An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 44 (2002) 1297–1306
www.elsevier.com/locate/camwa

Exact Transient Analysis of a Circulant Queuing Network

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(Received April 2001; revised and accepted January 2002)

Abstract—Circulant matrices possess unusual and interesting properties. These properties have been exploited to obtain the transient solution in closed form for a circulant queuing network that models a distributed query processing system. The sojourn time of a customer in the circulant queuing network is determined. A semi-Markov generalisation of this network is also studied. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Queuing networks, Circulants, Sojourn times, Semi-Markov process.

1. INTRODUCTION

Queuing networks are extensively used to analyse and design computer, communication, manufacturing, and transportation systems [1,2]. In these models, *jobs or customers* move through various *stations* of the network, queuing up in a *buffer* at each station until they are processed by the server. Queuing networks have been analysed and solved under different assumptions and constraints. Various classes of queuing networks have product form solutions, and efficient algorithms have been developed [3,4] to exactly study their performance. Jackson networks belong to this class of product-form networks wherein the stationary solution of the network is expressed in the form of a product of the marginal distributions of each node.

In this paper, we consider a circulant queuing network where the nodes are placed in a circular manner. The customers arrive to the system through any of the nodes, and the customers are routed probabilistically to other nodes or out of the system and these transition probabilities of the Markov chain form the routing matrix of the network. This routing matrix has a specific form called a circulant matrix. Such a matrix possesses a rich mathematical structure [5], and this

P. R. Parthasarathy thanks École Polytechnique Fédérale de Lausanne (EPFL), Switzerland, and N. Selvaraju is grateful to the Council for Scientific and Industrial Research (CSIR), India, for financial assistance during the preparation of this paper.

PII: S0898-1221(02)00257-2

makes it possible to derive the transient measures explicitly for the queuing network. Circulant matrices have many connections with problems in physics, image processing, probability and statistics, communication systems, number theory, and geometry.

As an application of this circulant queuing network, we consider a distributed query processing system wherein data are horizontally partitioned, vertically partitioned, and/or mirrored among a group of distributed servers. A typical query processing requires data access from multiple servers. In an unstructured distributed data processing environment over the Internet, it may be required to search all the servers, in the group, to service a given query which is often expensive when the group size increases. However, an important observation is that a typical querying user is often interested in partial data rather than complete data. In such environments, probabilistic query processing plays a vital role wherein the query is processed by a subset of servers selected at random rather than by all the servers. As a result, the outcome of a query is either no data, partial data, or complete data depending on the availability of the requested data at these servers. The servers can be modelled as nodes of a queuing network and can be analysed to obtain various performance measures such as the query processing time.

Our main objective is to study the time-dependent behaviour of this circulant queuing network. Studying the transient behaviour of queuing systems analytically is usually difficult. Although in the study of queuing systems and networks, the emphasis had been on obtaining steady-state solutions, in many potential applications, steady-state measures of system performance simply do not make sense when the practitioner needs to know how the system will operate up to some specified time. Transient solutions are available for a wider class of problems and contribute to a more fine tuned analysis of the costs and benefits of the systems. For example, when buffers are allocated in real time by a central processor, the equilibrium distribution of buffer content may be used to determine the required number of buffers, but the fluctuations will determine the load on the central processor for buffer allocation [6]. Many types of applications in computer and communication systems require time-dependent analysis [7].

The rest of the paper is organised as follows. Section 2 gives some properties of the circulants, Section 3 obtains the transient solution for the circulant network, Section 4 computes sojourn time of a customer in the network, and finally in Section 5, a generalisation of this network is studied.

2. CIRCULANTS

The built-in periodicity indicates that circulants tie in with Fourier analysis and group theory. Practically every matrix theoretic question involving circulants may be resolved in a closed form [5,8]. For example, in the study of time series, it is often desirable to be able to diagonalize a covariance matrix in a simple manner [9].

A circulant of order N is of the form

$$\mathbf{A} = \operatorname{circ}(a_0, a_1, \dots, a_{N-1}) \equiv \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \cdots & a_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}.$$

Due to their inherent pattern, circulants possess a number of interesting properties. If $\mathbf{A}$ and $\mathbf{B}$ are circulants of order N and α_k are scalars, then $\mathbf{A}^\top$, $\mathbf{A}^*$ (conjugate transpose of $\mathbf{A}$), $\alpha_1\mathbf{A}+\alpha_2\mathbf{B}$, $\mathbf{AB}$, and $\sum_{k=0}^{r} \alpha_k \mathbf{A}^k$ are circulants. Moreover, $\mathbf{A}$ and $\mathbf{B}$ commute. If $\mathbf{A}$ is nonsingular, its inverse is a circulant.

