Dynamics and stability of a thin liquid film on a heated rotating disk film with variable viscosity

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A theoretical analysis of the thermal effects on the dynamics of a thin nonuniform film of a nonvolatile incompressible viscous fluid on a heated rotating disk has been considered and the effects of temperature-dependent viscosity and surface tension have been analyzed. A nonlinear evolution equation describing the shape of the film interface has been derived as a function of space and time and its stability characteristics have been examined using linear theory. It has been observed that the infinitesimal disturbances decay for small wave numbers and are transiently stable for large wave numbers, for both zero and nonzero values of Biot number. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2099007]

I. INTRODUCTION

The generation of thin free-surface liquid films is of practical importance to many processes arising in the field of chemical engineering and has attracted the attention of several investigators in science and engineering due to its enormous applications in many industrial processes. The production of a thin axisymmetric film on a horizontal disk by spinning has been employed in several spin-coating applications such as phosphor coatings on television screens, photoresist films on silicon wafers for integrated circuits, magnetic storage disks, magnetic paint coating on substrates, fabrication on thin uniform layers of plastic scintillator on supporting aluminized Mylar™, and so on. The possibility of controlling local accelerations in the flow of a thin film on a rotating disk has resulted in such technological exploitations, despite the complications in modeling these flows mathematically. The final thickness of the film and the uniformity in the thickness are central issues in these applications and they are observed to be dependent on several factors such as the viscosity of the liquid film, different spin-up protocols, heat and mass transfer processes, surface tension effects, and so on.

A number of theoretical and experimental studies of the spin-coating process have been reported in the literature. These include the investigations on the modeling of the flow over a rotating disk, wave generation in the liquid film moving on the surface of a rotating disk and the stability characteristics of a thin film on a rotating substrate. The hydrodynamic analysis of the flow of a Newtonian liquid on a spinning disk has been first theoretically modeled by Emslie *et al.*[1] Since then, modeling of the flow over a rotating disk has received considerable attention and they cover fundamental aspects of the fluid dynamics for Newtonian and non-Newtonian liquids.[2] A special mention must be made of the investigation by Higgins,[3] on the effect of inertia on a thin film on a disk rotating with constant angular velocity in the limit of small film thickness or small time by perturbing the full Navier-Stokes equations. Later, Rehg and Higgins[4] have obtained a numerical solution of the film flow on a rotating disk for a wide range of Reynolds numbers and different spin-up protocols by including inertial effects and interfacial shear caused by an overlying phase. Wang *et al.*[5] have examined the accelerating phase of spin coating and has obtained similarity solution of the unsteady Navier-Stokes equation. Stillwagon and Larson[6,7] have examined the leveling of film over uneven substrate topography during spin coating. Kitamura[8] has extended the investigation of matched asymptotic solution of Higgins[3] to a thin nonplanar film. The transient peripheral profile of the film has been considered by Moriarity *et al.*[9] by assuming the existence of a precursor film on a disk, McKinley *et al.*[10] (slip model), Melo *et al.*[11] (experimental), and Spaid and Homsy[12] (experimental). Further, the stationary axisymmetric waveless flow in the limit of large Eckman number has been investigated by Shkadov,[13] Rauscher *et al.*,[14] and Woods.[15]

The majority of experimental investigations of flow over a spinning disk have dealt with measurements of local maximum or local mean film thickness. Thomas *et al.*[16] have examined the wave formation at all values of the flow rate and rotational speed and have distinguished two wave regimes: wave-laminar flows involving irregular wave patterns and radial wave flows occurring at large values of the rotational speed and are accompanied by the formation of well-defined radially propagating waves travelling on a thin liquid substrate, carrying the main bulk of the liquid to the disk periphery. Their results suggest that the film thickness initially increases with radius upto a local maximum value beyond which the film exhibits a monotone decrease. Sisoev *et al.*[17] have extended the methods and results developed for the falling film problem to model the axisymmetric flow regimes in the flow of a thin liquid film over a spinning disk for a wide range of system parameters. Their results show that the steady quasiperiodic waves with largest amplitude compare well with experimentally observed wave profiles.

The stability of the limiting stationary solution for a rela-

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tively thin film has been investigated using an asymptotic method[13,18,19] or a numerical method.[20] The numerical solution by Sisoev and Shkadov,[20] under the assumption of local plane parallelism, on the stability of a steady axisymmetric flow of a film with a relatively small thickness on the surface of a rotating disk, has indicated a stabilizing effect of the surface tension. The spatial stability of a liquid film on a rotating disk has been examined by Eliseev[21] within a framework of linear theory and his results have shown that the wavelength of the perturbation determines the radial domain of the instability, which becomes wider with increasing wavelength. Charwat *et al.*[18] have examined the linear stability of the flow in the case of large Eckman number in the small wave number limit. Their results show that axisymmetirc perturbations have the largest amplification factors and coriolis forces exert a stabilizing influence. Needham and Merkin[19] have examined the unsteady flow driven by perturbations applied to the flux. Woods[15] has performed a linear stability analysis of spiral perturbations using lubrication approach. The above investigations[13–21] describe the film flow over a spinning disk in the case of constant fluid feeding to the center of the disc. In all these studies, steady or unsteady wave regimes exist in time, and their parameters strongly depend on the surface tension.

The above-mentioned investigations, however, have not incorporated the influence of heat transfer in the liquid phase and the thermocapillary force at the liquid-gas interface on the development of flow and thickness of the liquid film on a spinning disk, which arise due to temperature difference between the disk and the ambient gas and the initial liquid. Dandapat and Ray[22,23] have analyzed the flow of a thin liquid film of uniform thickness over a cold/hot rotating disk when the disk is either cooled or heated axisymmetrically from below. Usha and Ravindran[24] have considered the development of a heat conducting fluid film of nonuniform thickness on a rotating disk that is cooled axisymmetrically from below and have solved the evolution equation for transient film thickness numerically. Recently, Kitamura[25] has investigated the film flow of nonuniform thickness on a heated rotating disk and has focused on the effects of heat transfer on a nonvolatile fluid film whose viscosity and surface tension vary with temperature. A uniform temperature on the disk has been imposed and the thermal effects on the transient film profile has been examined. An analytical expression for transient film thickness of a nonuniform film profile has been obtained by a perturbation method. Kitamura's solution, obtained at small radius shows that viscosity decreasing due to heating leads to the film thinning.

