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Equivariant Abelian principal bundles on nonsingular toric varieties

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ABSTRACT

We give a classification of the holomorphic (resp. algebraic) torus equivariant principal G-bundles on a nonsingular toric variety X when G is an Abelian, closed, holomorphic (resp. algebraic) subgroup of the complex general linear group. We prove that any such bundle splits, that is, admits a reduction of structure group to a torus. We give an explicit parametrization of the isomorphism classes of such bundles for a large family of G when X is complete.

1. Introduction

Denote the algebraic torus $(\mathbb{C}^*)^n$ by T. A T-equivariant principal G-bundle on a complex manifold X is a locally trivial, principal G-bundle $\pi : \mathcal{E} \to X$ such that $\mathcal{E}$ and X are left T-spaces, the map π is T-equivariant and the actions of T and G commute:

$$t(e \cdot g) = (te) \cdot g \text{ for all } t \in T,\ g \in G \text{ and } e \in \mathcal{E}.$$

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If, in addition, the bundle $\pi : \mathcal{E} \to X$ and the actions of T and G are holomorphic, we say that $\mathcal{E}$ is a holomorphic T-equivariant principal G-bundle.

Let X_Ξ be a nonsingular toric variety of dimension n corresponding to a fan Ξ. Denote the set of d-dimensional cones in Ξ by $\Xi(d)$. For a cone σ in Ξ, denote the corresponding affine variety by X_σ and the corresponding T-orbit by O_σ. Note that each orbit O_σ has a natural group structure and the principal orbit O is identified with T (see [7, Proposition 1.6]). Let T_σ denote the stabilizer of any point in O_σ. Then T admits a decomposition, $T \cong T_\sigma \times O_\sigma$. Let $\pi_\sigma : T \to T_\sigma$ be the associated projection.

Let G be an Abelian, closed, holomorphic subgroup of $GL_k(\mathbb{C})$ for some positive integer k. (Note that an Abelian linear algebraic group satisfies this hypothesis, but not every Abelian, closed, holomorphic subgroup of $GL_k(\mathbb{C})$ is algebraic.) Then our main theorem is the following (same as Theorem 4.1).

Theorem 1.1. *The isomorphism classes of holomorphic T-equivariant principal G-bundles on X_Ξ are in one-to-one correspondence with collections of holomorphic group homomorphisms $\{\rho_\sigma : T_\sigma \to G \mid \sigma$ is a maximal cone in $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a G-valued holomorphic function over $X_\sigma \cap X_\tau$.*

A similar classification for algebraic T-equivariant principal G-bundles over X_Ξ is given in Theorem 4.3. In this case G is assumed to be an Abelian linear algebraic group. However, using the classification of Abelian subgroups of the general linear group in [1], one can show that any holomorphic homomorphism $\rho : T \to G$ is algebraic. This follows from Lemma 5.2. Therefore, the isomorphism classes of algebraic and holomorphic T-equivariant principal G-bundles coincide when G is an Abelian linear algebraic group.

These theorems provide a partial analogue to Klyachko's classification of vector bundles on toric varieties in [6]. In particular, we show that any T-equivariant principal G-bundle over a nonsingular affine toric variety is trivializable (see Lemma 3.4). The main tool used by Klyachko for the local classification (i.e. classification on affine toric variety) is representation theory. In analogous situation, our main tool is complex analysis. However, our approach does not work when G is not Abelian since the proof of Theorem 3.2 relies heavily on this assumption. In this article, we are restricted to the case of nonsingular toric variety because we use Oka–Grauert theory [4].

At this time we should point out the work of Heinzner and Kutzschebauch [5]. Their work is in much more general setting where T is any complex reductive group and the structure group G is a complex Lie group. They prove the T-equivariant version of Grauert's Oka principle. In particular they show that any T-equivariant principal G-bundle over $\mathbb{C}^n$ is equivariantly isomorphic to $\mathbb{C}^n \times G$ where the latter has diagonal T-action corresponding to some homomorphism $T \to G$. Hence as a special case we get our Theorem 3.2. In other words, we give an alternative simple proof of their theorem in a very special case.

As a corollary of the main theorem(s), we prove that any such T-equivariant principal G-bundle splits, that is, admits a reduction of structure group to the intersection of G

with a torus, see Theorems 5.1 and 5.4. In Corollary 5.5, we give an explicit parametrization of the isomorphism classes of T-equivariant principal G-bundles, for a large family of Abelian groups G, when Ξ is complete. This generalizes the classical description of the isomorphism classes of T-equivariant line bundles.

We also prove that if G is any discrete group, then any holomorphic T-equivariant principal G-bundle on X_Ξ is trivial with trivial T-action (Theorem 4.2). Our method may be used to prove a similar result for Γ^n-equivariant principal G-bundles over a topological toric manifold [2] where G is discrete and $\Gamma = S^1 \times \mathbb{R}$ acts smoothly.

2. Local action functions

Let $X = X_\sigma$ where $\sigma \in \Xi$. For any sub-cone $\delta \leq \sigma$ we denote the corresponding T-orbit by O_δ. As T is Abelian, the stabilizer of x in T is the same for all $x \in O_\delta$. Denote this stabilizer subgroup by T_δ.

Let G denote a holomorphic group. Suppose $\mathcal{E}$ is a T-equivariant principal G-bundle over X. Assume that $\mathcal{E}$ is trivial (we will show in Lemma 3.4 that this holds for suitable G). Let $s : X \to \mathcal{E}$ be any holomorphic section. We encode the T-action on $\mathcal{E}$ as follows.

Definition 2.1. For any $x \in X$ and $t \in T$, define $\rho_s(x,t) \in G$ by

$$ts(x) = s(tx) \cdot \rho_s(x,t). \tag{2.1}$$

We say that $\rho_s : X \times T \to G$ is a local action function.

Since the action of G on each fiber of $\mathcal{E}$ is free, it follows that $\rho_s(x,t)$ is well-defined and holomorphic in x and t.