Let $w = \exp(2\pi i/N) = \cos 2\pi/N + i \sin 2\pi/N$ be a primitive N^{th} root of unity. We use the following properties in our analysis: $w^N = 1$, $w\bar{w} = 1$, $\bar{w}^k = w^{-k} = w^{N-k}$, $1 + w + w^2 + \cdots +$

$w^{N-1} = 0$, and if N is even, $w^{N/2} = -1$. Also,

$$\sum_{j=0}^{N-1} w^{(n-k)j} = \begin{cases} N, & k = n, \\ 0, & k \neq n. \end{cases} \tag{2.1}$$

Associate with the N-tuple $(a_0, a_1, \ldots, a_{N-1})$ the polynomial, known as the *representer polynomial* of the circulant,

$$p(x) = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}.$$

Then the eigenvalues of $\mathbf{A}$ are $p(1), p(w), p(w^2), \ldots, p(w^{N-1})$.

Circulants have an interesting diagonalization in terms of Fourier matrices, *viz.*,

$$\mathbf{A} = \mathbf{F}^* \mathbf{\Lambda} \mathbf{F}, \tag{2.2}$$

where $\mathbf{\Lambda} = \operatorname{diag}(p(1), p(w), p(w^2), \ldots, p(w^{N-1}))$, and

$$\mathbf{F}^* = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2N-2} & \cdots & w^{(N-1)^2} \end{pmatrix} \tag{2.3}$$

$$= \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{N-2} & \cdots & w \end{pmatrix}. \tag{2.4}$$

Observe that $\mathbf{F}$ is symmetric and unitary, i.e., $\mathbf{F}^{-1} = \mathbf{F}^*$. Conversely, if $\mathbf{\Lambda} = \operatorname{diag}(a_0, a_1, \ldots, a_{N-1})$, then $\mathbf{F}^* \mathbf{\Lambda} \mathbf{F}$ is a circulant, and this may be useful to study inverse problems.

3. TRANSIENT SOLUTION

We consider an open Markovian network where the N nodes, numbered as $0, 1, 2, \ldots, N-1$, are placed on a circle to which the customers arrive, which then move from node to node or out of system according to the following continuous time Markov chain: a customer at any node at time t will have moved to a node that is *j steps to it clockwise*, for $j = 1, 2, \ldots, N-1$, by time $t + \Delta t$ with probability $\lambda_j \Delta t + o(\Delta t)$ (moving j steps clockwise is the same as moving $N - j$ steps counterclockwise), and a customer at node i at time t will have moved out of the system by time $t + \Delta t$ with probability $\mu \Delta t + o(\Delta t)$. Suppose that the system starts with a_i customers at node i at time zero, and in addition, suppose that new customers arrive at the system as a Poisson stream with rate α_i at node i and let $\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_{N-1}$. We assume that the capacities of the nodes are infinite and all the customers in the system are assumed to behave independently of one another in the sense that the original customers present in the system at the initial time zero will subsequently behave independently of all the new customers who will enter the system after time zero. The analysis is similar to the one used by Purdue [10] for stochastic compartmental models.

The solution for the state of the system at time $t (> 0)$ can be written as a sum of two independent random vectors,

$$\begin{bmatrix} Z_0(t) \\ Z_1(t) \\ \vdots \\ Z_{N-1}(t) \end{bmatrix} = \begin{bmatrix} X_0(t) \\ X_1(t) \\ \vdots \\ X_{N-1}(t) \end{bmatrix} + \begin{bmatrix} Y_0(t) \\ Y_1(t) \\ \vdots \\ Y_{N-1}(t) \end{bmatrix}, \tag{3.1}$$

where $Z_i(t)$ is the *total* number of customers at node i at time t, $X_i(t)$ is the number of *original* customers at node i at time t, and $Y_i(t)$ is the number of *new* customers (those who entered the system after initial time) at node i at time t.

