EQUIVARIANT PRINCIPAL BUNDLES AND LOGARITHMIC CONNECTIONS ON TORIC VARIETIES

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Abstract. Let M be a smooth complex projective toric variety equipped with an action of a torus T, such that the complement D of the open T–orbit in M is a simple normal crossing divisor. Let G be a complex reductive affine algebraic group. We prove that an algebraic principal G–bundle $E_G \longrightarrow M$ admits a T–equivariant structure if and only if E_G admits a logarithmic connection singular over D. If $E_H \longrightarrow M$ is a T–equivariant algebraic principal H–bundle, where H is any complex affine algebraic group, then E_H in fact has a canonical integrable logarithmic connection singular over D.

1. Introduction

Our aim here is to give characterizations of the equivariant principal bundles on smooth complex projective toric varieties.

Let M be a smooth complex projective toric variety equipped with an action of a torus T

$$\rho : T \times M \longrightarrow M .$$

For any point $t \in T$, define the automorphism

$$\rho_t : M \longrightarrow M , \ x \longmapsto \rho(t , x) .$$

We assume that the complement D of the open T–orbit in M is a simple normal crossing divisor.

Let G be a complex reductive affine algebraic group, and let E_G be an algebraic principal G–bundle on M. In Proposition 4.1 we prove the following:

The principal G–bundle E_G admits a T–equivariant structure if and only if the pulled back principal G–bundle $\rho_t^* E_G$ is isomorphic to E_G for every $t \in T$.

When $G = \mathrm{GL}(n, \mathbb{C})$, this result was proved by Klyachko [Kl, p. 342, Proposition 1.2.1].

Using the above characterization of T–equivariant principal G–bundles on M, we prove the following (see Theorem 4.2):

The principal G–bundle E_G admits a logarithmic connection singular over D if and only if E_G admits a T–equivariant structure.

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The "if" part of Theorem 4.2 does not require G to be reductive. More precisely, any T–equivariant principal H–bundle $E_H \longrightarrow M$, where H is any complex affine algebraic group, admits a canonical integrable logarithmic connection singular over D (see Proposition 3.2).

2. Equivariant bundles

Let $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ be the multiplicative group. Take a complex algebraic group T which is isomorphic to a product of copies of $\mathbb{G}_m$. Let M be a smooth irreducible complex projective variety equipped with an algebraic action of T

$$\rho : T \times M \longrightarrow M \tag{2.1}$$

such that

- there is a Zariski open dense subset $M^0 \subset M$ with $\rho(T, M^0) = M^0$,
- the action of T on M^0 is free and transitive, and
- the complement $M \setminus M^0$ is a simple normal crossing divisor of M.

In particular, M is a smooth projective toric variety. Note that M^0 is the unique T–orbit in M with trivial isotropy.

Let G be a connected complex affine algebraic group. A T*–equivariant* principal G–bundle on M is a pair $(E_G, \widetilde{\rho})$, where

$$p : E_G \longrightarrow M$$

is an algebraic principal G–bundle, and

$$\widetilde{\rho} : T \times E_G \longrightarrow E_G$$

is an algebraic action of T on the total space of E_G, such that

- $p \circ \widetilde{\rho} = \rho \circ (\mathrm{Id}_T \times p)$, where ρ is the action in (2.1), and
- the actions of T and G on E_G commute.

Fix a point $x_0 \in M^0 \subset M$. Let

$$\iota : \rho(T, x_0) = M^0 \hookrightarrow M \tag{2.2}$$

be the inclusion map. Let $M^0 \times G$ be the trivial principal G–bundle on M^0. It has a tautological integrable algebraic connection given by its trivialization.

Let $(E_G, \widetilde{\rho})$ be a T–equivariant principal G–bundle on M. Fix a point $z_0 \in (E_G)_{x_0}$. Using z_0, the action $\widetilde{\rho}$ produces an isomorphism of principal G–bundles between $M^0 \times G$ and the restriction $E_G|_{M^0}$. This isomorphism of principal G–bundles is uniquely determined by the following two conditions:

- this isomorphism is T–equivariant (the action of T on $M^0 \times G$ is given by the action of T on M^0), and
- it takes the point $z_0 \in E_G$ to $(x_0, e) \in M^0 \times G$.