It is easy to check that if $s'(x) = s(x) \cdot \gamma(x)$ is another section, then

$$\rho_{s'}(x,t) = \gamma(tx)^{-1}\rho_s(x,t)\gamma(x). \tag{2.2}$$

Lemma 2.1. *For any t_1, t_2 in T, $\rho_s(x,t_1t_2) = \rho_s(t_2x,t_1)\rho_s(x,t_2)$.*

Proof. Obviously, $t_1t_2s(x) = s(t_1t_2x) \cdot \rho_s(x,t_1t_2)$. On the other hand,

$$\begin{aligned} t_1t_2s(x) &= t_1(s(t_2x) \cdot \rho_s(x,t_2)) = (t_1s(t_2x)) \cdot \rho_s(x,t_2) \\ &= s(t_1t_2x) \cdot \rho_s(t_2x,t_1)\rho_s(x,t_2). \qquad \square \end{aligned}$$

It follows that if $x \in O_\delta$ then the restriction

$$\rho_s(x,\cdot) : T_\delta \to G \tag{2.3}$$

is a group homomorphism.

Lemma 2.1 implies that the value of ρ_s on any T-orbit O_δ may be determined from the value of ρ_s at a point $x \in O_\delta$. However a stronger statement holds.

Lemma 2.2. *The map ρ_s is completely determined by its restriction $\rho_s(x_0, \cdot) : T \to G$ at any point x_0 in the principal T-orbit O.*

Proof. Let $\delta \neq \{0\}$ be a sub-cone of σ. Let x_δ be any point in O_δ. Then there exist a point $x_1 \in O$ and a one-parameter subgroup $\lambda^r(z)$ of T such that $\lim_{z\to 0} \lambda^r(z)x_1 = x_\delta$ (see [3, Section 2.3]). Then using Lemma 2.1, we get $\rho_s(x_1, t\lambda^r(z)) = \rho_s(\lambda^r(z)x_1, t) \times \rho_s(x_1, \lambda^r(z))$. Hence $\rho_s(\lambda^r(z)x_1, t) = \rho_s(x_1, t\lambda^r(z))\rho_s(x_1, \lambda^r(z))^{-1}$. Taking limit as z approaches 0, we get

$$\rho_s(x_\delta, t) = \lim_{z\to 0} \rho_s(x_1, t\lambda^r(z))\rho_s(x_1, \lambda^r(z))^{-1}. \tag{2.4}$$

Since $\rho_s(x_1, \cdot)$ is determined by $\rho_s(x_0, \cdot)$, the lemma follows. □

Lemma 2.3. *Let X_1 and X_2 be affine toric varieties. Let $\alpha : X_1 \to X_2$ be an isomorphism of T-spaces up to an automorphism $a : T \to T$, i.e. $\alpha \circ t = a(t) \circ \alpha$. Suppose $\pi_i : \mathcal{E}_i \to X_i$ is a T-equivariant trivial principal G-bundle for $i = 1, 2$. Let $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ be an isomorphism of T-equivariant principal G-bundles over X compatible with α and a:*

$$\pi_2 \circ \phi = \alpha \circ \pi_1 \quad \textit{and} \quad \phi \circ t = a(t) \circ \phi.$$

Let s_1 be any section of $\mathcal{E}_1$ and let s_2 be the section of $\mathcal{E}_2$ defined by $s_2(\alpha(x)) = \phi(s_1(x))$ for $x \in X_1$. Then $\rho_{s_1}(x, t) = \rho_{s_2}(\alpha(x), a(t))$ for every $x \in X_1$, $t \in T$. In particular, if α and a are both identities then $\rho_{s_1} = \rho_{s_2}$.

Proof. The lemma follows from the following calculation.

$$\begin{aligned} & s_2(a(t)\alpha(x)) \cdot \rho_{s_2}(\alpha(x), a(t)) \\ & = a(t)s_2(\alpha(x)) = a(t)\phi(s_1(x)) \\ & = \phi(ts_1(x)) = \phi(s_1(tx) \cdot \rho_{s_1}(x, t)) \\ & = \phi(s_1(tx)) \cdot \rho_{s_1}(x, t) = s_2(\alpha(tx)) \cdot \rho_{s_1}(x, t) \\ & = s_2(a(t)\alpha(x)) \cdot \rho_{s_1}(x, t). \qquad \square \end{aligned}$$

Lemma 2.4. *If $\rho_s(x, \cdot)$ is independent of x, then it defines a group homomorphism $\rho_s : T \to G$. Conversely if $\rho_s(x_0, \cdot)$ is a group homomorphism for some $x_0 \in O$, then $\rho_s(x, t)$ is independent of x.*

Proof. The first claim follows immediately from Lemma 2.1. On the other hand if $\rho_s(x_0, \cdot)$ is a group homomorphism, then we have

$$\rho_s(x_0, t)\rho_s(x_0, u) = \rho_s(x_0, tu) = \rho_s(ux_0, t)\rho_s(x_0, u)$$

for any $u, t \in T$. Therefore for any $u \in T$

$$\rho_s(ux_0, \cdot) = \rho_s(x_0, \cdot).$$

Now the result follows either by using holomorphicity of $\rho_s(x, t)$ in x, or equation (2.4). □

Lemma 2.5. *Suppose G is a discrete group and $X \cong \mathbb{C}^d$. Let $\mathcal{E}$ be a T-equivariant holomorphic principal G-bundle over X. Then $\mathcal{E}$ is trivial and for any section s of $\mathcal{E}$, $\rho_s(x, \cdot) : T \to G$ is independent of x and is the trivial homomorphism.*

Proof. Note that as X is contractible, the bundle $\mathcal{E}$ is topologically trivial. Therefore, it admits a continuous section s. But as X is connected and G is discrete, s is constant and hence holomorphic.

Fix $t \in T$. Since X is connected and $\rho_s(x, t)$ is continuous in x, it is a constant map due to the discreteness of G. This implies that $\rho_s(x, \cdot)$ is independent of x and therefore is a group homomorphism $\rho_s(\cdot) : T \to G$ by Lemma 2.4. Continuity in t and connectedness of T then imply that $\rho_s(t)$ is a constant. This completes the proof. □

3. Local action homomorphisms

Suppose $\mathcal{E}$ is a holomorphic principal G-bundle over $X \cong \mathbb{C}^n$ where G is a holomorphic, closed subgroup of $GL_k(\mathbb{C})$. By Oka–Grauert theory [4], $\mathcal{E}$ is trivial and admits a holomorphic section s. We will show that if G is Abelian, then s can be chosen so that the local action function ρ_s is a homomorphism.