Elliott and Hockey,[26] Rehg,[27] and Rehg and Higgins[28] have postulated that the radial striations are a consequence of a Marangoni flow, the secondary flow driven by temperature induced surface tension gradients. In spin coating, surface tension gradients can be induced either by temperature gradients (due to evaporative cooling) or by concentration gradients. Rehg[27] has demonstrated experimentally that the spin-coating conditions that give rise to striations are consistent with Marangoni convection. His results have shown that only solutions of intermediate viscosity show the striations. In the case of high viscosity solutions, Marangoni convection

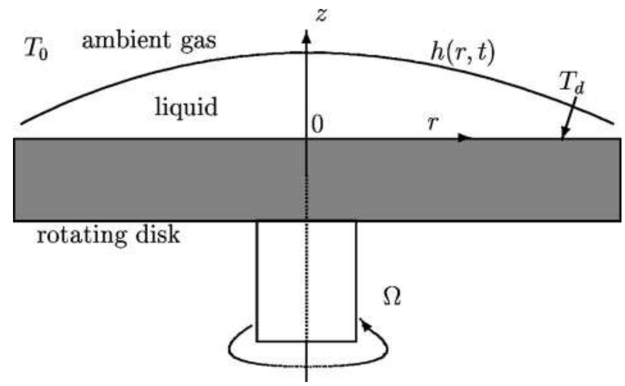

FIG. 1. Schematic representation of film flow on a heated rotating disk.

is suppressed by large viscous forces. Although the Marangoni number is initially large for low viscosity films and the film is initially susceptible to Marangoni convection, Rehg[27] and Rehg and Higgins[28] have argued that the films spin coated from low viscosity coating solutions are stabilized by the dynamic thinning process. The striations do not grow to appreciable magnitude before the flow is restabilized by viscous forces that become increasingly important as the film rapidly thins. Reisfeld *et al.*[29] have predicted an analogous phenomenon, the formation and suppression of inertial circumferential waves. Using a linear stability analysis, they have shown that spin-coated films are unstable to formation of these waves during spin-up, but as the film thins, the stabilizing effect of viscosity intervenes and the disturbances decay.

In the present study, the dynamics and stability characteristics of the film on a heated rotating disk is examined using linear theory. It is observed that infinitesimal disturbances decay for small wave numbers and are transiently stable for larger wave numbers. The temperature-dependent viscosity has a stabilizing effect on the flow in the film. The results show that the film flow of nonuniform thickness on a heated rotating disk is unstable to formation of waves during spin-up but as the film thins, the stabilizing effect of viscosity intervenes and the disturbance decays. The decay is faster for a film with temperature-dependent viscosity.

II. MATHEMATICAL FORMULATION

An axisymmetric flow of a nonvolatile incompressible, viscous liquid with an initial temperature of T_0 (the same temperature as the ambient gas) on a horizontal rotating disk with a constant temperature T_d and a constant angular velocity Ω_0 is considered (Fig. 1). A variation of temperature in the liquid film produces changes in the values of the physical properties of the liquid. When this variation is small, the changes in physical properties of common liquids, other than (kinematic) viscosity can be assumed to be small. In the formulation, kinematic viscosity and surface tension are assumed to depend on temperature and the other properties are taken to be independent of temperature.[30] The relative variation of surface tension with temperature is small but the resulting shear stress is taken into account in order to incorporate the thermocapillary effect. Further, it is assumed that

$T_d - T_0$ is small in magnitude, the shear stress at the free surface of the fluid caused by the air flow[31] and the evaporation at the liquid-gas interface[32] are negligible. The cylindrical polar coordinates (r,θ,z) in a frame of reference rotating with the disk, where r measures the radial distance from the center of the disk, θ is the angle from some fixed radial line in the horizontal plane and z measures the distance vertically upward from the solid surface of the disk, is used. The liquid-gas interface is located at $z=h(r,t)$, where h is the film thickness as a function of r and t. The outward unit normal vector, $\mathbf{n}$, and unit tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are given by

$$\mathbf{n} = (-h_r,0,1)(1+h_r^2)^{-1/2}, \tag{1}$$

$$\mathbf{t}_1 = (1,0,h_r)(1+h_r^2)^{-1/2}, \tag{2}$$

$$\mathbf{t}_2 = (0,1,0). \tag{3}$$

The equations of motion, continuity, and energy conservation are

$$\rho(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla p + \nabla\cdot\bar{\tau} - \rho[2\boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \mathbf{g}], \tag{4}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{5}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \alpha\nabla^2 T, \tag{6}$$

where ρ, p, $\bar{\tau}$, and α are the density, the pressure, the viscous stress tensor, the thermal diffusivity of the liquid, respectively, and $\mathbf{v}$ and $\boldsymbol{\omega}$ are the fluid velocity vectors with components (u,v,w) and angular velocity vector $(0,0,\Omega_0)$, respectively; $\mathbf{g}=(0,0,g)$ is the gravitational vector, $\mathbf{r}=(r,0,0)$ is the radius vector, and T is the temperature of the fluid. The boundary conditions are

$$\mathbf{v}=0, \quad T=T_d \quad \text{on} \quad z=0 \tag{7}$$

(no-slip condition, constant temperature at the disk),

$$\mathbf{n}\cdot\bar{\mathbf{T}}\cdot\mathbf{n} = -\nabla\cdot\mathbf{n}\sigma \quad \text{on} \quad z=h(r,t) \tag{8}$$

(balance of normal stress at the free surface),

$$\mathbf{n}\cdot\bar{\mathbf{T}}\cdot\mathbf{t}_1 = (1+h_r^2)^{-1/2}(\sigma_r + h_r\sigma_z) \quad \text{on} \quad z=h(r,t) \tag{9}$$

(balance of shear stress at the free surface),

$$\mathbf{n}\cdot\bar{\mathbf{T}}\cdot\mathbf{t}_2 = 0 \quad \text{on} \quad z=h(r,t) \tag{10}$$

(balance of shear stress at the free surface),

$$-k\mathbf{n}\cdot\nabla T = L_h(T-T_0) \quad \text{on} \quad z=h(r,t) \tag{11}$$

(continuity of heat flux; Newton's law of cooling at the free surface),

$$w = h_t + uh_r \quad \text{on} \quad z=h(r,t) \tag{12}$$

(kinematic boundary condition at the free surface),

where $\bar{\mathbf{T}}$ is the stress tensor, σ is the coefficient of surface tension, k is the thermal conductivity, and L_h is the heat transfer coefficient.