Using this trivialization of $E_G|_{M^0}$, the tautological integrable algebraic connection on $M^0 \times G$ produces an integrable algebraic connection $\mathcal{D}^0$ on $E_G|_{M^0}$. We note that this connection $\mathcal{D}^0$ is independent of the choice of the points x_0 and z_0. Indeed, the flat sections for $\mathcal{D}^0$ are precisely the orbits of T in $E_G|_{M^0}$. Note that this description of $\mathcal{D}^0$ does not require choosing base points in M^0 and $E_G|_{M^0}$.

In Proposition 3.2 it will be shown that $\mathcal{D}^0$ extends to a logarithmic connection on E_G over M singular over the simple normal crossing divisor $M \setminus M^0$.

3. Logarithmic connections

3.1. A canonical trivialization.

The Lie algebra of T will be denoted by $\mathfrak{t}$. Let

$$\mathcal{V} := M \times \mathfrak{t} \longrightarrow M \tag{3.1}$$

be the trivial vector bundle with fiber $\mathfrak{t}$. The holomorphic tangent bundle of M will be denoted by TM. Consider the action of T on M in (2.1). It produces a homomorphism of $\mathcal{O}_M$–coherent sheaves

$$\beta : \mathcal{V} \longrightarrow TM\,. \tag{3.2}$$

Let

$$D := M \setminus M^0$$

be the simple normal crossing divisor of M. Let

$$TM(-\log D) \subset TM \tag{3.3}$$

be the corresponding logarithmic tangent bundle. We recall that $TM(-\log D)$ is characterized as the maximal coherent subsheaf of TM that preserves $\mathcal{O}_M(-D) \subset \mathcal{O}_M$ for the derivation action of TM on $\mathcal{O}_M$.

Lemma 3.1.

(1) *The image of* β *in* (3.2) *is contained in the subsheaf* $TM(-\log D) \subset TM$.
(2) *The resulting homomorphism* $\beta : \mathcal{V} \longrightarrow TM(-\log D)$ *is an isomorphism.*

Proof. The divisor D is preserved by the action of T on M. Therefore, the action of T on $\mathcal{O}_M$, given by the action of T on M, preserves the subsheaf $\mathcal{O}_M(-D)$. From this it follows immediately that the subsheaf $\mathcal{O}_M(-D) \subset \mathcal{O}_M$ is preserved by the derivation action of the subsheaf

$$\beta(\mathcal{V}) \subset TM\,.$$

Therefore, we conclude that $\beta(\mathcal{V}) \subset TM(-\log D)$.

It is known that the vector bundle $TM(-\log D)$ is holomorphically trivial [Fu, p. 87, Proposition 2]. We note that Proposition 2 of [Fu, p. 87] says that $\Omega^1_M(\log D)$ is holomorphically trivial. But $\Omega^1_M(\log D)^* = TM(-\log D)$, and hence $TM(-\log D)$ is also holomorphically trivial.

So, both $\mathcal{V}$ and $TM(-\log D)$ are trivial vector bundles, and β is a homomorphism between them which is an isomorphism over the open subset M^0. From this it can be deduced that β is an isomorphism over entire M. To see this, consider the homomorphism

$$\bigwedge^r \beta \,:\, \bigwedge^r \mathcal{V} \longrightarrow \bigwedge^r TM(-\log D)$$

induced by β, where $r = \dim_{\mathbb{C}} T = \text{rank}(\mathcal{V})$. So $\bigwedge^r \beta$ is a holomorphic section of the line bundle $(\bigwedge^r TM(-\log D)) \bigotimes (\bigwedge^r \mathcal{V})^*$. This line bundle $(\bigwedge^r TM(-\log D)) \bigotimes (\bigwedge^r \mathcal{V})^*$ is holomorphically trivial because both $\mathcal{V}$ and $TM(-\log D)$ are holomorphically trivial. Fixing a trivialization of $(\bigwedge^r TM(-\log D)) \bigotimes (\bigwedge^r \mathcal{V})^*$, we consider $\bigwedge^r \beta$ as a holomorphic function on M. This function is nowhere vanishing because it does not vanish on M^0 and holomorphic functions on M are constants. Since $\bigwedge^r \beta$ is nowhere vanishing, the homomorphism β is an isomorphism. □

3.2. **A canonical logarithmic connection on equivariant bundles.** The Lie algebra of G will be denoted by $\mathfrak{g}$.