First, let us determine the distribution of $X_i(t)$. Let $X_{ij}(t)$ be the number of customers which were at node i at time zero and which are at node j at time t. Then

$$X_j(t) = \sum_{i=0}^{N-1} X_{ij}(t), \qquad j = 0, 1, 2, \dots, N-1,$$

with $X_{ij}(t)$ being binomial with parameters a_i and $P_{ij}(t)$, where $P_{ij}(t)$ is the probability that a customer who was at i at time zero is at j at time t.

Next, we have to determine the distribution of $Y_j(t)$. If $P_j(t)$ is the probability that a customer is, t units of time after having entered the system, at node j, then

$$P_j(t) = \sum_{i=0}^{N-1} \frac{\alpha_i}{\alpha} P_{ij}(t).$$

If we let

$$M_j(t) = \alpha \int_0^t P_j(u)\, du,$$

then $Y_0(t), Y_1(t), \dots, Y_{N-1}(t)$ are independent and $Y_j(t)$ has a Poisson distribution with mean $M_j(t)$.

The major difficulty is the determination of $P_{ij}(t)$. The $P_{ij}(t)$s when considered all together satisfy the matrix differential equations

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}(t)\mathbf{U},$$

where $\mathbf{P}(t)$ is the $N \times N$ matrix of $P_{ij}(t)$s (with $\mathbf{P}(0) = \mathbf{I}$) and the other matrix $\mathbf{U}$ being defined as

$$\mathbf{U} = \begin{bmatrix} \lambda_0 - \mu & \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} \\ \lambda_{N-1} & \lambda_0 - \mu & \lambda_1 & \cdots & \lambda_{N-2} \\ \lambda_{N-2} & \lambda_{N-1} & \lambda_0 - \mu & \cdots & \lambda_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_0 - \mu \end{bmatrix}, \tag{3.2}$$

with λ_0 defined implicitly by $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \cdots + \lambda_{N-1} = 0$. The solution is given by

$$\begin{aligned} \mathbf{P}(t) &= e^{\mathbf{U}t} \\ &= e^{-\mu t}\mathbf{F}^* \operatorname{diag}\left(1, e^{p(w)t}, e^{p(w^2)t}, \dots, e^{p(w^{N-1})t}\right)\mathbf{F} \\ &= \left(\left(\frac{1}{N} e^{-\mu t} \sum_{k=0}^{N-1} w^{(i-j)k} e^{p(w^k)t}\right)\right), \end{aligned}$$

where the polynomial $p(x)$ is defined as

$$p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_{N-1} x^{N-1}.$$

Thus,

$$\begin{aligned} P_{ij}(t) &= \frac{1}{N} e^{-\mu t} \sum_{k=0}^{N-1} w^{(i-j)k} e^{p(w^k)t} \\ &= \frac{1}{N} e^{-\mu t} \left[1 + S_{ij}(t)\right], \qquad \text{using } p(1) = 0, \end{aligned} \tag{3.3}$$

where

$$S_{ij}(t) = \sum_{k=1}^{N-1} w^{(i-j)k} e^{p(w^k)t}.$$

Putting $t = 0$ in (3.3), we retrieve the initial condition using (2.1).

The probabilities in (3.3) are indeed real. If N is odd,

$$\begin{aligned} S_{ij}(t) &= \left(\sum_{k=1}^{(N-1)/2} + \sum_{k=(N+1)/2}^{N-1} \right) w^{(i-j)k} e^{p(w^k)t} \\ &= \sum_{k=1}^{(N-1)/2} \left(w^{(i-j)k} e^{p(w^k)t} + \bar{w}^{(i-j)k} e^{p(\bar{w}^k)t} \right) \\ &= 2 \sum_{k=1}^{(N-1)/2} \mathrm{Rl}\left[w^{(i-j)k} e^{p(w^k)t} \right] \\ &= 2 \sum_{k=1}^{(N-1)/2} \cos\left[\frac{2\pi(i-j)k}{N} + t \sum_{r=0}^{N-1} \lambda_r \sin\frac{2\pi kr}{N} \right] \exp\left[t \sum_{r=0}^{N-1} \lambda_r \cos\frac{2\pi kr}{N} \right]. \end{aligned}$$