Nondimensionalizing the governing equations and boundary conditions using dimensionless (asterisks) variables

$$h^* = \frac{h}{h_0}, \quad z^* = \frac{z}{h_0}, \quad r^* = \frac{r}{L}, \quad t^* = \frac{t\Omega_0^2 h_0^2}{\nu_0},$$

$$u^* = \frac{u\nu_0}{\Omega_0^2 L h_0^2}, \quad v^* = \frac{v\nu_0^2}{\Omega_0^3 L h_0^4}, \quad w^* = \frac{w\nu_0}{\Omega_0^2 h_0^3}, \quad p^* = \frac{p}{\rho\Omega_0^2 L^2}, \tag{13}$$

$$T^* = \frac{T-T_0}{T_d - T_0}, \quad \nu^* = \frac{\nu}{\nu_0}, \quad \sigma^* = \frac{\sigma}{\sigma_0}, \quad Q^* = \frac{Q}{Q_0},$$

where h_0, L, ν_0, and σ_0 are the initial film thickness at $r=0$, the disk radius, the kinematic viscosity, and surface tension coefficient of the fluid when $T=T_0$, respectively, and Q_0 is the initial fluid volume on the disk, the equations are obtained as (after dropping asterisks)

$$\epsilon\mathrm{Re}[u_t + uu_r + wu_z] - \epsilon^2\,\mathrm{Re}^2\frac{v^2}{r} = -p_r + 2\epsilon\,\mathrm{Re}\,v + r + 2\epsilon^2\left[\nu\frac{u_r}{r} - \nu\frac{u}{r^2} + (\nu u_r)_r\right] + [\nu(u_z + \epsilon^2 w_r)]_z, \tag{14}$$

$$\epsilon\,\mathrm{Re}\left[v_t + uv_r + \frac{uv}{r} + wv_z\right] = \frac{\epsilon^2}{r^2}[\nu r^3(v/r)_r]_r + (\nu v_z)_z - 2u, \tag{15}$$

$$\epsilon^3\,\mathrm{Re}[w_t + uw_r + ww_z] = -p_z - \epsilon\,\mathrm{Fr} + \frac{\epsilon^2}{r}[\nu r(u_z + \epsilon^2 w_r)]_r + 2\epsilon^2[\nu w_z]_z, \tag{16}$$

$$\frac{1}{r}(ru)_r + w_z = 0, \tag{17}$$

$$\epsilon\mathrm{Pe}[T_t + uT_r + wT_z] = \epsilon^2\left[T_{rr} + \frac{T_r}{r}\right] + T_{zz}, \tag{18}$$

$$u=v=w=0; \quad T=1 \quad \text{on} \quad z=0, \tag{19}$$

$$-p + \frac{2\epsilon^2\nu}{(1+\epsilon^2 h_r^2)}[w_z + \epsilon^2 u_r h_r^2 - h_r u_z - \epsilon^2 h_r w_r] = \frac{\epsilon\overline{\mathrm{We}}}{r(1+\epsilon^2 h_r^2)^{3/2}}[h_r + \epsilon^2 h_r^3 + rh_{rr}] \quad \text{on} \quad z=h(r,t), \tag{20}$$

$$\nu[2\epsilon^2 h_r(w_z-u_r)+(1-\epsilon^2 h_r^2)(u_z+\epsilon^2 w_r)] = \bar{\alpha}\epsilon\sqrt{1+\epsilon^2 h_r^2}(T_r+h_r T_z) \quad \text{on} \quad z=h(r,t), \tag{21}$$

$$v_z-\epsilon^2 h_r\left(v_r-\frac{v}{r}\right)=0 \quad \text{on} \quad z=h(r,t), \tag{22}$$

$$T_z-\epsilon^2 h_r T_r=-\mathrm{Bi}(1+\epsilon^2 h_r^2)^{1/2}T \quad \text{on} \quad z=h(r,t), \tag{23}$$

$$-h_t-uh_r+w=0 \quad \text{on} \quad z=h(r,t), \tag{24}$$

where $\epsilon=h_0/L$ is the aspect ratio, $\mathrm{Re}=\Omega_0^2 L h_0^3/\nu_0^2$ is the Reynolds number, $\mathrm{Fr}=g/\Omega_0^2 L$ is the modified Froude number, $\overline{\mathrm{We}}=\sigma/\rho\Omega_0^2 L^3$ is the modified Weber number, $\mathrm{Pe}=\Omega_0^2 L h_0^3/\nu_0\alpha$ is the Peclet number, $\mathrm{Bi}=L_h h_0/k$ is the Biot number, $\nu=1-\delta T$, $\delta=-\nu_0^{-1}(d\nu/dT)_{T=T_0}(T_d-T_0)$, $\sigma=1-\hat{\delta}T$, $\hat{\delta}=-\sigma_0^{-1}(d\sigma/dT)_{T=T_0}(T_d-T_0)$, and $\bar{\alpha}=\sigma_T(T_d-T_0)/\rho\Omega_0^2 L h_0^2$ is the thermocapillary parameter. Here, δ and $\hat{\delta}$ are positive constants and ν and σ are taken in this form because ν and σ decrease with temperature for common liquids. The order of magnitude of the nondimensional parameters can be estimated by taking

$$h_0=O(10^{-4})-O(10^{-3})\text{ m}, \quad L=O(10^{-2})\text{ m},$$

$$T=O(10^0)\text{ deg}, \quad \Omega_0=O(10^1) \quad \text{rps}$$

(rps—rotations per second) as Kitamura[25]

$$\begin{aligned} &\epsilon=O(10^{-2})-O(10^{-1}), \quad \delta=O(10^{-2})-O(10^{-1}),\\ &\hat{\delta}=O(10^0)-O(10^1),\\ &\mathrm{Re}=O(10^0), \quad \mathrm{Fr}=O(10^0),\\ &\overline{\mathrm{We}}=O(10^{-1})-O(10^0), \quad \mathrm{Pe}=O(10^0). \end{aligned} \tag{25}$$