Let $p : E_G \longrightarrow M$ be an algebraic principal G–bundle. Consider the differential

$$dp \,:\, TE_G \longrightarrow p^*TM\,, \tag{3.4}$$

where TE_G is the algebraic tangent bundle of E_G. The kernel of dp will be denoted by $T_{E_G/M}$. Using the action of G on E_G, this subbundle $T_{E_G/M} \subset TE_G$ is identified with the trivial vector bundle over E_G with fiber $\mathfrak{g}$.

The action of G on E_G produces an action of G on TE_G. So we get an action of G on the quasicoherent sheaf p_*TE_G on M. The invariant part

$$\text{At}(E_G) \,:=\, (p_*TE_G)^G \,\subset\, p_*TE_G$$

is a locally free coherent sheaf; its coherence property follows from the fact that the action of G on the fibers of p is transitive, implying that a G–invariant section of $(TE_G)|_{p^{-1}(x)}$, $x \in M$, is uniquely determined by its evaluation at just one point of the fiber $p^{-1}(x)$. Also note that $\text{At}(E_G) = (TE_G)/G$. This $\text{At}(E_G)$ is known as the *Atiyah bundle* for E_G. Since $T_{E_G/M}$ is identified with $E_G \times \mathfrak{g}$, the invariant direct image $(p_*T_{E_G/M})^G$ is identified with the adjoint vector bundle

$$\text{ad}(E_G) \,:=\, E_G \times^G \mathfrak{g} \longrightarrow M$$

associated to E_G for the adjoint action of G on $\mathfrak{g}$. We note that $\text{ad}(E_G) = T_{E_G/M}/G$. Now the differential dp in (3.4) produces a short exact sequence of holomorphic vector bundles on M

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \stackrel{\phi}{\longrightarrow} TM \longrightarrow 0\,, \tag{3.5}$$

which is known as the Atiyah exact sequence. A holomorphic connection on E_G over M is a holomorphic splitting

$$TM \longrightarrow \text{At}(E_G)$$

of (3.5) [At].

As before, setting $D = M \setminus M^0$, define

$$\mathrm{At}(E_G)(-\log D) := \phi^{-1}(TM(-\log D)) \subset \mathrm{At}(E_G)\,,$$

where ϕ is the projection in (3.5) and $TM(-\log D)$ is the subsheaf in (3.3). So (3.5) gives the following short exact sequence of holomorphic vector bundles on M

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G)(-\log D) \stackrel{\phi}{\longrightarrow} TM(-\log D) \longrightarrow 0\,. \tag{3.6}$$

A *logarithmic connection* on E_G, with singular locus D, is a holomorphic homomorphism

$$\delta : TM(-\log D) \longrightarrow \mathrm{At}(E_G)(-\log D)$$

such that $\phi \circ \delta$ is the identity automorphism of $TM(-\log D)$, where ϕ is the homomorphism in (3.6). Just like the curvature of a connection, the curvature of a logarithmic connection δ on E_G is the obstruction for the homomorphism δ to preserve the Lie algebra structure of the sheaf of sections of $TM(-\log D)$ and $\mathrm{At}(E_G)(-\log D)$ given by the Lie bracket of vector fields. In particular, δ is called *integrable* (or *flat*) if it preserves the Lie algebra structure of the sheaf of sections of $TM(-\log D)$ and $\mathrm{At}(E_G)(-\log D)$ given by the Lie bracket of vector fields.

Proposition 3.2. *Let $(E_G\,,\widetilde{\rho})$ be a T–equivariant principal G–bundle on M. Then E_G admits an integrable logarithmic connection that restricts to the connection $\mathcal{D}^0$ on M^0 constructed in Section 2.*

Proof. Let

$$\widetilde{\mathcal{V}} := E_G \times \mathfrak{t} \longrightarrow E_G$$

be the trivial vector bundle over E_G with fiber $\mathfrak{t}$. Note that $p^*\mathcal{V} = \widetilde{\mathcal{V}}$, where $\mathcal{V}$ is the vector bundle in (3.1), and p, as before, is the projection of E_G to M.