Proposition 3.1. *Let G be a holomorphic, closed subgroup of $GL_k(\mathbb{C})$. Suppose $f : \mathbb{C}^* \to G$ is a holomorphic map such that $\lim_{z\to 0} f(zt)f(z)^{-1} = I$ for every $t \in \mathbb{C}^*$. Then f admits a holomorphic extension $f : \mathbb{C} \to G$.*

Proof. Let

$$f(z) = \sum_{n=-\infty}^{\infty} C_n z^n \tag{3.1}$$

be the Laurent series representation of f for $z \in \mathbb{C}^*$, where each $C_n \in M_k(\mathbb{C})$. Note that $f(\cdot)^{-1} : \mathbb{C}^* \to G$ is also holomorphic. Let its Laurent series be

$$f(z)^{-1} = \sum_{n=-\infty}^{\infty} A_n z^n \tag{3.2}$$

where each $A_n \in M_k(\mathbb{C})$. Then we have

$$f(zt)f(z)^{-1} = \sum_{n=-\infty}^{\infty} z^n t^n \Big(\sum_{m=-\infty}^{\infty} C_{n-m} A_m t^{-m} \Big). \tag{3.3}$$

As the limit $\lim_{z\to 0} f(zt)f(z)^{-1}$ exists by assumption, the Riemann extension theorem shows that $f(zt)f(z)^{-1}$ is holomorphic at $z = 0$ for every $t \in \mathbb{C}^*$. Therefore,

$$\sum_{m=-\infty}^{\infty} C_{n-m} A_m t^{-m} = 0 \tag{3.4}$$

for every $n < 0$ and every $t \in \mathbb{C}^*$. This implies that

$$C_{n-m} A_m = 0 \tag{3.5}$$

for every $n < 0$ and every m. Setting $j = n - m$ and varying n, we obtain

$$C_j A_m = 0 \tag{3.6}$$

for every $j < -m$ for every m.

Let S be the set of all column vectors of the matrices A_m, $m \in \mathbb{Z}$. Since every subspace of $\mathbb{C}^k$ is closed, the span of S is closed in $\mathbb{C}^k$. Therefore the columns of $f(z)^{-1}$ belong to the span of S. Since $f(z)^{-1} \in GL_k(\mathbb{C})$, the span of S must equal $\mathbb{C}^k$. Therefore there exists a finite set of values of m, say $m_1, \dots, m_p$, such that the column spaces of the corresponding A_m's span $\mathbb{C}^k$. Let $m_0 = \min\{-m_1, \dots, -m_p\}$. Consider any $j < m_0$. Then $j < -m_i$ for each $1 \le i \le p$. Hence by equation (3.6) $C_j A_{m_i} = 0$ for each $1 \le i \le p$. Hence $C_j = 0$.

Assume that n_0 is the minimum value of n such that C_{n_0} is nonzero. Then $f(z) = z^{n_0}\phi(z)$ where $\phi : \mathbb{C} \to GL_k(\mathbb{C})$ is holomorphic near zero. Then

$$I = \lim_{z\to 0} f(zt)f(z)^{-1} = \lim_{z\to 0} \phi(zt)\phi(z)^{-1} t^{n_0}. \tag{3.7}$$

Since $\lim\limits_{z\to 0} \phi(z) = C_{n_0}$ and $\lim\limits_{z\to 0} \phi(tz) = C_{n_0}$, multiplying both sides of (3.7) by $\lim\limits_{z\to 0} \phi(z)$ yields

$$C_{n_0} = \lim_{z\to 0} \phi(z) = \lim_{z\to 0} \phi(zt) t^{n_0} = C_{n_0} t^{n_0} \tag{3.8}$$

for every $t \in \mathbb{C}^*$. Therefore we must have $n_0 = 0$. Hence f is holomorphic at 0 and $f(0) = C_0$.

Let $g(z) = \det(f(z))$. It follows that $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\lim\limits_{z\to 0} g(zt)g(z)^{-1} = 1$. If g has a zero of order n at zero, we get $\lim_{z\to 0} g(zt)g(z)^{-1} = t^n$. Therefore $n = 0$ and $g(0) \neq 0$. Hence $f(0)$ is nonsingular. Thus f defines a holomorphic map $f : \mathbb{C} \to GL_k$.

Since G is closed in $GL_k(\mathbb{C})$, $f(0) \in G$. So f defines a holomorphic map $f : \mathbb{C} \to G$. □

Theorem 3.2. *Consider the standard action of T on $\mathbb{C}^n$. Suppose $\mathcal{E}$ is a holomorphic T-equivariant principal G-bundle over $\mathbb{C}^n$ where G is a holomorphic, closed, Abelian subgroup of $GL_k(\mathbb{C})$. Then there exists a holomorphic section s' of $\mathcal{E}$ such that $\rho_{s'}(x,\cdot) : T \to G$ equals $\rho_{s'}(0,\cdot)$ for any x in $\mathbb{C}^n$.*

Proof. Let s be a holomorphic section of $\mathcal{E}$ over $\mathbb{C}^n$. Fix a point x_0 in $O = T$. Note that $\rho_s(0,\cdot) : T \to G$ is a group homomorphism as the origin 0 is fixed by T. Define a holomorphic function $F : T \to G$ by

$$F(t) = \rho_s(0,t)^{-1}\rho_s(x_0,t). \tag{3.9}$$

We may therefore write

$$\rho_s(x_0,t) = \rho_s(0,t)F(t). \tag{3.10}$$

For any $y, x \in \mathbb{C}^n$, define yx to be coordinate-wise multiplication of y and x. Let z denote an element of T. For any $y \in \mathbb{C}^n$,

$$\begin{aligned}\rho_s(yx_0,t) &= \lim_{z\to y}\rho_s(zx_0,t) = \lim_{z\to y}\rho_s(x_0,tz)\rho_s(x_0,z)^{-1}\\ &= \lim_{z\to y}\rho_s(0,tz)F(tz)F(z)^{-1}\rho_s(0,z)^{-1}.\end{aligned}$$

Therefore,

$$\lim_{z\to y}F(tz)F(z)^{-1} = \lim_{z\to y}\rho_s(0,tz)^{-1}\rho_s(yx_0,t)\rho_s(0,z) = \rho_s(yx_0,t)\rho_s(0,t)^{-1}, \tag{3.11}$$

where the last equality follows from the assumption that G is Abelian, and the fact that $\rho_s(0,\cdot)$ is a homomorphism.