III. DERIVATION OF EVOLUTION EQUATION

The order of magnitude of the parameters ϵ, δ, and $\hat{\delta}$ under consideration [given by Eq. (25)] imply that the relative variation of surface tension with temperature is very small as compared to that of viscosity with temperature and therefore, the evolution equation is derived under the assumption that the surface tension is a constant. The dependent variables u, v, w, p, and T are expanded in powers of ϵ and δ as

$$(u,v,w,p,T)=\sum_{\hat{s},\hat{q}}\delta^{\hat{s}}\epsilon^{\hat{q}}[u_{\hat{s},\hat{q}},v_{\hat{s},\hat{q}},w_{\hat{s},\hat{q}},p_{\hat{s},\hat{q}},T_{\hat{s},\hat{q}}] \tag{26}$$

and substituted in Eqs. (14)–(23). This results in zero-order and first-order equations in δ and ϵ as follows;

zero-order equations:

$$\begin{aligned} &-p_{00r}+r+u_{00zz}=0,\\ &v_{00zz}-2u_{00}=0,\\ &-p_{00z}=0,\\ &u_{00r}+\frac{u_{00}}{r}+w_{00z}=0,\\ &T_{00zz}=0, \end{aligned} \tag{27}$$

$$u_{00}=0, v_{00}=0, \quad w_{00}=0; \quad T_{00}=1 \quad \text{on} \quad z=0, \tag{28}$$

$$\begin{aligned} &-p_{00}=0, \quad u_{00z}=0, \quad v_{00z}=0,\\ &T_{00z}=-\mathrm{Bi}\,T_{00} \quad \text{on} \quad z=h. \end{aligned} \tag{29}$$

First-order equations in δ:

$$\begin{aligned} &-p_{10r}+u_{10zz}-T_{00z}u_{00z}-T_{00}u_{00zz}=0,\\ &v_{10zz}-T_{00}v_{00zz}-T_{00z}v_{00z}-2u_{10}=0,\\ &-p_{10z}=0,\\ &u_{10r}+\frac{u_{10}}{r}+w_{10z}=0,\\ &T_{10zz}=0, \end{aligned} \tag{30}$$

$$u_{10}=0, \quad v_{10}=0, \quad w_{10}=0; \quad T_{10}=0 \quad \text{on} \quad z=0, \tag{31}$$

$$\begin{aligned} &-p_{10}=0, \quad -T_{00}u_{00z}+u_{10z}=0, \quad v_{10z}=0,\\ &T_{10z}=-\mathrm{Bi}\,T_{10} \quad \text{on} \quad z=h. \end{aligned} \tag{32}$$

First-order equations in ϵ:

$$\begin{aligned} &\mathrm{Re}[u_{00t}+u_{00}u_{00r}+w_{00}u_{00z}]=-p_{01r}+2\,\mathrm{Re}\,v_{00}+u_{01zz},\\ &\mathrm{Re}\left[v_{00t}+u_{00}v_{00r}+\frac{u_{00}v_{00}}{r}+w_{00}v_{00z}\right]=v_{01zz}-2u_{01},\\ &-p_{01z}-\mathrm{Fr}=0,\\ &u_{01r}+\frac{u_{01}}{r}+w_{01z}=0,\\ &T_{01zz}=\mathrm{Pe}[T_{00t}+u_{00}T_{00r}+w_{00}T_{00z}], \end{aligned} \tag{33}$$

$$u_{01}=0, \quad v_{01}=0, \quad w_{01}=0; \quad T_{01}=0 \quad \text{on} \quad z=0, \tag{34}$$

$$\begin{aligned} &p_{01}=-\mathrm{We}\left[h_{rr}+\frac{h_r}{r}\right], \quad u_{01z}=\bar{\alpha}[T_{00r}+h_r T_{00z}] \quad \text{on} \quad z=h,\\ &v_{01z}=0, \quad T_{01z}=-\mathrm{Bi}\,T_{01} \quad \text{on} \quad z=h, \end{aligned} \tag{35}$$

where $\mathrm{We}=\sigma_0/\rho\Omega_0^2 L^3$ is the Weber number. The solutions of Eqs. (28)–(35) give the radial and axial velocities as

$$u = \frac{1}{2}rz(2h-z) - \frac{\delta r}{6}\left[\frac{-2\,\mathrm{Bi}}{(1+\mathrm{Bi}\,h)}z^3 + 3\frac{(1+2\,\mathrm{Bi}\,h)}{(1+\mathrm{Bi}\,h)}z^2 - 6hz\right] + \epsilon\left[\bar{\alpha}\frac{\mathrm{Bi}}{(1+\mathrm{Bi}\,h)^2}h_r z + \mathrm{Re}\left(\frac{rz^6}{360} - \frac{rhz^5}{60} + \frac{r^2h_rhz^4}{24} + \frac{2rh^3z^3}{9} + \frac{rh_tz^3}{6} - \frac{3rh^5z}{5} - \frac{r^2h_rh^4z}{6} - \frac{rh_th^2z}{2}\right) + \mathrm{Fr}\left(\frac{h_rz^2}{2} - hh_rz\right) + \mathrm{We}\left(h_{rrr} + \frac{h_{rr}}{r} - \frac{h_r}{r^2}\right)\left(\frac{-z^2}{2} + hz\right)\right], \tag{36}$$

$$w = \frac{z^2}{6}(2z - 3rh_r - 6h) + \frac{\delta}{r}\left[\frac{\mathrm{Bi}}{(1+\mathrm{Bi}\,h)}\frac{r^2z^4}{12} - \frac{1}{6}r^2z^3 \times\left(1 + \frac{\mathrm{Bi}\,h}{(1+\mathrm{Bi}\,h)}\right) + \frac{r^2hz^2}{2}\right]_r - \frac{\epsilon}{r}\left\{\bar{\alpha}\frac{\mathrm{Bi}}{2(1+\mathrm{Bi}h)^2}rh_rz^2 + \mathrm{Re}\left[\frac{r^2z^7}{1260} - \frac{r^2hz^6}{360} + \frac{r^3hh_rz^5}{120} + \frac{r^2h^3z^4}{18} + \frac{r^2h_tz^4}{24} - \frac{3r^2h^5z^2}{10} - \frac{r^3h_rh^4z^2}{12} - \frac{r^2h_th^2z^2}{4}\right] + \mathrm{Fr}\left[\frac{rh_rz^3}{6} - \frac{rhh_rz}{6}\right] + \mathrm{We}\left[\left(rh_{rrr} + h_{rr} - \frac{h_r}{r}\right)\left(-\frac{z^3}{6} + \frac{hz^2}{2}\right)\right]\right\}_r. \tag{37}$$