If N is even, there will be another term without trigonometric functions, and thus,

$$\begin{aligned} S_{ij}(t) &= \left(\sum_{k=1}^{N/2-1} + \sum_{k=N/2+1}^{N-1} \right) w^{(i-j)k} e^{p(w^k)t} + w^{(i-j)N/2} e^{p(w^{N/2})t} \\ &= 2 \sum_{k=1}^{N/2-1} \cos\left[\frac{2\pi(i-j)k}{N} + t \sum_{r=0}^{N-1} \lambda_r \sin\frac{2\pi kr}{N} \right] \exp\left[t \sum_{r=0}^{N-1} \lambda_r \cos\frac{2\pi kr}{N} \right] + (-1)^{i-j} e^{P(-1)t}, \end{aligned}$$

where $p(-1) = \lambda_0 - \lambda_1 + \lambda_2 - \cdots - \lambda_{N-1}$ and we have used $w^{N/2} = -1$.

It is interesting to note that the complete solution can be written in terms of the probabilities $P_{ij}(t)$, $i, j = 0, 1, 2, \ldots, N-1$. The steady-state distribution of the system is

$$\lim_{t\to\infty} \Pr\left[\mathbf{Z}(t) = \mathbf{n}\right] = \prod_{i=0}^{N-1} e^{-\theta} \frac{\theta^{n_i}}{n_i!},$$

where $\mathbf{n} = [n_0, n_1, \ldots, n_{N-1}]^{\top}$ and $\theta = \alpha/(\mu + \lambda_1 + \lambda_2 + \cdots + \lambda_{N-1})$.

4. SOJOURN TIME OF A CUSTOMER

The sojourn time of a customer in queuing networks is a vital performance measure as it may represent, for example, the query processing time in the circulant queuing network. In this section, we find the probability distribution that a typical customer having entered the system stays within the network at time t using the concept of phase distributions.

A continuous *phase distribution* is the distribution of time until absorption in an absorbing Markov process [11]. If we consider the movement of a customer in the above analysis, we can see that eventually the customer leaves the system. If we introduce a state, say N, representing the absorption state, then the states $0, 1, 2, \ldots, N-1$ are transient states. To calculate an equation for the distribution function of a phase distribution, let the generator matrix be partitioned as

$$\left[\begin{array}{c|c} \mathbf{U} & \mathbf{A} \\ \hline \mathbf{0} & 0 \end{array} \right],$$

where $\mathbf{U}$ is given in (3.2), and $\mathbf{A}$ is the $N \times 1$ vector of all μ. If $\bar{\mathbf{P}}(0) = [\beta_0, \beta_1, \dots, \beta_{N-1}, \beta_N] = [\beta, \beta_N]$ and $\bar{\mathbf{P}}(t) = [P_0(t), P_1(t), \dots, P_{N-1}(t), P_N(t)]$, then the solution to the Kolmogorov forward equations can be written as

$$\bar{\mathbf{P}}(t) = \beta e^{\mathbf{U}t}.$$

The probability that the customer has not gone out by time t is given by $\bar{\mathbf{P}}(t)\mathbf{e}$. We can write the distribution function of the absorption time, a random variable denoted by X, for the transient matrix $\mathbf{U}$ as

$$\begin{aligned} F_X(t) &= 1 - \bar{\mathbf{P}}(t)\mathbf{e} = 1 - \beta e^{\mathbf{U}t}\mathbf{e} \\ &= 1 - \sum_{i=0}^{N-1} \beta_i \sum_{j=0}^{N-1} P_{ij}(t), \end{aligned}$$

which can be rewritten as

$$F_X(t) = 1 - \sum_{i=0}^{N-1} \beta_i \sum_{j=0}^{N-1} \frac{1}{N} e^{-\mu t} \sum_{k=0}^{N-1} w^{(i-j)k} e^{p(w^k)t}.$$

5. A SEMI-MARKOV GENERALISATION

One of the major disadvantages of the time-homogeneous open Markovian network model discussed in previous sections is that the time spent in each node of the network is an exponential random variable. It is desirable to study a more general setup in which the delays at the nodes are general random variables. Such a degree of generality is introduced by using semi-Markov processes (see [12] for basic definitions and properties).

We consider a queuing network with the same setup as in the previous sections except that here the sojourn time of a customer at the nodes follows a general distribution. On entering the network, a customer's progress is assumed to be a semi-Markov process. A customer who leaves the network is said to be in node N and may not reenter.