Define $P_k = \mathbb{C}^n - \cup_{i=k}^n\{z_i = 0\}$ where $1 \le k \le n$. Note that $P_1 = (\mathbb{C}^*)^n$ and $P_n = \mathbb{C}^n$. We will now show by induction over k that F admits a holomorphic G-valued extension over $\mathbb{C}^n$. Note that F admits an extension over P_1 by definition. Now assume that F has a G-valued holomorphic extension over P_k. Take any point $p = (p_1,\dots,p_n) \in P_k$. Note that $tp \in P_k$ for any $t \in T$. Setting $y = p$ in (3.11), we get

$$F(tp)F(p)^{-1} = \rho_s(px_0,t)\rho_s(0,t)^{-1}. \tag{3.12}$$

Let $\pi_k : \mathbb{C}^n \to \{z_k = 0\}$ be the standard projection. Taking limit of (3.12) as $p_k \to 0$, we get

$$\lim_{p_k\to 0}F(tp)F(p)^{-1} = \rho_s(\pi_k(p)x_0,t)\rho_s(0,t)^{-1}. \tag{3.13}$$

Let $t_k \in \mathbb{C}^*$. Write $\iota(t_k)$ for $(1,\dots,1,t_k,1,\dots,1)$ where t_k occupies the k-th position.

In the case $p_i \neq 0$ for each $i \neq k$, $t(p) = (p_1, \dots, p_{k-1}, 1, p_{k+1}, \dots, p_n)$ defines an element of T. Then by Lemma 2.1, and using G is Abelian, we have

$$\begin{aligned} &\rho_s(\pi_k(p)x_0, \iota(t_k)) \\ &\quad = \rho_s(t(p)\pi_k(x_0), \iota(t_k)) \\ &\quad = \rho_s(\pi_k(x_0), \iota(t_k)t(p))\, \rho_s(\pi_k(x_0), t(p))^{-1} \\ &\quad = \rho_s(\pi_k(x_0), t(p)\iota(t_k))\, \rho_s(\pi_k(x_0), t(p))^{-1} \\ &\quad = \rho_s(\iota(t_k)\pi_k(x_0), t(p))\, \rho_s(\pi_k(x_0), \iota(t_k))\, \rho_s(\pi_k(x_0), t(p))^{-1} \\ &\quad = \rho_s(\pi_k(x_0), t(p))\, \rho_s(\pi_k(x_0), \iota(t_k))\, \rho_s(\pi_k(x_0), t(p))^{-1} \\ &\quad = \rho_s(\pi_k(x_0), \iota(t_k)). \end{aligned} \tag{3.14}$$

The set $\{\pi_k(p)x_0 \mid p_i \neq 0\ \forall i \neq k\}$ is dense in the hyperplane $z_k = 0$. Hence by holomorphicity of $\rho_s(w, \iota(t_k))$ in w, we obtain $\rho_s(w, \iota(t_k)) = \rho_s(\pi_k(x_0), \iota(t_k))$ for every w in the hyperplane $z_k = 0$. Therefore $\rho_s(w, \iota(t_k))$ is constant for $w \in \{z_k = 0\}$ and equals $\rho_s(0, \iota(t_k))$. In particular,

$$\rho_s(\pi_k(p)x_0, \iota(t_k)) = \rho_s(0, \iota(t_k)) \tag{3.15}$$

for any p in P_k, since $\pi_k(p)x_0 \in \{z_k = 0\}$.

Fix p_i for each $i \neq k$, and define

$$f(p_k) = F(p_1, \dots, p_k, \dots, p_n). \tag{3.16}$$

Then setting $t = \iota(t_k)$ in (3.13), we get

$$\lim_{p_k \to 0} f(t_k p_k) f(p_k)^{-1} = \rho_s(\pi_k(p)x_0, \iota(t_k))\, \rho_s(0, \iota(t_k))^{-1}. \tag{3.17}$$

Therefore by (3.15),

$$\lim_{p_k \to 0} f(t_k p_k) f(p_k)^{-1} = I$$

for any $t_k \in \mathbb{C}^*$. Therefore Proposition 3.1 applies to f. Hence

$$\lim_{p_k \to 0} F(p_1, \dots, p_n) = \lim_{p_k \to 0} f(p_k)$$

exists and takes value in G. As this argument works for every $p \in P_k$, by the Riemann extension theorem, F has a unique G-valued extension over $P_{k+1} = \mathbb{C}^n - \cup_{i=k+1}^n \{z_i = 0\}$. Therefore, by induction, F admits a unique G-valued holomorphic extension over $\mathbb{C}^n$.

Now define a new section s' of $\mathcal{E}$ by

$$s'(zx_0) = s(zx_0) \cdot F(z) \tag{3.18}$$

for every $z \in \mathbb{C}^n$. Note that $F(I) = I$ as $\rho_s(x_0, I) = I = \rho_s(0, I)$. Then

$$ts'(x_0) = t(s(x_0) \cdot F(I)) = ts(x_0) = s(tx_0) \cdot \rho_s(x_0, t). \tag{3.19}$$

On the other hand,

$$ts'(x_0) = s'(tx_0) \cdot \rho_{s'}(x_0, t) = s(tx_0) \cdot F(t)\rho_{s'}(x_0, t). \tag{3.20}$$

Therefore, using (3.10), we have

$$\rho_{s'}(x_0, t) = F(t)^{-1}\rho_s(x_0, t) = F(t)^{-1}\rho_s(0, t)F(t) = \rho_s(0, t). \tag{3.21}$$

Therefore $\rho_{s'}(x_0, t)$ is a homomorphism and the theorem follows from Lemma 2.4. □

Remark 3.3. Note that the homomorphism $\rho_s(0, \cdot)$ is independent of the choice of the section s by (2.2) when G is Abelian.