Substituting Eqs. (36) and (37) into the kinematic boundary condition at the free surface given by Eq. (24), the evolution equation is obtained as

$$h_t = -\frac{2}{3}h^3 - rh^2h_r - \delta\left[\frac{-\mathrm{Bi}}{6(1+\mathrm{Bi}\,h)}(2rh^3h_r + h^4) + \frac{\mathrm{Bi}^2\,rh^4h_r}{12(1+\mathrm{Bi}\,h)^2} + \frac{2h^3}{3} + rh^2h_r\right] + \epsilon\left[\mathrm{Re}\left(\frac{68h^7}{315} + \frac{16rh^6h_r}{45} - \frac{2r^2h^6h_{rr}}{15} - \frac{4r^2h^5h_r^2}{5}\right) + \mathrm{Fr}\left(\frac{h^3h_{rr}}{3} + \frac{h^3h_r}{3r} + h^2h_r^2\right) - \mathrm{We}\left(\left(h_{rrr} + \frac{h_{rr}}{r} - \frac{h_r}{r^2}\right)h^2h_r + \frac{h^3}{3}\left(h_{rrrr} + \frac{2h_{rrr}}{r} - \frac{h_{rr}}{r^2} + \frac{h_r}{r^3}\right)\right) - \bar{\alpha}\frac{\mathrm{Bi}\,h^2}{2}\left(\frac{h_{rr}}{(1+\mathrm{Bi}\,h)^2} - \frac{2\mathrm{Bi}\,h_r^2}{(1+\mathrm{Bi}\,h)^3} + \frac{h_r}{r(1+\mathrm{Bi}\,h)^2}\right) - \frac{\bar{\alpha}\,\mathrm{Bi}\,hh_r^2}{(1+\mathrm{Bi}\,h)^2}\right]. \tag{38}$$

In deriving Eq. (38), the term h_t contained within the Re term is replaced by its leading order representation in terms of r, h and its spatial derivative.[29,33] Equation (38) describes the shape of a thin liquid film interface on a rotating substrate [Eq. (38) is a corrected version of Eq. (11) in Kitamura's[25] paper obtained after correcting the mistakes in the sign in certain terms and the error in Eq. (7b) of Kitamura[25]]. Kitamura[25] has obtained an analytical expression for transient film thickness of a nonuniform film profile by solving Eq. (38) by a perturbation method and has examined the thermal effects on the transient film profile. In his analysis, he has neglected the peripheral effects of the liquid film. The results reveal the effects of inertia, gravitational, surface tension, thermocapillary forces and of variable viscosity on the film planarization and thinning. It has been observed that the variable viscosity has the most profound effect on the transient film thickness.[25]

The boundary conditions at the center of the disk $r=0$ are determined from the symmetry conditions as

$$\frac{\partial h(r=0,t)}{\partial r} = 0, \quad \frac{\partial^3 h(r=0,t)}{\partial r^3} = 0. \tag{39}$$

The boundary condition at the edge of the finite disk cannot be prescribed in a simple mathematical form due to the complicated edge effect, known as the "teapot" effect;[34,35] further, Reisfeld *et al.*[29] have pointed out that specifying appropriate boundary conditions at the edge of the disk is difficult and that it requires an intimate understanding of the complicated time-dependent film rupture process taking place in that region. For thin films, this effect is regarded as a local phenomenon occurring at the edge and is thought to have a negligible influence on the solution away from the edge.[3,34] One of the ways of prescribing the boundary conditions at the edge of the disk suggested by Tu[34] and Hwang and Ma[35] and implemented in their analysis of thin liquid film on a spinning disk is as follows: They have assumed that the disk hypothetically extends beyond the edge as a smooth flat surface and have taken the boundary condition at the edge of the disk as follows: Let the edge of the disk be $r_{\max}$. Then a boundary condition at $r=r_{\max}$ is

$$\frac{\partial H}{\partial r}(r = r_{\max}, T) = 0. \tag{40}$$

The second way of imposing the boundary condition at the edge of the disk is to use the asymptotic properties of Eq. (38) at large values of the radius and use the results of Higgins[3] as the condition at large radial distances (that is, at the edge of the disk). In view of the difficulties in specifying appropriate boundary conditions for the evolution equation, as expressed previously, in what follows, the efforts are towards investigation on the time-dependent basic state and the determination of its stability characteristics using linear theory. The analysis extends the study by Reisfeld *et al.*[29] to the case when the rotating disk is heated from below with the model equation for further analysis as the simplified evolution equation given by

$$h_t = -\frac{2}{3}h^3 - rh^2h_r - \delta\left[\frac{-\mathrm{Bi}}{6(1+\mathrm{Bi}\,h)}(2rh^3h_r + h^4) + \frac{\mathrm{Bi}^2\,rh^4h_r}{12(1+\mathrm{Bi}\,h)^2} + \frac{2h^3}{3} + rh^2h_r\right] + \epsilon\left[\mathrm{Re}\left(\frac{68h^7}{315} + \frac{16rh^6h_r}{45} - \frac{2r^2h^6h_{rr}}{15} - \frac{4r^2h^5h_r^2}{5}\right)\right]. \tag{41}$$

It is reasonable to examine the linear instability of a nonstationary problem since there are two sets of solutions to the governing system that are of particular interest. One is spatially uniform and time dependent. It describes a long flat layer of fluid film spreading radially due to centrifugal effects. The second basic state is steady and develops in space.

It describes a steady layer, which thins in the flow (radial) direction and may terminate in a contact line. In the present study, only the time-dependent spatially uniform basic state is examined and its stability to long wave disturbances is analyzed. Such an analysis of the linear instability of nonstationary problems has been performed by several researchers including Joo *et al.*[36]

IV. DESCRIPTION OF TIME-DEPENDENT BASIC STATE

As the film is draining due to centrifugation, the basic state is time dependent and it is assumed to be flat. The film thickness is independent of the radial position and therefore the dependence on r is removed by using the following transformation:[3,37]

$$h=\bar{h}(t),\quad u=r\bar{u}(z,t),\quad v=r\bar{v}(z,t),\quad w=\bar{w}(z,t), \tag{42}$$

where quantities with an overbar denote the basic-state quantities. The resulting nonlinear ordinary differential equation governing the basic state behavior is obtained as

$$\bar{h}_t=-\frac{2}{3}\bar{h}^3(1+\delta)+\frac{\delta}{6}\left(\frac{\mathrm{Bi}}{1+\mathrm{Bi}\,\bar{h}}\right)\bar{h}^4+\epsilon\,\mathrm{Re}\frac{68}{315}\bar{h}^7, \tag{43}$$

where $\bar{h}(0)=1$. For fixed values of the Biot number Bi, Eq. (43) is integrated to give the time-dependent basic state behavior.

Case i: Biot number $\mathrm{Bi}=0$.

The governing basic state equation is obtained from Eq. (43) as

$$\bar{h}_t=-\frac{2}{3}\bar{h}^3(1+\delta)+\epsilon\,\mathrm{Re}\frac{68}{315}\bar{h}^7. \tag{44}$$

The leading order solution to Eq. (44) is given by

$$\bar{h}(t)=\frac{1}{\sqrt{1+\frac{4}{3}(1+\delta)t}}, \tag{45}$$

which shows that the thickness of the film decreases monotonically in time and that the thickness goes to zero as $t\to\infty$. The effect of δ is to enhance the thinning rate of the film.