For a given customer, let X_n denote the node entered at the n^{th} transition of the customer within the network, and let T_n be the epoch of this transition. Then we have by the assumption

$$\begin{aligned} &\Pr\{X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_0, X_1, T_1, \dots, X_n = i, T_n\} \\ &= \Pr\{X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_n = i\} \\ &= A_{ij}(t), \qquad \text{(say) } 0 \le i,\ j \le N. \end{aligned}$$

The matrix $\mathbf{A}(t)$ with $(i,j)^{\text{th}}$ element $A_{ij}(t)$ is the semi-Markov matrix associated with the network. We assume that

$$A_{ij}(t) = \begin{cases} \theta_{j-i} H(t), & i < j \text{ and } i, j = 0, 1, 2, \dots, N-1, \\ \theta_{N+j-i} H(t), & i > j \text{ and } i, j = 0, 1, 2, \dots, N-1, \\ \left(1 - \sum_{m=1}^{N-1} \theta_m\right) H(t), & i = 0, 1, 2, \dots, N-1 \text{ and } j = N, \\ 0, & i = N \text{ and } j = 0, 1, 2, \dots, N-1, \\ H(t), & i = N \text{ and } j = N, \end{cases}$$

where $H(t)$ is the distribution function of the length of time the customer spends at a node and θ_m is the conditional probability of moving m steps to the right given that the customer leaves the present node.

Consider now a customer who enters the system at time zero and let $P_{ij}(t)$ be defined, as before, to be the probability that the customer who was at node i at time zero is at node j at time t. Then we have [12]

$$P_{ij}(t) = \int_0^t [1 - H(t-x)] R_{ij}(dx),$$

where $R_{ij}(x)$ is the $(i,j)^{\text{th}}$ element of

$$\mathbf{R}(t) = \sum_{n=0}^{\infty} \mathbf{A}^{*(n)}(t),$$

with $\mathbf{A}^{*(n)}(t)$ defined as

$$\mathbf{A}^{*(0)}(t) = \mathbf{I}, \qquad \mathbf{A}^{*(1)}(t) = \mathbf{A}(t), \qquad \mathbf{A}^{*(n)}(t) = \mathbf{A} * \mathbf{A}^{*(n-1)}(t),$$

and

$$\mathbf{A} * B_{ij}(t) = \int_0^t \sum_{l=0}^{N} A_{il}(t-x) B_{lj}\,(dx).$$

In other words,

$$R_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{n}(t),$$

where

$$Q_{ij}^{n}(t) = \Pr\left\{X_n = j,\ T_n - T_{n-1} \le t \mid X_0 = i\right\}.$$

Obviously, $Q_{ij}^{1}(t) = A_{ij}(t)$ and

$$Q_{ij}^{0}(t) = I_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The functions $t \to R_{ij}(t)$ are called Markov renewal functions, and the collection $\{R_{ij},\ i,j = 0,1,2,\dots,N\}$ of these functions is called a Markov renewal kernel.

The matrix $\mathbf{A}(t)$ here is given by

$$\begin{aligned}\mathbf{A}(t) &= H(t)\begin{pmatrix} 0 & \theta_1 & \theta_2 & \cdots & \theta_{N-1} & 1 - \sum\limits_{m=1}^{N-1} \theta_m \\ \theta_{N-1} & 0 & \theta_1 & \cdots & \theta_{N-2} & 1 - \sum\limits_{m=1}^{N-1} \theta_m \\ \theta_{N-2} & \theta_{N-1} & 0 & \cdots & \theta_{N-3} & 1 - \sum\limits_{m=1}^{N-1} \theta_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_1 & \theta_2 & \theta_3 & \cdots & 0 & 1 - \sum\limits_{m=1}^{N-1} \theta_m \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ &= H(t)\begin{pmatrix} \mathbf{C} & \mathbf{d} \\ \mathbf{0} & 1 \end{pmatrix},\end{aligned}$$

where $\mathbf{C}$ is the cyclic matrix of order N given by

$$\mathbf{C} = \begin{pmatrix} 0 & \theta_1 & \theta_2 & \cdots & \theta_{N-1} \\ \theta_{N-1} & 0 & \theta_1 & \cdots & \theta_{N-2} \\ \theta_{N-2} & \theta_{N-1} & 0 & \cdots & \theta_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_1 & \theta_2 & \theta_3 & \cdots & 0 \end{pmatrix},$$