Lemma 3.4. *Let G be a holomorphic, closed, Abelian subgroup of $GL_k(\mathbb{C})$. Suppose σ is any cone of an n-dimensional nonsingular fan Ξ and $\mathcal{E}$ a T-equivariant holomorphic principal G-bundle on the affine toric variety X_σ. Then $\mathcal{E}$ is trivial and admits a section s^* for which the local action function ρ_{s^*} is a homomorphism. Moreover, there exists a canonical homomorphism $\rho_\sigma : T_\sigma \to G$ and a choice of s^* such that $\rho_{s^*}(t) = \rho_\sigma(\pi_\sigma(t))$, where $\pi_\sigma : T \to T_\sigma$ is the projection associated with the decomposition $T \cong T_\sigma \times O_\sigma$.*

Proof. Let d denote the dimension of the cone σ. Note that there exists an isomorphism $\alpha : X_\sigma \to \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$ where the latter space has the standard T-action, such that α is equivariant up to an automorphism a_σ of T:

$$\alpha(tx) = a_\sigma(t)\alpha(x).$$

Define $H = a_\sigma(T_\sigma)$ and $K = a_\sigma(O_\sigma)$. Note $H \cong (\mathbb{C}^*)^d$ and $K \cong (\mathbb{C}^*)^{n-d}$.

Let $\mathcal{F}$ be the pull-back of $\mathcal{E}$ with respect to α^{-1}. Let $\phi : \mathcal{E} \to \mathcal{F}$ be the natural isomorphism. Note that $\mathcal{F}$ inherits natural actions of T and G that satisfy

$$\phi(te) = a_\sigma(t)\phi(e), \quad \phi(e \cdot g) = \phi(e) \cdot g$$

for every $e \in \mathcal{E}$. This makes $\mathcal{F}$ a T-equivariant principal G-bundle over $\mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$.

Fix a point y_0 in $(\mathbb{C}^*)^{n-d}$. Then by Oka–Grauert theory $\mathcal{F}$ is trivial on $\mathbb{C}^d \times \{y_0\}$. Let s be a section of this restricted bundle. We extend s to a section of $\mathcal{F}$ over $\mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$ by defining

$$s(x, ky_0) = ks(x, y_0) \tag{3.22}$$

for every $x \in \mathbb{C}^d$ and $k \in K$. This shows that $\mathcal{F}$, and consequently $\mathcal{E}$, is trivial.

By Theorem 3.2 and Remark 3.3 we may assume that the local action function of the section s over $\mathbb{C}^d \times \{y_0\}$ satisfies

$$\rho_s((x, y_0), h) = \rho_s((0, y_0), h) \tag{3.23}$$

for all $h \in H$, and defines a homomorphism $H \to G$ that is independent of s.

Since

$$\begin{aligned} hs(x, ky_0) &= hks(x, y_0) = khs(x, y_0) \\ &= ks(hx, y_0) \cdot \rho_s((x, y_0), h) = s(hx, ky_0) \cdot \rho_s((x, y_0), h), \end{aligned} \tag{3.24}$$

we deduce that

$$\rho_s((x, ky_0), h) = \rho_s((x, y_0), h) \tag{3.25}$$

for every $k \in K$. This shows that the homomorphism $\rho_s : H \to G$ is independent of the choice of y_0 as well. Recall that $O_\sigma \cong (\mathbb{C}^*)^{n-d}$ and $a_\sigma(T_\sigma) = H$. We define $\rho_\sigma : T_\sigma \to G$ by

$$\rho_\sigma(t) = \rho_s((0, y_0), a_\sigma(t)). \tag{3.26}$$

It follows easily from (3.22) that $ks(x, y) = s(x, ky)$ for any $y \in (\mathbb{C}^*)^{n-d}$. Then,

$$\begin{aligned} hks(x, y) &= hs(x, ky) = s(hx, ky) \cdot \rho_s((x, ky), h) \\ &= s(hx, ky) \cdot \rho_s((x, y_0), h) = s(hx, ky) \cdot \rho_s((0, y_0), h) \end{aligned} \tag{3.27}$$

for every $(h, k) \in H \times K$ and $(x, y) \in \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. Therefore,

$$\rho_s((x, y), hk) = \rho_s((0, y_0), h). \tag{3.28}$$

Then $\rho_s((x, y), \cdot)$ is a homomorphism as $\rho_s((0, y_0), \cdot)$ is so.

We set $s^* = \phi^{-1}(s)$. Then by Lemma 2.3 and (3.28),

$$\rho_{s^*}(x, t) = \rho_s((\alpha(x), a_\sigma(t)) = \rho_s((0, y_0), pr(a_\sigma(t))), \tag{3.29}$$

where $pr : T \to H$ is the projection corresponding to the decomposition $T = H \times K$. Then by definition of H and K we have $pr \circ a_\sigma = a_\sigma \circ \pi_\sigma$. Therefore from (3.29) and (3.26) we have

$$\rho_{s^*}(x, t) = \rho_s((0, y_0), a_\sigma(\pi_\sigma(t))) = \rho_\sigma(\pi_\sigma(t)). \qquad \square$$

4. Gluing condition

The main idea of this section is borrowed from [6].

Let X be a nonsingular toric variety of dimension n corresponding to a fan Ξ. Suppose $\mathcal{E}$ is a holomorphic T-equivariant principal G-bundle over X where G is a holomorphic, closed, Abelian subgroup of $GL_k(\mathbb{C})$.

Let σ be any maximal cone in Ξ. Let $\widetilde{\rho}_\sigma = \rho_\sigma \circ \pi_\sigma : T \to G$ where $\rho_\sigma : T_\sigma \to G$ is the homomorphism obtained by applying Lemma 3.4 to the bundle $\mathcal{E}_\sigma := \mathcal{E}|_{X_\sigma}$. Let s_σ be a section of $\mathcal{E}_\sigma$ whose local action homomorphism is $\widetilde{\rho}_\sigma$.