Case ii: Biot number $\mathrm{Bi}\neq 0$.

The differential equation for the basic state is given by Eq. (43) and the leading order solution is obtained as

$$t-\frac{3}{4(1+\delta)}\left(\frac{1}{\bar{h}^2}-1\right)-\frac{3\delta\,\mathrm{Bi}}{8(1+\delta)^2}\left(\frac{1}{\bar{h}}-1\right)-3\,\mathrm{Bi}^2\frac{(4\delta+3\delta^2)}{32(1+\delta)^3}\ln\bar{h}+3\,\mathrm{Bi}^2\frac{(4\delta+3\delta^2)}{32(1+\delta)^3}\times\ln\left[\frac{(4+4\delta)+(4+3\delta)\mathrm{Bi}\,\bar{h}}{(4+4\delta)+(4+3\delta)\mathrm{Bi}}\right]=0. \tag{46}$$

V. LINEAR STABILITY ANALYSIS AND DISCUSSION

The linearized disturbance equation is obtained from Eq. (41) by perturbing the basic state by a small amount $h'[h=\bar{h}(t)+\bar{a}h'(r,t)]$ and substituting in Eq. (41) and linearizing in disturbance amplitude "$\bar{a}$" as

$$h'_t+a_1(t)r^2h'_{rr}+a_2(t)rh'_r+a_3(t)h'=0, \tag{47}$$

where

$$a_1(t)=\frac{2}{15}\epsilon\,\mathrm{Re}\,\bar{h}^6, \tag{48}$$

$$a_2(t)=(1+\delta)\bar{h}^2-\frac{\delta\,\mathrm{Bi}\,\bar{h}^3}{3(1+\mathrm{Bi}\,\bar{h})}+\frac{\delta\,\mathrm{Bi}^2\,\bar{h}^4}{12(1+\mathrm{Bi}\,\bar{h})^2}-\frac{16\epsilon\,\mathrm{Re}\,\bar{h}^6}{45}, \tag{49}$$

$$a_3(t)=\frac{3\mathrm{Bi}\,\bar{h}_t}{(1+\mathrm{Bi}\,\bar{h})}+\frac{2\,\mathrm{Bi}\,\bar{h}^3(1+\delta)}{(1+\mathrm{Bi}\,\bar{h})}+2(1+\delta)\bar{h}^2-\frac{68\epsilon\,\mathrm{Re}\,\bar{h}^6}{45}-\frac{\delta\,\mathrm{Bi}^2\,\bar{h}^4}{3(1+\mathrm{Bi}\,\bar{h})^2}-\frac{2\delta\,\mathrm{Bi}\,\bar{h}^3}{3(1+\mathrm{Bi}\,\bar{h})}-\frac{68\epsilon\,\mathrm{Re}\,\mathrm{Bi}\,\bar{h}^7}{105(1+\mathrm{Bi}\,\bar{h})}. \tag{50}$$

Transforming Eq. (47) into an equation with constant coefficients by the transformation $r=e^{\xi}, \xi=\ln r$, gives

$$h'_t+a_1(t)h'_{\xi\xi}+[a_2(t)-a_1(t)]h'_{\xi}+a_3(t)h'=0. \tag{51}$$

Using normal mode analysis and assuming the disturbance quantity h' in the form

$$h'(\xi,t)=H_d(t)e^{ik_0\xi}, \tag{52}$$

where $i=\sqrt{-1}$, the equation for normal mode amplitude $H_d(t)$ is obtained as

$$\frac{\dot{H}_d}{H_d}=\frac{\mathrm{Re}\,\bar{h}^6}{45}[6\bar{k}^2+22\epsilon^{1/2}i\bar{k}+68\epsilon]+\frac{\delta\,\mathrm{Bi}\,\bar{h}^3}{3(1+\mathrm{Bi}\,\bar{h})}\times[2+i\bar{k}\epsilon^{-1/2}]-\frac{\delta\,\mathrm{Bi}^2\,\bar{h}^4}{12(1+\mathrm{Bi}\,\bar{h})^2}[2+i\bar{k}\epsilon^{-1/2}]-\bar{h}^2(1+\delta)[2+i\bar{k}\epsilon^{-1/2}], \tag{53}$$

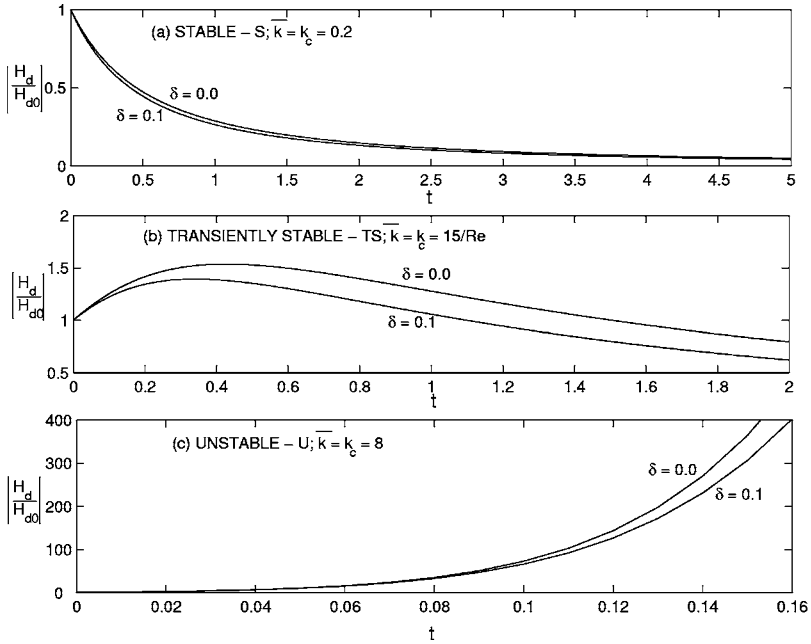

FIG. 2. Normal mode amplitude for a disturbance when Bi=0.0, Re=6.2, and ϵ=0.01.

where $\bar{k}=\sqrt{\epsilon}k_0$. The equation can be rewritten as

$$\frac{d}{d\bar{h}}(\ln H_d)=\frac{1}{45\bar{h}_t}\Bigg\{(6\bar{k}^2+22\epsilon^{1/2}i\bar{k}+68\epsilon)\mathrm{Re}\,\bar{h}^6+\frac{15\delta\,\mathrm{Bi}\,\bar{h}^3}{(1+\mathrm{Bi}\,\bar{h})}[2+i\bar{k}\epsilon^{-1/2}]-\frac{15\delta\,\mathrm{Bi}^2\,\bar{h}^4}{4(1+\mathrm{Bi}\,\bar{h})^2}\times[2+i\bar{k}\epsilon^{-1/2}]-45\bar{h}^2(1+\delta)[2+i\bar{k}\epsilon^{-1/2}]\Bigg\}. \tag{54}$$

In what follows, attention is focused on the region away from the origin where inertial effects are important and where the equations are valid.