The action $\widetilde{\rho}$ of T on E_G produces a homomorphism

$$\widetilde{\beta} : \widetilde{\mathcal{V}} \longrightarrow TE_G\,. \tag{3.7}$$

Since $p^{-1}(D)$ is preserved by the action of T on E_G, the induced action of T on $\mathcal{O}_{E_G}$ preserves the subsheaf $\mathcal{O}_{E_G}(-p^{-1}(D))$. Hence the image of $\widetilde{\beta}$ lies inside the subsheaf

$$TE_G(-\log p^{-1}(D)) \subset TE_G\,.$$

Note that $p^{-1}(D)$ is a simple normal crossing divisor on E_G because D is a simple normal crossing divisor on M.

In Lemma 3.1(2) we saw that β is an isomorphism. Consider

$$p^*\beta^{-1} : p^*(TM(-\log D)) \longrightarrow p^*\mathcal{V} = \widetilde{\mathcal{V}}\,.$$

Pre-composing this with $\widetilde{\beta}$ in (3.7), we have

$$\widetilde{\beta} \circ (p^*\beta^{-1}) : p^*(TM(-\log D)) \longrightarrow TE_G(-\log p^{-1}(D))\,.$$

We observe that the homomorphism $\widetilde{\beta} \circ (p^*\beta^{-1})$ is G–equivariant for the trivial action of G on $p^*(TM(-\log D))$ and the action of G on $TE_G(-\log p^{-1}(D))$ induced by the action of G on E_G. Therefore, taking the G–invariant parts of the direct images by p, the above homomorphism $\widetilde{\beta}\circ(p^*\beta^{-1})$ produces a homomorphism

$$\beta' \,:\, TM(-\log D) \,=\, (p_*p^*(TM(-\log D)))^G$$
$$\longrightarrow\, (p_*TE_G(-\log p^{-1}(D)))^G \,=\, \mathrm{At}(E_G)(-\log D)\,.$$

It is now straightforward to check that the above homomorphism β' produces a holomorphic splitting of the exact sequence in (3.6). Therefore, β' defines a logarithmic connection on E_G singular on D. The restriction of this logarithmic connection to M^0 clearly coincides with the connection $\mathcal{D}^0$ constructed in Section 2. □

4. A criterion for equivariance

For each point $t\in T$, define the automorphism

$$\rho_t \,:\, M \longrightarrow M\,,\ x\longmapsto \rho(t\,,x)\,,$$

where ρ is the action in (2.1). If $(E_G\,,\widetilde{\rho})$ is a T–equivariant principal G–bundle on M, then clearly the map

$$E_G \longrightarrow E_G\,,\ z\longmapsto \widetilde{\rho}(t\,,z)$$

is an isomorphism of the principal G–bundle $\rho_t^*E_G$ with E_G. The aim in this section is to prove a converse of it.

Take an algebraic principal G–bundle

$$p\,:\, E_G \longrightarrow M\,.$$

Let $\mathcal{G}$ be the set of all pairs of the form $(t\,,f)$, where $t\in T$ and

$$f\,:\, E_G \longrightarrow E_G$$

is an algebraic automorphism of the variety E_G satisfying the following two conditions:

(1) $p\circ f \,=\, \rho_t\circ p$, and
(2) f intertwines the action of G on E_G.

Note that the above two conditions imply that f is an algebraic isomorphism of the principal G–bundle $\rho_t^*E_G$ with E_G.

We have the following composition on the above defined set $\mathcal{G}$:

$$(t_1\,,f_1)\cdot(t_2\,,f_2) \,:=\, (t_1\circ t_2\,,f_1\circ f_2)\,.$$

The inverse of $(t\,,f)$ is $(t^{-1}\,,f^{-1})$. These operations make $\mathcal{G}$ a group. In fact, $\mathcal{G}$ has the structure of an affine algebraic group defined over $\mathbb{C}$. Let $\mathcal{A}$ denote the group of all algebraic automorphisms of the principal G–bundle E_G. So $\mathcal{A}$ is a subgroup of $\mathcal{G}$ with the inclusion map being $f\longmapsto (e\,,f)$. We have a natural projection

$$h\,:\,\mathcal{G}\longrightarrow T\,,\quad (t\,,f)\longmapsto t$$

which fits in the following exact sequence of complex affine algebraic groups:

$$0\longrightarrow \mathcal{A}\longrightarrow \mathcal{G}\stackrel{h}{\longrightarrow} T\,. \tag{4.1}$$