Let $\psi_\sigma : \mathcal{E}_\sigma \to X_\sigma \times G$ be the trivialization induced by the section s_σ,

$$\psi_\sigma(s_\sigma(x) \cdot h) = (x, h).$$

Note that

$$\psi_\sigma(ts_\sigma(x) \cdot h) = (tx, \widetilde{\rho}_\sigma(t)h).$$

So the T action on the trivialization $X_\sigma \times G$ is defined by

$$t(x, h) = (tx, \widetilde{\rho}_\sigma(t)h).$$

Let σ, τ be any two maximal cones. Let $\phi_{\tau\sigma} : X_\sigma \cap X_\tau \to G$ denote the transition function defined as follows,

$$\psi_\tau \psi_\sigma^{-1}(x, h) = (x, \phi_{\tau\sigma}(x)h).$$

By equivariance, we have

$$t(\psi_\tau \psi_\sigma^{-1}(x, h)) = \psi_\tau \psi_\sigma^{-1}(t(x, h)).$$

This implies that

$$(tx, \widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)h) = (tx, \phi_{\tau\sigma}(tx)\widetilde{\rho}_\sigma(t)h).$$

Therefore,

$$\phi_{\tau\sigma}(tx) = \widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)\widetilde{\rho}_\sigma(t)^{-1} = \phi_{\tau\sigma}(x)\widetilde{\rho}_\tau(t)\widetilde{\rho}_\sigma(t)^{-1}. \tag{4.1}$$

Consider the point $x_0 = (1, \dots, 1)$ in the principal T-orbit O of X. For any maximal cones τ and σ, we have

$$\widetilde{\rho}_\tau(t)\widetilde{\rho}_\sigma(t)^{-1} = \phi_{\tau\sigma}(x_0)^{-1}\phi_{\tau\sigma}(tx_0).$$

Therefore the function $\widetilde{\rho}_\tau \widetilde{\rho}_\sigma^{-1} : T \to G$ admits a holomorphic extension to a function from $X_\tau \cap X_\sigma$ to G, namely $\phi_{\tau\sigma}(x_0)^{-1}\phi_{\tau\sigma}(\cdot)$.

Theorem 4.1. *Let X be an n-dimensional nonsingular toric variety with fan Ξ. Let G be a holomorphic, closed, Abelian subgroup of $GL_k(\mathbb{C})$. Then the isomorphism classes of holomorphic T-equivariant principal G-bundles on X are in one-to-one correspondence with collections of holomorphic group homomorphisms $\{\rho_\sigma : T_\sigma \to G \mid \sigma$ is a maximal cone of $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a G-valued holomorphic function over $X_\sigma \cap X_\tau$.*

Proof. Given a T-equivariant principal G-bundle $\mathcal{E}$ on X, we have a canonical collection of homomorphisms $\{\rho_\sigma : T_\sigma \to G\}$ by Lemma 3.4. We have shown above that this collection satisfies the extension condition. Moreover, the collection of homomorphisms is invariant under an isomorphism of the bundle by Lemma 2.3.

Conversely given a collection of homomorphisms $\{\rho_\sigma\}$ satisfying the extension condition, define $\widetilde{\rho}_\sigma = \rho_\sigma \circ \pi_\sigma$. Let $\phi_{\tau\sigma} : X_\sigma \cap X_\tau \to G$ denote the extension of $\widetilde{\rho}_\tau \widetilde{\rho}_\sigma^{-1}$. Note that $\{\phi_{\tau\sigma}\}$ satisfies the cocycle condition. Therefore we may construct a principal G-bundle $\mathcal{E}$ over X with $\{\phi_{\tau\sigma}\}$ as transition functions,

$$\mathcal{E} = (\bigsqcup_{\sigma} X_\sigma \times G)/\sim$$

where $(x, g) \sim (y, h)$ for $(x, g) \in X_\sigma \times G$ and $(y, h) \in X_\tau \times G$ if and only if

$$x = y, \; x \in X_\sigma \cap X_\tau \text{ and } h = \phi_{\tau\sigma}(x)g. \tag{4.2}$$

Define T action on each $X_\sigma \times G$ by $t(x, g) = (tx, \widetilde{\rho}_\sigma(t)g)$. Then note that if $(y, h) \in X_\tau \times G$ is equivalent to $(x, g) \in X_\sigma \times G$, then

$$t(y, h) = t(x, \phi_{\tau\sigma}(x)g) = (tx, \widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)g). \tag{4.3}$$

Now if x belongs to the open orbit $O = T \subset X_\sigma \cap X_\tau$, then

$$\phi_{\tau\sigma}(tx)\widetilde{\rho}_\sigma(t) = \widetilde{\rho}_\tau(tx)\widetilde{\rho}_\sigma(tx)^{-1}\widetilde{\rho}_\sigma(t) = \widetilde{\rho}_\tau(t)\widetilde{\rho}_\tau(x)\widetilde{\rho}_\sigma(x)^{-1} = \widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x). \tag{4.4}$$

Since both $\phi_{\tau\sigma}(tx)\widetilde{\rho}_\sigma(t)$ and $\widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)$ are continuous in x on $X_\sigma \cap X_\tau$ and O is dense in $X_\sigma \cap X_\tau$,

$$\phi_{\tau\sigma}(tx)\widetilde{\rho}_\sigma(t) = \widetilde{\rho}_\tau(t)\phi_{\tau\sigma}(x) \text{ for all } x \in X_\sigma \cap X_\tau. \tag{4.5}$$

From (4.3) and (4.5), we have

$$t(y, h) = (tx, \phi_{\tau\sigma}(tx)\widetilde{\rho}_\sigma(t)g)$$

whenever $(x, g) \sim (y, h)$. Since $t(x, g) = (tx, \widetilde{\rho}_\sigma(t)g)$, by (4.2) this implies that $t(y, h) \sim t(x, g)$ whenever $(x, g) \sim (y, h)$. In other words, the T-actions on the $X_\sigma \times G$ are compatible and define an action on $\mathcal{E}$. It is obvious that $\{\rho_\sigma\}$ are the local homomorphisms associated with $\mathcal{E}$. □

Theorem 4.2. *If G is a discrete group, then any holomorphic T-equivariant principal G-bundle $\mathcal{E}$ over a nonsingular toric variety is trivial with trivial T-action.*

Proof. By using Lemma 2.5 and mimicking the proof of Lemma 3.4, we obtain that $\mathcal{E}$ is trivial over any nonsingular affine toric variety X_σ. We also obtain canonical homomorphisms $\rho_\sigma : T_\sigma \to G$. Then, we mimic the proof of Theorem 4.1 and obtain an analogous result. Note that a holomorphic homomorphism from any T_σ to the discrete group G must be trivial. The result follows. □

4.1. Algebraic case

Suppose that $\mathcal{E}$ is an algebraic T-equivariant principal G-bundle over $\mathbb{C}^n$ where G is a linear algebraic group. It follows from [8] that $\mathcal{E}$ is trivial. Then the local action function corresponding to a section s of $\mathcal{E}$ is a regular map $\rho_s : \mathbb{C}^n \times T \to G$. Therefore, the function $F : T \to G$ of (3.9) is regular.