It is observed from Eq. (53) that the extremum point in time for $|H_d|$ occurs whenever

$$(6\bar{k}^2+68\epsilon)\frac{\mathrm{Re}\,\bar{h}^6}{90}+\frac{\delta\,\mathrm{Bi}\,\bar{h}^3}{3(1+\mathrm{Bi}\,\bar{h})}-\frac{\delta\,\mathrm{Bi}^2\,\bar{h}^4}{12(1+\mathrm{Bi}\,\bar{h})^2}-\bar{h}^2(1+\delta)=0, \tag{55}$$

which gives

$$k_c^2=\frac{15(1+\delta)}{\mathrm{Re}\,\bar{h}^4}-\frac{34}{3}\epsilon+\frac{5\delta\,\mathrm{Bi}^2}{4(1+\mathrm{Bi}\,\bar{h})^2\,\mathrm{Re}\,\bar{h}^2}-\frac{5\delta\mathrm{Bi}}{(1+\mathrm{Bi}\,\bar{h})\mathrm{Re}\,\bar{h}^3}, \tag{56}$$

where k_c is the cutoff wave number. The cutoff wave number for the relative disturbance amplitude is obtained by determining the location of an extremum point for the relative disturbance amplitude $|H_d|/\bar{h}$ and is given by

$$k_{rc}^2=\frac{10(1+\delta)}{\mathrm{Re}\,\bar{h}^4}-\frac{68}{7}\epsilon+\frac{5\delta\,\mathrm{Bi}^2}{4(1+\mathrm{Bi}\,\bar{h})^2\mathrm{Re}\,\bar{h}^2}-\frac{15\delta\mathrm{Bi}}{4(1+\mathrm{Bi}\,\bar{h})\mathrm{Re}\bar{h}^3}. \tag{57}$$

Case i: Biot number Bi=0.

In this case, Eq. (44) governs the behavior of the thinning basic state film and this is substituted into the normal mode equation and integrated. This yields amplitude of the disturbance as

$$H_d=H_{d0}\bar{h}^3\exp\left[\mathrm{Re}\left(\frac{\bar{k}^2}{20(1+\delta)}+\frac{17\epsilon}{30(1+\delta)}-\frac{17\epsilon}{70(1+\delta)}\right)\times(1-\bar{h}^4)+\frac{17\epsilon\bar{k}^2\,\mathrm{Re}^2}{2100(1+\delta)^2}(1-\bar{h}^8)\right]\times\exp\left[i\bar{k}\left\{\epsilon^{-1/2}\,\mathrm{Re}\left[\frac{11}{60(1+\delta)}-\frac{17}{140(1+\delta)}\right]+\frac{187\epsilon^{3/2}\,\mathrm{Re}^2}{6300(1+\delta)^2}(1-\bar{h}^8)+\frac{3}{2}\epsilon^{-1/2}\ln\bar{h}\right\}\right]. \tag{58}$$

Figure 2 presents the normal mode amplitude for a disturbance that is (a) stable—$S(\bar{k}^2<k_c^2)$, (b) transiently stable—$TS(\bar{k}^2=k_c^2)$, and (c) unstable—$U(\bar{k}^2>k_c^2)$. The real exponent in Eq. (58) indicates that $|H_d/H_{d0}|$ will increase to a finite amplitude for all $\bar{k}$ but the presence of $\bar{h}^3$ shows that $|H_d/H_{d0}|\to 0$ as $\bar{h}\to 0$.

Davis[38] has pointed out that the product of these terms leads to a disturbance (called transiently stable: TS) that corresponds to an increase in magnitude of $|H_d/H_{d0}|$ to a finite amplitude for sufficiently large wave numbers and a decay to zero ultimately [Fig. 2(b)]. For lower wave numbers, the magnitude of the amplitude disturbance remains stable [Fig.

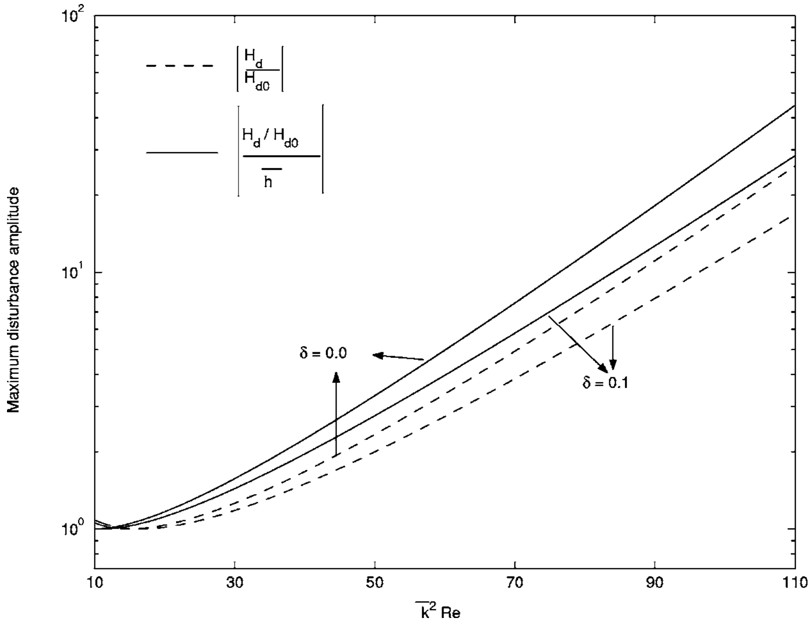

FIG. 3. Maximum disturbance amplitude and relative disturbance amplitude for transient stability when Bi =0, Re=6.2, and ϵ=0.01.