We note that there is a tautological action of $\mathcal{G}$ on E_G; the action of any $(t\,,f)\,\in\,\mathcal{G}$ on E_G is given by the map defined by $y\,\longmapsto\, f(y)$.

Now assume that E_G satisfies the condition that for every $t\,\in\, T$, the pulled back principal G–bundle $\rho_t^*E_G$ is isomorphic to E_G. This assumption is equivalent to the statement that the homomorphism h in (4.1) is surjective.

In view of the above assumption, the sequence in (4.1) becomes the following short exact sequence of complex affine algebraic groups

$$0\,\longrightarrow\,\mathcal{A}\,\longrightarrow\,\mathcal{G}\,\stackrel{h}{\longrightarrow}\,T\,\longrightarrow\,0\,. \tag{4.2}$$

Let $\mathcal{G}^0\,\subset\,\mathcal{G}$ be the connected component containing the identity element. Since T is connected and h is surjective, the restriction of h to $\mathcal{G}^0$ is also surjective. Therefore, from (4.2) we have the following short exact sequence of affine complex algebraic groups

$$0\,\longrightarrow\,\mathcal{A}^0\,\stackrel{\iota_{\mathcal{A}}}{\longrightarrow}\,\mathcal{G}^0\,\stackrel{h^0}{\longrightarrow}\,T\,\longrightarrow\,0\,, \tag{4.3}$$

where $\mathcal{A}^0\,:=\,\mathcal{A}\bigcap\mathcal{G}^0$, and $h^0\,:=\,h|_{\mathcal{G}^0}$.

Take a maximal torus $T_{\mathcal{G}}\,\subset\,\mathcal{G}^0$. From (4.3) it follows that the restriction

$$h'\,:=\,h^0|_{T_{\mathcal{G}}}\,:\,T_{\mathcal{G}}\,\longrightarrow\,T$$

is surjective. Define $T_{\mathcal{A}}\,:=\,\mathcal{A}^0\bigcap T_{\mathcal{G}}\,\subset\,T_{\mathcal{G}}$ using the homomorphism $\iota_{\mathcal{A}}$ in (4.3). Therefore, from (4.3) we have the following short exact sequence of algebraic groups

$$0\,\longrightarrow\,T_{\mathcal{A}}\,\stackrel{\iota_{\mathcal{A}}|_{T_{\mathcal{A}}}}{\longrightarrow}\,T_{\mathcal{G}}\,\stackrel{h'}{\longrightarrow}\,T\,\longrightarrow\,0\,. \tag{4.4}$$

Recall that $\mathcal{G}$ has a tautological action on E_G. Therefore, the subgroup $T_{\mathcal{G}}$ has a tautological action on E_G which is the restriction of the tautological action of $\mathcal{G}$.

Now we assume that the group G is reductive.

A parabolic subgroup of G is a connected Zariski closed subgroup $P\,\subset\, G$ such that the variety G/P is projective. For a parabolic subgroup P, its unipotent radical will be denoted by $R_u(P)$. A Levi subgroup of P is a connected reductive subgroup $L(P)\,\subset\, P$ such that the composition

$$L(P)\,\hookrightarrow\, P\,\longrightarrow\, P/R_u(P)$$

is an isomorphism. Levi subgroups exists, and any two Levi subgroups of P differ by conjugation by an element of $R_u(P)$ [Hu, p. 184–185, § 30.2], [Bo, p. 158, 11.22, 11.23].

Let $\text{Ad}(E_G)\,:=\,E_G\times^G G\,\longrightarrow\, M$ be the adjoint bundle associated to E_G for the adjoint action of G on itself. The fibers of $\text{Ad}(E_G)$ are groups identified with G up to an inner automorphism; the corresponding Lie algebra bundle is $\text{ad}(E_G)$. We note that $\mathcal{A}$ in (4.2) is the space of all algebraic sections of $\text{Ad}(E_G)$.