and $\mathbf{d}$ is a column vector. In order to find $\mathbf{R}(t)$, we first have to compute $\mathbf{C}^n$ which, with the help of the representer polynomial $r(x) = \theta_1 x + \theta_2 x^2 + \cdots + \theta_{N-1} x^{N-1}$ of the matrix $\mathbf{C}$, can be simplified to

$$\begin{aligned}\mathbf{C}^n &= \mathbf{F}^* \operatorname{diag}\left((r(1))^n, (r(w))^n, \ldots, \left(r\left(w^{N-1}\right)\right)^n\right) \mathbf{F} \\ &= \frac{1}{N}\left(\left(\sum_{k=0}^{N-1} w^{(i-j)k}\left(r\left(w^k\right)\right)^n\right)\right),\end{aligned}$$

where $\mathbf{F}^*$ and w are as defined earlier. Using this result, we finally deduce that

$$\begin{aligned}\mathbf{P}(t) &= (1 - H(t)) * \mathbf{R}(t) \\ &= \begin{pmatrix} P_{00}(t) & P_{01}(t) & \cdots & P_{0,N-1}(t) & 1 - \sum_{m=1}^{N-1} P_{0m}(t) \\ P_{10}(t) & P_{11}(t) & \cdots & P_{1,N-1}(t) & 1 - \sum_{m=1}^{N-1} P_{1m}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N-1,0}(t) & P_{N-1,1}(t) & \cdots & P_{N-1,N-1}(t) & 1 - \sum_{m=1}^{N-1} P_{N-1,m}(t) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},\end{aligned}$$

where, for $i, j = 0, 1, 2, \ldots, N-1$,

$$P_{ij}(t) = \frac{1}{N}\sum_{k=0}^{N-1} w^{(i-j)k} \sum_{n=0}^{\infty}\left(r\left(w^k\right)\right)^n \int_0^t [1 - H(t-x)] H^{*(n)}(dx). \tag{5.1}$$

The quantities $P_{ij}(t)$ enable us to find the distribution function, $G(t)$ (say), of the time spent in the network by a customer. Accordingly, we get

$$\begin{aligned}G(t) &= \sum_{i=0}^{N-1} \frac{\alpha_i}{\alpha} G_i(t) \\ &= 1 - \sum_{i=0}^{N-1} \frac{\alpha_i}{\alpha} \sum_{j=0}^{N-1} P_{ij}(t).\end{aligned}$$

We let $X(t)$ denote the number of customers in the network at time t. Then, by recognizing $X(t)$ as the number of customers in an $M/G/\infty$ queue with arrival rate α and service time density function $G(t)$, for $t > 0$, we get

$$\Pr\{X(t) = n \mid X(0) = 0\} = \frac{\left[\alpha \int_0^t [1 - G(x)]\, dx\right]^n}{n!} \exp\left\{-\alpha \int_0^t [1 - G(x)]\, dx\right\}, \qquad n = 0, 1, 2, \ldots. \tag{5.2}$$

We now consider the number of customers at node i at time t, say, $X_i(t)$. Let

$$\begin{aligned}P_j(t) &= \sum_{i=0}^{N-1} \frac{\alpha_i}{\alpha} P_{ij}(t), \qquad \text{and} \\ M_j(t) &= \alpha \int_0^t P_j(x)\, dx, \qquad j = 0, 1, 2, \ldots, N-1,\end{aligned}$$

as before. Then, for each $t > 0$, $X_0(t), X_1(t), \ldots, X_{N-1}(t)$ are independent random variables and $X_j(t)$ has a Poisson distribution with parameter $M_j(t)$.

In order to find $\lim_{t\to\infty} \Pr[X(t) = n]$, we note that we must evaluate $\int_0^\infty [1 - G(x)]\,dx$. This, of course, is the mean first passage time to state N which we denote by γ. Let Φ_i denote the mean first passage time to N starting in state i and let Φ_i' denote the mean time spent in node i. Letting $\mathbf{\Phi} = (\Phi_0, \Phi_1, \ldots, \Phi_{N-1})^\top$, $\mathbf{\Phi}' = (\Phi_0', \Phi_1', \ldots, \Phi_{N-1}')^\top$, and

$$\Psi = \begin{pmatrix} 0 & \theta_1 & \theta_2 & \cdots & \theta_{N-1} \\ \theta_{N-1} & 0 & \theta_1 & \cdots & \theta_{N-2} \\ \theta_{N-2} & \theta_{N-1} & 0 & \cdots & \theta_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_1 & \theta_2 & \theta_3 & \cdots & 0 \end{pmatrix},$$