2(a)]. The maximum disturbance amplitude predicted by linear theory is given by

$$\left|\frac{H_d}{H_{d0}}\right|_{\max} = \left(\frac{15(1+\delta)}{\bar{k}^2\,\mathrm{Re}}\right)^{3/4} \exp\left[\frac{1}{20}\left(\frac{\bar{k}^2\,\mathrm{Re}^2}{1+\delta} - 15\right)\right]. \tag{59}$$

The maximum relative disturbance amplitude is obtained as

$$\left|\frac{H_d/H_{d0}}{\bar{h}}\right|_{\max} = \left(\frac{10(1+\delta)}{\bar{k}^2\,\mathrm{Re}}\right)^{1/2} \exp\left[\frac{1}{20}\left(\frac{\bar{k}^2\,\mathrm{Re}^2}{1+\delta} - 10\right)\right]. \tag{60}$$

Figure 3 shows the maximum disturbance and relative disturbance amplitude for transient stability and the result reveals that both increase monotonically with the wave number.

The role of temperature-dependent viscosity (parameter δ) is evident from Figs. 2 and 3. It is observed that in all cases, the magnitude of disturbance amplitude (when $\delta \neq 0$) is always less than the corresponding case of constant viscosity of fluid. The temperature-dependent viscosity has a tendency to stabilize the flow.

Case ii: Biot number Bi$\neq 0$.

Figure 4 shows the relative disturbance amplitude as a function of time for Re=7.4 and 6.0 when $\bar{k}$=4.5 for the transiently stable behavior. The basic state of the film governed by Eq. (43) is used in the normal mode equation and integrated numerically to study the dynamic behavior of the

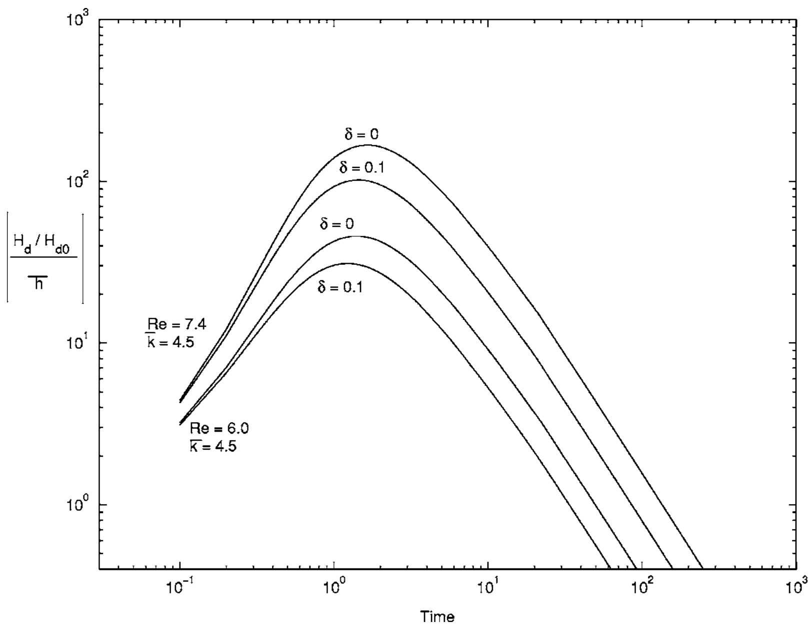

FIG. 4. Normalized disturbance amplitude over time when Bi=10.0 and ϵ=0.01.

perturbed basic state. It is observed that $|H_d/H_{d0}|$ exhibits two regions—one in which the disturbance is stable and the other in which it is transiently stable. The disturbance amplitude increases initially, reaches a maximum at around $t=2$ ($\simeq 0.1$ s) and then decays monotonically. The disturbance amplitude takes a value below its initial value at $t=250$ ($\simeq 7-8$ s). It is observed that at around $t=18$ ($\simeq 0.7$ s), the normalized disturbance amplitude is still significant and takes one fourth of its maximum value. It is worth mentioning here that the choice of the wave number is subjective in the sense that a larger wave number might have resulted in larger disturbance amplitude at the outset of Phase II and a smaller wave number might have produced a smaller disturbance amplitude at the start of Phase II, where Phase II begins at a time t when disturbance amplitude equals one fourth of the initial amplitude.

The same scenario has been observed by Reisfeld *et al.*[39] in their investigation of linear stability theory for an evaporating fluid consisting of a dissolved or suspended solute in a volatile solvent. Reisfeld *et al.*[39] have reported that their predictions based on linear stability theory are well supported by the observations by Spangler *et al.*[40] It has not been possible to compare the results of the present paper with experimental predictions due to the nonavailability of such experimental results incorporating thermal effects.

VI. CONCLUSIONS

The thermal effects on the planarization and thinning process of a nonvolatile thin viscous incompressible liquid film whose viscosity varies with temperature on a heated disk during spin coating is investigated. The evolution equation describing the transient film thickness is derived on the basis of lubrication theory and small aspect ratio for the film. The linear stability analysis is performed in the region lying away from the origin where inertial effects are important and where the equations are valid. The results show that the infinitesimal disturbances decay for small wave numbers and are transiently stable for large wave numbers, for both zero and nonzero values of Biot number.

It is worth mentioning that in the investigation on axisymmetric wave regimes in viscous liquid film flow over a spinning disk by Sisoev *et al.*[17] the results of the amplification factors $\omega_{k,i}$ of unstable perturbations for different wave numbers α_k have been presented[17] [Fig. (6b); p. 398; JFM 2003]. The solutions of the eigenvalue problem and those of linearized Navier-Stokes equations and boundary conditions have been presented. It is observed that the localized solutions adequately capture the properties of the full problem. In the present case, the solutions of the eigenvalue problem also show an analogous behavior namely, the disturbance grows to a maximum, then decays and exhibits a transiently stable behavior. It is hoped that the corresponding linearized solution of Navier-Stokes equations and energy equation would predict the same behavior as the localized solution for the same parameter values and this forms a part of our future work. It is important to note that, in the investigation by Sisoev *et al.*[17] and in the studies,[13–21] the steady or unsteady wave regimes exist in time, with the parameters depending strongly on surface tension. On the other hand, in the case of flow considered, any perturbation is vanishing with time, and the film thickness becomes monotonically decreasing in accordance with Eq. (43). Moreover, the linearized Eq. (47) does not contain the Weber number because the stabilizing mechanism is just the spreading due to centrifugal forces.

Further, the numerical solution of the evolution equation for transient film thickness with boundary conditions corresponding to either the flow over the infinite disk when asymptotic boundary conditions are applied or to coating the disk with the edge may throw more light on the role of temperature-dependent viscosity on the planarization and rate of thinning of the film on a spinning disk. It is expected that such an analysis would help in concluding whether the linear stability analysis presented in this investigation captures the essential features of the thinning film or not and this is of considerable importance to researchers in the field of spin coating and forms a part of future work.

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