Using the action of $T_{\mathcal{A}}$ on E_G, we have

- a Levi subgroup $L(P)$ of a parabolic subgroup P of G, and
- an algebraic reduction of structure group $E_{L(P)}\,\subset\, E_G$ of E_G to $L(P)$ which is preserved by the tautological action of $T_{\mathcal{G}}$ on E_G,

such that the image of $T_{\mathcal{A}}$ in $\mathrm{Ad}(E_G)$ (recall that the elements of $\mathcal{A}$ are sections of $\mathrm{Ad}(E_G)$) lies in the connected component, containing the identity element, of the center of each fiber of $\mathrm{Ad}(E_{L(P)}) \subset \mathrm{Ad}(E_G)$ (see [BBN], [BP] for the construction of $E_{L(P)}$). The construction of $E_{L(P)}$ requires fixing a point z_0 of E_G, and $E_{L(P)}$ contains z_0. Using z_0, the fiber $(E_{L(P)})_{p(z_0)}$ is identified with $L(P)$. Moreover, the evaluation, at $p(z_0)$, of the sections of $\mathrm{Ad}(E_G)$ corresponding to the elements of $T_{\mathcal{A}}$ makes $T_{\mathcal{A}}$ a subgroup of the connected component, containing the identity element, of the center of $E_{L(P)}$; in particular, this evaluation map on $T_{\mathcal{A}}$ is injective (see the second paragraph in [BBN, p. 230, Section 3]). We briefly recall (from [BBN], [BP]) the argument that the evaluation map on semisimple elements of $\mathcal{A}$ is injective. Let ξ be a semisimple element of $\mathcal{A} = \Gamma(M, \mathrm{Ad}(E_G))$. Since ξ is semisimple, for each point $x \in M$, the evaluation $\xi(x)$ is a semisimple element of $\mathrm{Ad}(E_G))_x$. The group $\mathrm{Ad}(E_G))_x$ is identified with G up to an inner automorphism of G. All conjugacy classes of semisimple element of G are parametrized by T_G/W_{T_G}, where T_G is a maximal torus in G, and $W_{T_G} = N(T_G)/T_G$ is the Weyl group with $N(T_G)$ being the normalizer of T_G in G. We note that T_G/W_{T_G} is an affine variety. Therefore, we get a morphism $\xi' : M \longrightarrow T_G/W_{T_G}$ that sends any $x \in M$ to the conjugacy class of $\xi(x)$. Since M is a projective variety and T_G/W_{T_G} is an affine variety, we conclude that ξ' is a constant map. So if $\xi(x) = e$ for some $x \in M$, then $\xi = e$ identically.

Let $Z^0_{L(P)} \subset L(P)$ be the connected component, containing the identity element, of the center. We note that $Z^0_{L(P)}$ is a product of copies of $\mathbb{G}_m$. Therefore, the above injective homomorphism $T_{\mathcal{A}} \longrightarrow Z^0_{L(P)}$ extends to a homomorphism

$$\eta : T_{\mathcal{G}} \longrightarrow Z^0_{L(P)}.$$

Define

$$\eta' := \tau \circ \eta, \tag{4.5}$$

where τ is the inversion homomorphism of $Z^0_{L(P)}$ defined by $g \longmapsto g^{-1}$.

Consider the action of $T_{\mathcal{G}}$ on $E_{L(P)}$; recall that $E_{L(P)}$ is preserved by the tautological action of $T_{\mathcal{G}}$ on E_G. We can twist this action on $E_{L(P)}$ by η' in (4.5), because the actions of $Z^0_{L(P)}$ and $L(P)$ on $E_{L(P)}$ commute. For this new action, the group $T_{\mathcal{A}}$ clearly acts trivially on $E_{L(P)}$.