we have $\mathbf{\Phi} = (I - \Psi)^{-1}\mathbf{\Phi}'$, where $[I - \Psi]^{-1}$ exists since the maximal eigenvalue of Ψ is less than 1. In our case, noting that the representer polynomial for the matrix $[I - \Psi]$ is $1 - r(x)$ ($r(x)$ is the representer polynomial of $\mathbf{C}$ defined earlier), we obtain

$$\begin{aligned} [I - \Psi]^{-1} &= \left[\mathbf{F}^* \operatorname{diag}\left(1 - r(1), 1 - r(w), \ldots, 1 - r\left(w^{N-1}\right)\right) \mathbf{F}\right]^{-1} \\ &= \mathbf{F}^* \operatorname{diag}\left(\frac{1}{1 - r(1)}, \frac{1}{1 - r(w)}, \ldots, \frac{1}{1 - r(w^{N-1})}\right) \mathbf{F} \\ &= \frac{1}{N}\left(\left(\sum_{k=0}^{N-1} w^{(i-j)k} \frac{1}{1 - r(w^k)}\right)\right). \end{aligned}$$

Clearly, now $\Phi = \sum_{i=0}^{N-1} (\alpha_i/\alpha)\Phi_i$ and we see, as a corollary to result (5.2),

$$\lim_{t\to\infty} \Pr[X(t) = n] = e^{-\alpha\Phi} \frac{(\alpha\Phi)^n}{n!}, \qquad n = 0, 1, 2, \ldots.$$

We can use this result to determine the mean occupation time of the network. The system is occupied if there is at least one customer present in the system. Let Θ denote the mean occupation time. As a two-state system, we have an alternating renewal process with "on" state having mean Θ and "off" state having mean $1/\alpha$. Then

$$\begin{aligned} \lim_{t\to\infty} \Pr[X(t) = 0] &= \frac{1/\alpha}{\Theta + 1/\alpha} \\ &= e^{-\alpha\Phi}, \end{aligned}$$

which gives $\Theta = (1/\alpha)[e^{\alpha\Phi} - 1]$.

REMARK. If $H(t) = 1 - e^{\sigma t}$, letting $\lambda_i = \sigma\theta_i$, $i = 1, 2, 3, \ldots, N-1$, and $\mu = \sigma(1 - \sum_{m=1}^{N-1} \theta_m)$, it can be seen that this network reduces to the open Markovian network discussed in previous sections.

REFERENCES

1. G. Bolch, S. Greiner, H. de Meer and K.S. Trivedi, *Queueing Networks and Markov Chains*, John Wiley, New York, (1998).
2. J.H. Dshalalow, Editor, *Frontiers in Queueing: Models and Applications in Science and Engineering*, CRC Press, Boca Raton, FL, (1997).
3. N.M. van Dijk, *Queueing Networks and Product Forms: A Systems Approach*, Wiley, Chichester, (1993).
4. M. Lebah and J. Pellaumail, Transient behaviour for some Jackson networks, *Performance Evaluation* **17**, 115–122, (1993).
5. P.J. Davis, *Circulant Matrices*, John Wiley, New York, (1979).
6. W. Whitt, Untold horrors of the waiting room: What the equilibrium distribution will never tell about the queue-length process, *Management Science* **29**, 395–408, (1983).

7. S.K. Tripathi and A. Duda, Time-dependent analysis of queuing systems, *INFOR Journal* **24**, 199–220, (1986).
8. R. Vein and P. Dale, Determinants and their applications, In *Mathematical Physics*, Springer, New York, (1999).
9. P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods*, 2nd edition, Springer-Verlag, Heidelberg, (1991).
10. P. Purdue, Stochastic compartmental models: A review of the mathematical theory with ecological applications, In *Compartmental Analysis of Ecosystem Models*, (Edited by J.H. Matis *et al.*), pp. 223–260, International Cooperative Publishing House, Burtonsville, MD, (1979).
11. M.F. Neuts, *Matrix Geometric Solutions in Stochastic Models: An Algorithmic Approach*, Dover, New York, (1995).
12. E. Çinlar, *Introduction to Stochastic Processes*, Prentice-Hall, Englewood Cliffs, NJ, (1975).