As G is closed in the Zariski topology, it is closed in the complex topology. So, the proof of Theorem 3.2 shows that F admits a holomorphic extension $F : \mathbb{C}^n \to G$. A priori F is represented by $k \times k$ matrix A with entries in the ring $\mathbb{C}[z_i, z_i^{-1}, 1 \le i \le n]$. But as F is entire, the entries of A are convergent power series in $z_1, \dots, z_n$. Therefore the entries must be polynomials. Hence the extension $F : \mathbb{C}^n \to G$ is regular. This yields an algebraic analogue of Theorem 3.2 which in turn leads to an algebraic analogue of Lemma 3.4. Then the same proof as in the holomorphic case gives the following result.

Theorem 4.3. *Let X be an n-dimensional nonsingular toric variety with fan Ξ. Let G be an Abelian linear algebraic group. Then the isomorphism classes of algebraic T-equivariant principal G-bundles on X are in one-to-one correspondence with collections of algebraic group homomorphisms $\{\rho_\sigma : T_\sigma \to G \mid \sigma$ is a maximal cone of $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a G-valued regular function over $X_\sigma \cap X_\tau$.*

5. Applications

Every complete nonsingular toric variety X admits an equivariant principal G-bundle: Let $\rho : T \to G$ be a homomorphism. Set $\rho_\sigma = \rho$ for every $\sigma \in \Xi(n)$. Then the extension condition is satisfied as $\rho_\tau \rho_\sigma^{-1}$ is the identity map. However, X may admit more equivariant principal G-bundles.

Theorem 5.1. *Suppose G is a holomorphic (resp. algebraic), closed, Abelian subgroup of $GL_k(\mathbb{C})$. Then a holomorphic (resp. algebraic) T-equivariant principal G-bundle over a nonsingular toric variety X admits a reduction of structure group to the intersection of G with a maximal torus of $GL_k(\mathbb{C})$.*

Proof. Let $\sigma_1, \ldots, \sigma_m$ be the maximal cones of the fan of X. A homomorphism $\rho_{\sigma_i} : T \to G$ may be regarded as a representation of T on $V := \mathbb{C}^k$. We denote ρ_{σ_i} by ρ_i.

The representation ρ_i decomposes into one-dimensional representations with not necessarily distinct characters. However we may collate all one-dimensional representations having the same character λ_i into a subspace $V(\lambda_i)$, and write $V = \oplus_{\lambda_i} V(\lambda_i)$ where λ_i varies over characters of ρ_i.

As ρ_1 and ρ_2 commute, $V(\lambda_1)$ is invariant under ρ_2. Let $V(\lambda_1, \lambda_2)$ denote the direct sum of the one-dimensional subspaces of $V(\lambda_1)$ which are irreducible components of $\rho_2|_{V(\lambda_1)}$ with character λ_2. Therefore

$$V = \bigoplus_{\lambda_1, \lambda_2} V(\lambda_1, \lambda_2).$$

It is easy to observe that $V(\lambda_1, \lambda_2) = V(\lambda_1) \cap V(\lambda_2)$. Therefore it is invariant under ρ_3. Proceeding inductively, we have

$$V = \bigoplus_{\lambda_1, \ldots, \lambda_m} V(\lambda_1) \cap \ldots \cap V(\lambda_m).$$

Any one-dimensional subspace of $V(\lambda_1) \cap \ldots \cap V(\lambda_m)$ is an eigenspace of $\rho_i(t)$ for every i and t. Therefore all these operators are simultaneously diagonalizable. So there exists a $g \in GL(V)$ such that $g^{-1}\rho_i(t)g \in (\mathbb{C}^*)^k \subset GL_k(\mathbb{C})$. In other words, each $\rho_i(t)$ belongs to the intersection of the torus $g(\mathbb{C}^*)^k g^{-1}$ with G. By closedness of G and $(\mathbb{C}^*)^k$ in $GL_k(\mathbb{C})$, the transition functions $\rho_i \rho_j^{-1}$ also belong to this intersection. □

Note that the above theorem implies that any T-equivariant vector bundle over X with an Abelian structure group splits equivariantly into a sum of line bundles.

A more precise result is obtained if we use the classification of Abelian subgroups of $GL_k(\mathbb{C})$ in [1]: Let $\eta = (k_1, \ldots, k_r)$ be a partition of k into positive integers. Let K_η^* be the subgroup of $GL_k(\mathbb{C})$ consisting of all block diagonal matrices such that the i-th block has size $k_i \times k_i$, and every block is lower triangular with identical nonzero diagonal elements. The diagonal elements of different blocks need not be the same. Then given an Abelian subgroup G of $GL_k(\mathbb{C})$, there exists an element $h \in GL_k(\mathbb{C})$ and a partition η of k such that hGh^{-1} is a subgroup of K_η^*.

Lemma 5.2. *Let L denote the subgroup of lower triangular matrices in $GL_p(\mathbb{C})$ having equal diagonal elements. Given any holomorphic homomorphism $\rho : \mathbb{C}^* \to L$, denote the (i, j)-th entry of the matrix $\rho(t)$ by $\rho_{ij}(t)$. Write $\mu(t)$ for the common value of $\rho_{ii}(t)$. Then $\mu : \mathbb{C}^* \to \mathbb{C}^*$ is an algebraic homomorphism and $\rho_{ij}(t) = 0$ if $i \neq j$.*

Proof. Comparing the $(1, 1)$ entries of $\rho(t_1 t_2)$ and $\rho(t_1)\rho(t_2)$, we have $\mu(t_1 t_2) = \mu(t_1)\mu(t_2)$. Hence $\mu(t) = t^a$ for some $a \in \mathbb{Z}$.

