the frequency parameter drastically increases with increase in stiffness and then reaches a saturation value with the system frequency parameter again becoming insensitive to additional increase in stiffness. This stiffness is called as saturation stiffness  $K_{s,i}$  and the associated frequency is called as the saturation frequency  $\beta_{s,i}$ .

A quick estimate of the natural frequency of a beam with single spring can obtained by fitting straight line equation to this log-linear plot.

$$\beta_i(K_1) = \frac{\beta_{s,i} - \beta_{t,i}}{\log K_{s,i} - \log K_{t,i}} (\log K_1 - \log K_{t,i}) + \beta_{t,i} \quad i = 1 \dots 4 \quad (12)$$

Such an equation is very valuable as a design tool. The values of  $K_{t,i}$  and  $K_{s,i}$  are obtained from the transition curves shown in figs. 5 and 6.

Table 1: Non-dimensional frequency parameter  $\beta_i$  for a beam mounted on a spring at  $\zeta_1 = \frac{1}{2}$ .

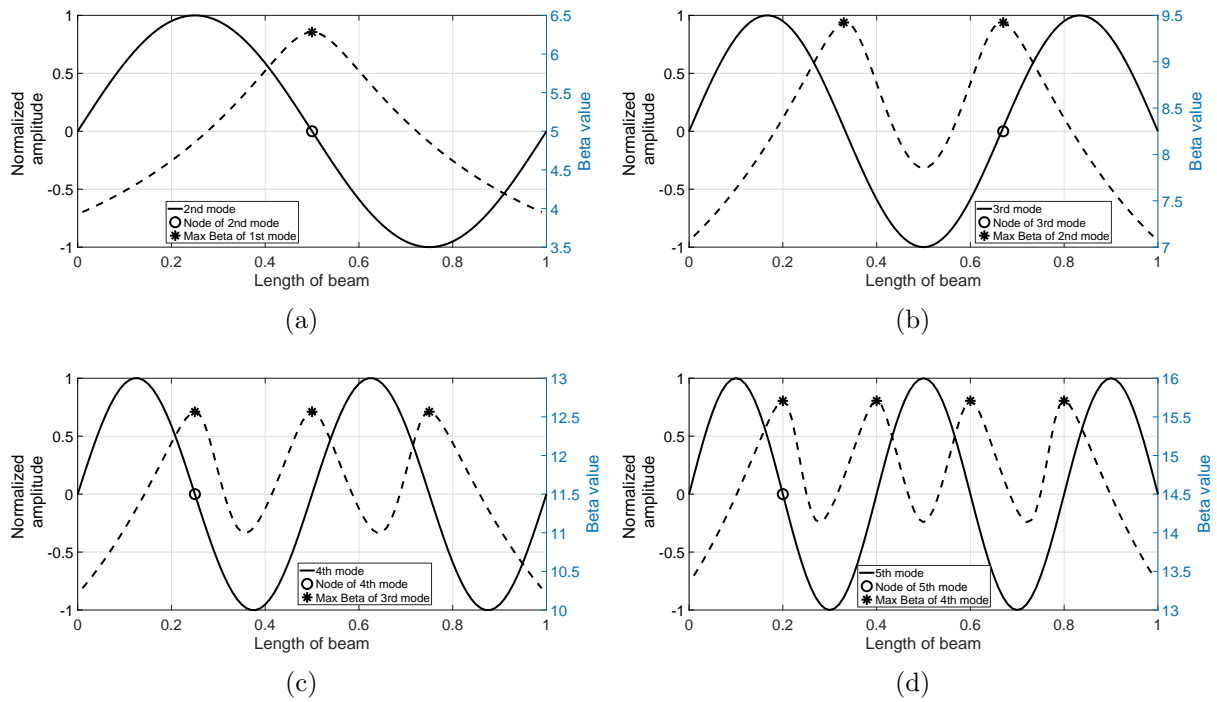
| Non dimensional stiffness $K_1$ | Mode 1( $\beta_1$ ) |       | Mode 2( $\beta_2$ ) |       | Mode 3( $\beta_3$ ) |        | Mode 4( $\beta_4$ ) |        |
|---------------------------------|---------------------|-------|---------------------|-------|---------------------|--------|---------------------|--------|
|                                 | Present method      | FEM   | Present method      | FEM   | Present method      | FEM    | Present method      | FEM    |
| 0.01                            | 3.141               | 3.141 | 6.283               | 6.283 | 9.424               | 9.424  | 12.566              | 12.566 |
| 1                               | 3.157               | 3.157 | 6.283               | 6.283 | 9.425               | 9.425  | 12.566              | 12.566 |
| 10                              | 3.291               | 3.305 | 6.283               | 6.283 | 9.430               | 9.431  | 12.566              | 12.566 |
| 100                             | 4.132               | 4.202 | 6.283               | 6.283 | 9.485               | 9.491  | 12.566              | 12.566 |
| $10^3$                          | 6.283               | 6.283 | 6.292               | 6.394 | 10.055              | 10.125 | 12.566              | 12.566 |
| $10^4$                          | 6.283               | 6.283 | 7.656               | 7.676 | 12.566              | 12.566 | 12.868              | 12.984 |
| $10^5$                          | 6.283               | 6.283 | 7.833               | 7.835 | 12.566              | 12.566 | 14.085              | 14.032 |
| $10^6$                          | 6.283               | 6.283 | 7.850               | 7.851 | 12.566              | 12.566 | 14.132              | 14.127 |