Consider the above action of $T_{\mathcal{G}}$ on $E_{L(P)}$ constructed using η'. Since $T_{\mathcal{A}}$ acts trivially on $E_{L(P)}$, the action of $T_{\mathcal{G}}$ on $E_{L(P)}$ descends to an action of T on $E_{L(P)}$ (see (4.4)). The principal G–bundle E_G is the extension of structure group of $E_{L(P)}$ using the inclusion of $L(P)$ in G. Therefore, the above action on T on $E_{L(P)}$ produces an action of T on E_G. More precisely, the total space of E_G is the quotient of $E_{L(P)} \times G$ where two elements (z_1, g_1) and (z_2, g_2) of $E_{L(P)} \times G$ are identified if there is an element $g \in L(P)$ such that $z_2 = z_1 g$ and $g_2 = g^{-1} g_1$. Now the action of T on $E_{L(P)} \times G$, given by the above action of T on $E_{L(P)}$ and the trivial action of T on G, descends to an action of T on the quotient space E_G. Consequently, E_G admits a T–equivariant structure.

Therefore, we have proved the following:

Proposition 4.1. *Let G be reductive, and let $E_G \longrightarrow M$ be a principal G–bundle such that for every $t \in T$, the pulled back principal G–bundle $\rho_t^* E_G$ is isomorphic to E_G. Then E_G admits a T–equivariant structure.*

For vector bundles on M, Proposition 4.1 was proved by Klyachko [Kl, p. 342, Proposition 1.2.1].

4.1. **Equivariance property from a logarithmic connection.**

Theorem 4.2. *Let G be reductive, and let $p : E_G \longrightarrow M$ be a principal G–bundle admitting a logarithmic connection whose singularity locus is contained in the divisor $D = M \setminus M^0$. Then E_G admits a T–equivariant structure.*

Proof. Since E_G admits a logarithmic connection, by definition, there is a homomorphism of coherent sheaves

$$\delta \,:\, TM(-\log D) \,\longrightarrow\, \mathrm{At}(E_G)(-\log D)$$

such that $\phi \circ \delta$ is the identity automorphism of $TM(-\log D)$, where ϕ is the homomorphism in (3.6). Let

$$\widehat{\delta} \,:\, H^0(M,\, TM(-\log D)) \,\longrightarrow\, H^0(M,\, \mathrm{At}(E_G)(-\log D))$$

be the homomorphism of global sections given by δ. From Lemma 3.1(2) we know that $H^0(M,\, TM(-\log D))$ is the Lie algebra $\mathfrak{t}$ of T.

We will now show that there is a natural injective homomorphism

$$\theta \,:\, H^0(M,\, \mathrm{At}(E_G)(-\log D)) \,\longrightarrow\, \mathrm{Lie}(\mathcal{G})\,, \tag{4.6}$$

where $\mathrm{Lie}(\mathcal{G})$ is the Lie algebra of the group $\mathcal{G}$ in (4.1).

The elements of $\mathrm{Lie}(\mathcal{G})$ are all holomorphic sections $s \in H^0(M,\, \mathrm{At}(E_G))$ such that the vector field $\phi(s)$, where ϕ is the projection in (3.5), is of the form $\beta(s')$, where $s' \in \mathfrak{t}$ and β is the homomorphism in (3.2). Now, if

$$s \,\in\, H^0(M,\, \mathrm{At}(E_G)(-\log D)) \,\subset\, H^0(M,\, \mathrm{At}(E_G))\,,$$

then $\phi(s)$ is a holomorphic section of $TM(-\log D)$ (see (3.6)). From Lemma 3.1(2) it now follows that $\phi(s)$ is of the form $\beta(s')$, where $s' \in \mathfrak{t}$. This gives us the injective homomorphism in (4.6).

Finally, consider the composition

$$\theta \circ \widehat{\delta} \,:\, \mathfrak{t} \,=\, H^0(M,\, TM(-\log D)) \,\longrightarrow\, \mathrm{Lie}(\mathcal{G})\,.$$

From its construction it follows that

$$(dh) \circ \theta \circ \widehat{\delta} \,=\, \mathrm{Id}_{\mathfrak{t}}\,,$$

where $dh \,:\, \mathrm{Lie}(\mathcal{G}) \,\longrightarrow\, \mathfrak{t}$ is the homomorphism of Lie algebras given by h in (4.1). In particular, dh is surjective. Since T is connected, this immediately implies that the homomorphism h is surjective. Now from Proposition 4.1 it follows that E_G admits a T–equivariant structure. □

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