Similarly, comparing $(1,2)$ entries, we have

$$\rho_{12}(t_1t_2) = \rho_{12}(t_1)\mu(t_2) + \mu(t_1)\rho_{12}(t_2). \tag{5.1}$$

For simplicity, write $g(t)$ for $\rho_{12}(t)$. Differentiating (5.1) with respect to t_1 holding t_2 constant, we get

$$g'(t_1t_2)t_2 = g'(t_1)\mu(t_2) + \mu'(t_1)g(t_2). \tag{5.2}$$

Setting $t_2 = 1$ in (5.2) and simplifying, we have

$$\mu'(t_1)g(1) = 0. \tag{5.3}$$

If $\mu'(t_1) = 0$, then $\mu(t) = 1$ for all t. Then from (5.1) we have $g(t_1t_2) = g(t_1)+g(t_2)$. This implies that $g(t) = c\ln(t)$ and by holomorphicity of g on $\mathbb{C}^*$, c must be zero. Therefore in this case $g(t) = 0$.

If $\mu'(t_1) \neq 0$, then $g(1) = 0$. Setting $t_1 = 1$, $t = t_2$ and using $\mu(t) = t^a$ in (5.2), we get

$$g'(t)t = g'(1)t^a + ag(t). \tag{5.4}$$

Solving this differential equation, and using $g(1) = 0$, we obtain $g(t) = g'(1)t^a \ln(t)$. For g to be holomorphic on $\mathbb{C}^*$, we must have $g'(1) = 0$ and $g(t) = 0$. Thus $\rho_{12}(t) = g(t) = 0$.

Suppose $a > b$. If $\rho_{ij}(t) = 0$ for all (i,j) such that $i \neq j$, and either $j < b$ or $j = b$ but $i < a$, then comparing (a,b) terms of $\rho(t_1t_2)$ and $\rho(t_1)\rho(t_2)$, we have

$$\rho_{ab}(t_1t_2) = \rho_{ab}(t_1)\mu(t_2) + \mu(t_1)\rho_{ab}(t_2). \tag{5.5}$$

Then by analogy with equation (5.1) we get $\rho_{ab}(t) = 0$. Therefore, the lemma follows by induction. □

Corollary 5.3. *If $\rho : \mathbb{C}^* \to G \subset h^{-1}K^*_\eta h$ is a holomorphic (or algebraic) homomorphism, then the image of ρ is contained in the intersection $h^{-1}(\mathbb{C}^*)^k h \cap G$.*

Theorem 5.4. *Suppose G is a holomorphic (resp. algebraic), closed, Abelian subgroup of $GL_k(\mathbb{C})$ that is contained in $h^{-1}K^*_\eta h$. Then a holomorphic (resp. algebraic) T-equivariant principal G-bundle over a nonsingular toric variety X admits a reduction of structure group to the intersection of G with the torus $h^{-1}(\mathbb{C}^*)^k h$.*

Proof. It follows easily from Corollary 5.3 that if $\rho : T \to G \subset h^{-1}K^*_\eta h$ is a homomorphism, then the image of ρ is contained in $h^{-1}(\mathbb{C}^*)^k h \cap G$. Apply this to the homomorphisms ρ_σ of Theorem 4.1 (resp. Theorem 4.3). By closedness of G and $h^{-1}(\mathbb{C}^*)^k h$, the transition maps, which are extensions of $\rho_\sigma\rho_\tau^{-1}$, also take values in $h^{-1}(\mathbb{C}^*)^k h \cap G$. □

Definition 5.1. For any element $g \in K_\eta^*$, let g_s denote the diagonal matrix having same diagonal entries as g. We say that a subgroup H of K_η^* is big, if the group $\{g_s \mid g \in H\}$ is isomorphic to $(\mathbb{C}^*)^r$ where r is the number of summands in the partition η.

Corollary 5.5. *Suppose G is a holomorphic (resp. algebraic), closed, Abelian subgroup of $GL_k(\mathbb{C})$ which is conjugate to a big subgroup of K_η^*. Let X be a complete nonsingular toric variety with fan Ξ. Then the isomorphism classes of holomorphic (resp. algebraic) T-equivariant principal G-bundles over X are parametrized by $\mathbb{Z}^{dr}$ where d is the number of one-dimensional cones of Ξ and r is the number of summands in the partition η.*

Proof. We may assume, without loss of generality, that G is a big subgroup of K_η^*. A homomorphism $\rho_\sigma : T \to G$ is determined by r homomorphisms $\mu_\sigma^i : T \to \mathbb{C}^*$ corresponding to the diagonal blocks. Moreover the extension condition on $\rho_\sigma \rho_\tau^{-1}$ in Theorem 4.1 (resp. Theorem 4.3) decomposes into similar condition on $\mu_\sigma^i (\mu_\tau^i)^{-1}$ for each block. Therefore we consider each block separately.

For a particular block we consider homomorphisms $\{\mu_\sigma : T \to \mathbb{C}^* \mid \sigma \in \Xi(n)\}$ that satisfy the extension condition. Such a collection corresponds to an isomorphism class of T-equivariant principal $\mathbb{C}^*$-bundle on X. Following the remark in the introduction, the isomorphism classes of holomorphic and algebraic T-equivariant principal $\mathbb{C}^*$-bundles on X coincide, since $\mathbb{C}^*$ is linear algebraic.

We may identify T-equivariant principal $\mathbb{C}^*$-bundles on X with T-equivariant line bundles on X. The latter have been studied by Oda in [7], for instance. In Proposition 2.1(i) of [7], Oda defines a homomorphism from the group of support functions to the group of isomorphism classes of equivariant line bundles. The homomorphism is injective when X is complete. Moreover, when X is nonsingular, the group of support functions is isomorphic to $\mathbb{Z}^d$, and Theorem 4.1 (resp. Theorem 4.3) shows that the above homomorphism is surjective. More precisely, we can set μ_σ equal to the character corresponding to l_σ in Oda's construction, when σ has maximal dimension. For lower dimensional cones τ, the l_τ's are obtained by restriction of l_σ where $\tau \subset \sigma$. Completeness of the fan and the extension condition ensures that we obtain a support function by this procedure. □

Conflict of interest statement

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