Table 2: Non-dimensional frequency parameter  $\beta_i$  of a beam mounted on a spring at  $\zeta_1 = \frac{1}{3}$ .

|                                 | Mode 1 ( $\beta_1$ ) |       | Mode 2( $\beta_2$ ) |       | Mode 3( $\beta_3$ ) |        | Mode 4( $\beta_4$ ) |        |
|---------------------------------|----------------------|-------|---------------------|-------|---------------------|--------|---------------------|--------|
| Non dimensional stiffness $K_1$ | Present method       | FEM   | Present method      | FEM   | Present method      | FEM    | Present method      | FEM    |
| 0.01                            | 3.141                | 3.141 | 6.283               | 6.283 | 9.424               | 9.424  | 12.566              | 12.566 |
| 1                               | 3.153                | 3.153 | 6.285               | 6.284 | 9.424               | 9.424  | 12.566              | 12.566 |
| 10                              | 3.254                | 3.264 | 6.298               | 6.300 | 9.424               | 9.424  | 12.568              | 12.568 |
| 100                             | 3.898                | 3.942 | 6.441               | 6.463 | 9.424               | 9.424  | 12.585              | 12.586 |
| $10^3$                          | 5.038                | 5.046 | 7.805               | 7.949 | 9.424               | 9.425  | 12.726              | 12.773 |
| $10^4$                          | 5.305                | 5.287 | 9.424               | 9.422 | 10.520              | 10.624 | 14.060              | 14.065 |
| $10^5$                          | 5.331                | 5.311 | 9.424               | 9.422 | 11.086              | 11.137 | 14.711              | 14.660 |
| $10^6$                          | 5.334                | 5.313 | 9.424               | 9.422 | 11.138              | 11.184 | 14.767              | 14.712 |

The effect of a hard spring on the frequency parameter  $\beta$  as a function of location on beam is studied. The results for the first four modes of vibrations are shown in fig. 7. It is observed that the  $i^{th}$  mode natural frequency can be maximized by connecting the hard spring at the node of the unrestrained  $(i + 1)^{th}$  mode.

Figure 7: Effect of hard spring location on natural frequencies  $\beta'_i$ s; the corresponding unrestrained  $(i + 1)^{th}$  mode is also shown. (a)  $i=1$  (b)  $i=2$ , (c)  $i=3$ , (d)  $i=4$ .

### 3.2. Beam with two springs

Next, a simply supported beam with two springs mount is analysed. The two springs are positioned symmetrically at  $\frac{1}{3}$  and  $\frac{2}{3}$  from the centre of the beam. The lowest three mode shapes extracted from the present method are compared with FEM in fig. 8. Results are presented for three different stiffness values. Good correlation of the results obtained from the present method and FEM simulation is noted. Modal assurance criteria (MAC) is also computed from the results

obtained from the present method and FEM. It is found that the MAC is always  $\geq 0.95$  for the entire range of the stiffness values. This validates the the present methodology.

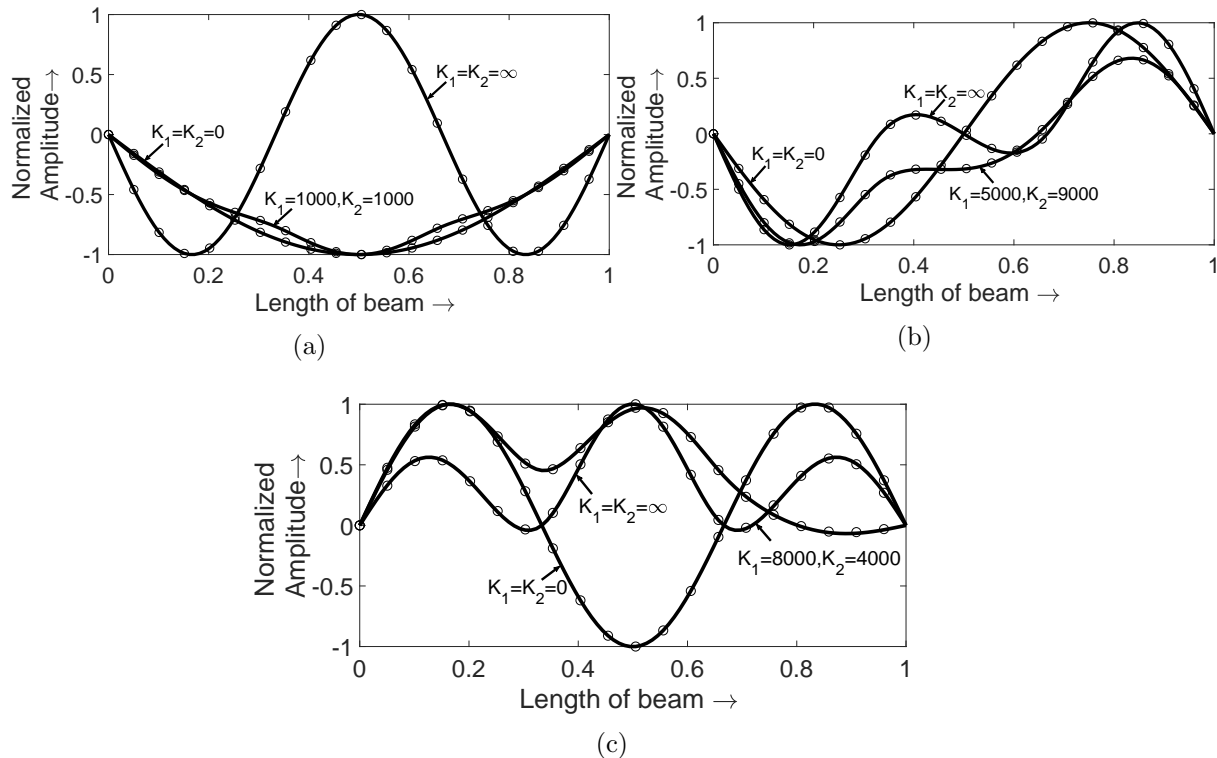


Figure 8: The first three mode shape of simply supported beam with springs attached at  $\zeta_1 = \frac{1}{3}$  and  $\zeta_2 = \frac{2}{3}$ , for different values of  $K_1, K_2$  key: present method —, FEM  $\circ$

Table. 3 presents the first four non dimensional natural frequencies obtained using the present method and FEM. Four admissible functions are generated from the extreme values of each spring. The effect of the variation of these non-dimensional stiffnesses on the frequency parameter is presented in fig. 9. It is observed that surface has four flat regions, with the first flat region associated with the lowest frequency indicating the negligible effect of both springs stiffnesses. The second and third flat regions represent either  $K_1$  or  $K_2$  being very stiff and both region have the same frequency. The fourth region is the region of highest frequency corresponds to both  $K_1$  and  $K_2$  being very stiff.

Table 3: The first four non-dimensional frequency parameter  $\beta_i$ 's of a beam mounted on two spring at  $\zeta_1 = \frac{1}{3}$  and  $\zeta_2 = \frac{2}{3}$ .

| Stiffness       | Mode 1 ( $\beta_1$ ) |       | Mode 2 ( $\beta_2$ ) |        | Mode 3 ( $\beta_3$ ) |        | Mode 4 ( $\beta_4$ ) |        |
|-----------------|----------------------|-------|----------------------|--------|----------------------|--------|----------------------|--------|
|                 | Present Method       | FEM   | Present Method       | FEM    | Present Method       | FEM    | Present Method       | FEM    |
| K1=0,K2=0       | 3.141                | 3.140 | 6.283                | 6.273  | 9.424                | 9.393  | 12.566               | 12.493 |
| K1=0,K2=1e12    | 5.334                | 5.326 | 9.424                | 9.393  | 11.143               | 11.073 | 14.773               | 14.639 |
| K1=1e12,K2=0    | 5.334                | 5.326 | 9.424                | 9.393  | 11.143               | 11.073 | 14.773               | 14.639 |
| K1=40,K2=1e3    | 5.247                | 5.242 | 7.810                | 7.797  | 9.424                | 9.393  | 12.770               | 12.698 |
| K1=2e3,K2=800   | 7.351                | 7.353 | 8.725                | 8.705  | 9.424                | 9.393  | 13.133               | 13.065 |
| K1=500,K2=8e3   | 6.876                | 6.869 | 9.424                | 9.374  | 10.369               | 10.320 | 14.003               | 13.917 |
| K1= 7e4,K2=4e4  | 9.424                | 9.393 | 10.575               | 10.516 | 12.551               | 12.436 | 18.849               | 18.608 |
| K1=1e12,K2=1e12 | 9.424                | 9.393 | 10.669               | 10.608 | 12.892               | 12.763 | 18.849               | 18.608 |

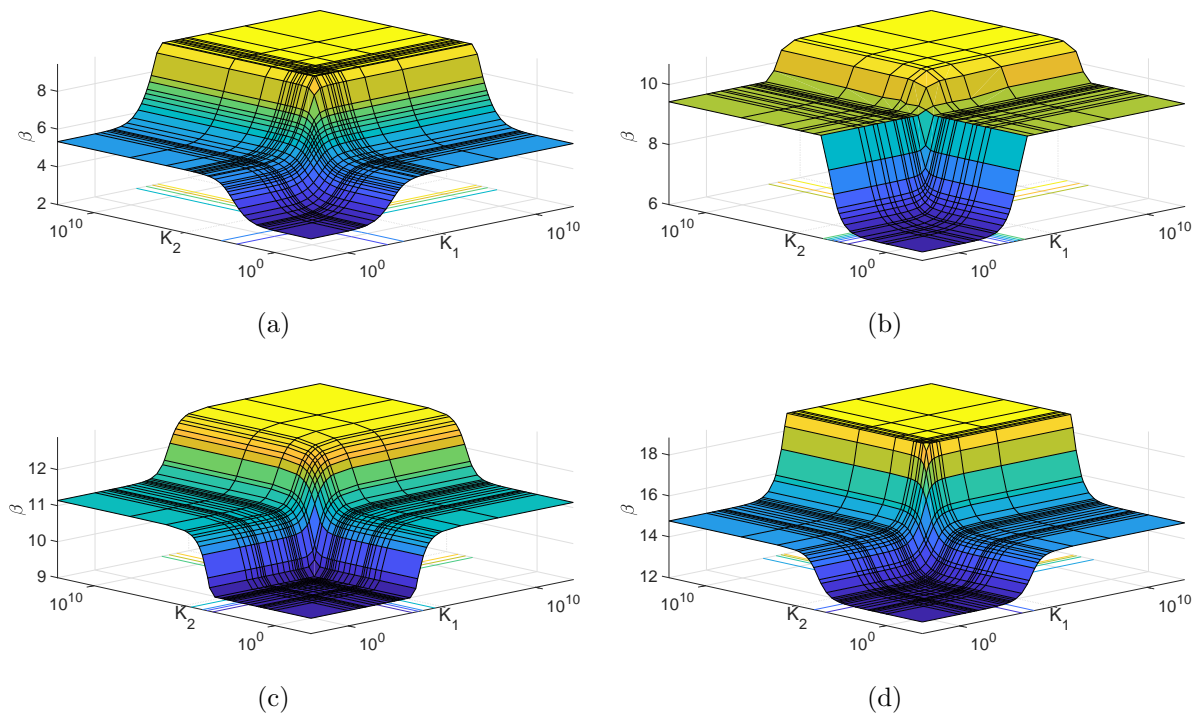


Figure 9: Variation of the natural frequency parameter ( $\beta_i$ ) as  $K_1$  and  $K_2$  varies from 0 to  $\infty$ . (a)  $\beta_1$ , (b)  $\beta_2$ , (c)  $\beta_3$ , (d)  $\beta_4$ .

#### 4. Conclusion

The transverse vibration of a spring-mounted beam is studied. A novel method to determine the eigensolutions based on the basis functions generated from the two extreme values of each spring stiffness has been proposed. The proposed method is much simpler to use and removes the need for solving the transcendental equation for every stiffness value as in the case of Green's function approach. Design guideline equations are presented for a spring mounted at middle and one third of beam length; these enables in estimating the first four natural frequencies efficiently.

#### 5. References

- [1] Nicholson J W and Bergman L A 1986 *Journal of engineering mechanics* **112** 1–13
- [2] Cha P and Wong W 1999 *Journal of Sound and Vibration* **219** 689–706
- [3] Lau J 1981 *Journal of Sound Vibration* **78** 154–157
- [4] Mohamad A 1994 *International journal of solids and structures* **31** 257–268
- [5] Wu J S and Lin T L 1990 *Journal of Sound and Vibration* **136** 201–213
- [6] Dowell E 1979 *Journal of Applied Mechanics* **46**
- [7] Gürgöze M 1996 *Journal of Sound and Vibration* **190** 149–162
- [8] Gürgöze M 1998 *Journal of Sound Vibration* **217** 585–595
- [9] Lin H P and Yang D 2013 *Journal of Mechanics* **29** 143–155
- [10] Rezaiee-Pajand M and Hozhabrossadati S M 2016 *Journal of Vibration and Control* **22** 3004–3017
- [11] Inceo G S and Gurgoze M 2001 *Journal of Sound and Vibration* **243** 370–379