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# Decentralized PID Controllers by Synthesis method for Multivariable Unstable Systems

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**Abstract:** The simple method of designing decentralised PID controllers for stable systems by the synthesis method is extended to unstable systems. The effectiveness of the proposed method is compared with that of the available decentralised controllers. The robustness of these controllers is evaluated by the inverse maximum singular value versus frequency plot for both the input multiplicative uncertainty and the output multiplicative uncertainty. Simulation results for two input - two output (TITO) systems demonstrate that the proposed method guarantees robust stability and it provides better servo and regulatory responses.



Keywords: PID Controller, Two-input two-output, Unstable system, Decentralized Controller, Maclaurin series, Synthesis method

## 1. INTRODUCTION

Methods of designing PI/PID controllers for scalar unstable systems include the pole placement, synthesis, IMC, gain and phase margin, equating coefficient, and optimization methods [1]. PID controller can stabilize the system if delay to time constant ratio is less than 1.2. A review of the control of unstable systems was recently made by Rao and Chidambaram [2]. Only few design methods are reported for multivariable controller for unstable systems. Georgiou et al. [3] have proposed an optimization method for the design of decentralized PID controllers for unstable multivariable systems. However, the system does not have any significant time delay. Agamennini et al. [4] have proposed a multivariable delay compensator using a least square method for unstable systems with time delay. The transfer functions relating to one of the two inputs considered are stable and the transfer functions relating to other input are unstable. Govindhakannan and Chidambaram [5] have applied the method of Tanttu and Lieslehto [6] method to design single stage multivariable PI controllers for unstable multivariable systems. Decentralized PI controllers are also designed by detuning method suggested by Luyben and Luyben [7]. Govindhakannan and Chidambaram [5] have shown that the decentralized PI controllers do not stabilize the unstable TITO system when all the transfer functions of the process are unstable. Only, the centralized PI controllers stabilize such systems. When the transfer functions of the system for one of the two inputs are stable, then the decentralized PI controllers stabilize the system and performance of the closed loop system is better than that of the centralized PI controllers. However, the overshoot is large for stabilized PI/PID controllers for both the control systems.

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Govindakannan and Chidambaram [8] extended the two stage design method [9] to multivariable unstable systems with time delay in order to get better performances. Multivariable PI/PID controllers were designed by TL method. The control system gave better performances than the single stage multivariable PI controllers. However, the method requires the transfer function matrix (gain, time delays and time constants) of the system. Ram et al. [10] proposed a method to design centralized PI controllers based on the steady state gain matrix (SSGM). Davison method [11] was modified to design a single stage PI controller matrix using the SSGM. A two stage P-PI centralized controller matrix was also designed. The centralized P controller matrix was first designed by the modified Davison method. For the stabilized system, a decentralized PI controller matrix was designed.

Lee et al. [12] have presented a simple method of designing decentralised PI/PID controllers by the synthesis method for stable systems. The main objective of the present work is to extend the synthesis method [12] to design decentralized controllers for unstable systems. The performance improvement of the present decentralized PID controllers over that of the reported decentralized PID controllers is to be evaluated along with the robustness of the control scheme.

## 2. MULTIVARIABLE SYSTEMS

Consider a Two-Input Two-Output (TITO) open-loop unstable multivariable system with time delays, as shown in Fig 1. $G_P(s)$ is the process transfer function matrix and $G_c(s)$ is the decentralized controllers transfer function matrix. The TITO transfer function matrix of the plant is given by:


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$$G_p(s) = \begin{bmatrix} g_{p,11}(s) & g_{p,12}(s) \\ g_{p,21}(s) & g_{p,22}(s) \end{bmatrix} \quad (1)$$

Let each diagonal elements of the process transfer function matrix be represented by an unstable first-order plus dead time (UFOPTD) model:

$$g_{p,ij} = \frac{k_{p,ij} e^{-\theta_{ij} s}}{\tau_{ij} s - 1} \quad (2)$$

The off-diagonal elements of the process transfer function matrix be represented by a stable first-order plus dead time (FOPTD) model, i.e.,

$$g_{p,ji} = \frac{k_{p,ij} e^{-\theta_{ij} s}}{\tau_{ij} s + 1} \quad \forall i, j \in 2 \quad j \neq i \quad (3)$$

The structure of the decentralized controller is given by:

$$G_c(s) = \begin{bmatrix} g_{c,1}(s) & 0 \\ 0 & g_{c,2}(s) \end{bmatrix} \quad (4)$$

The controller output and plant output are given by, $u_i = G_c e_i$ and $y_i = G_p u_i$, where $u_i$ *(i=1,2),* are inputs to the plant or the manipulated variables, *yi (i=1,2)* are the system outputs of plant, $r_i$ *(i=1,2)* are the reference inputs and $e_i = r_i - y_i$ *(i=1,2)* are the errors between feedback and reference /error signal to the controller. When a MIMO control system is closed, there exist interactions among loops as a result of the existence of non-zero off-diagonal elements in the transfer function matrix. The interactions can be measured by the concept of RGA[13]

## 3. CONTROLLER DESIGN

We consider the MIMO loop of the form shown in Fig 1, the closed loop transfer function matrix *H(s)* is defined by

$$y(s) = H(s) y_r(s) = [I + G_p(s) G_c(s)]^{-1} G_p(s) G_c(s) y_r(s) \quad (5)$$

where *y(s)* is controlled variable vector, $y_r(s)$ is set-point vector. The disturbance vector is entering the system along with the manipulated variable as shown in Figure 1.

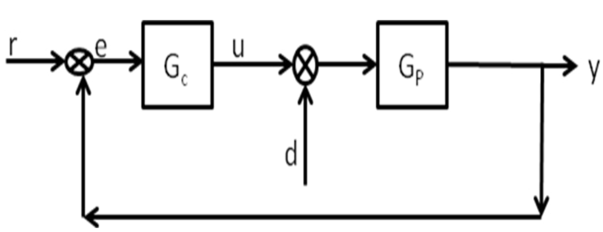


**Fig 1: Multivariable Control System with disturbance d.**

The design strategy of the IMC multivariable controllers is given by [14]. The closed loop response $R_i$ of the $i^{th}$ loop is typically chosen by

$$\frac{y_i}{y_{r,i}} = R_i = \frac{(\alpha_i s + 1) G_{p,i+}(s)}{(\tau_{ic} s + 1)^2} \quad (6)$$

where $G_{p,i+}(s)$ is the non-minimum part of $G_{p,i}(s)$ and is chosen to be the all-pass form, $\tau_{ic}$ is an adjustable constant for system performance and stability. Let the desired closed loop response matrix *R(s)* be

$$R(s) = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \quad (7)$$

To design the multivariable controller $G_c(s)$ such that all the diagonal elements of *H(s)* resemble of *R(s)* as close as possible over a frequency range relevant to control applications. $G_c(s)$ can be written in a Maclaurin series as

$$G_c(s) = \frac{K_0 + K_1 s + K_2 s^2}{s} = \begin{bmatrix} G_{c,1}(s) & 0 \\ 0 & G_{c,2}(s) \end{bmatrix} \quad (8)$$

$G_p(s)$ also can be written in Maclaurin series form as

$$G_p(s) = G_0 + G_1 s + G_2 s^2 = \begin{bmatrix} G_{p,11} & G_{p,12} \\ G_{p,21} & G_{p,22} \end{bmatrix} \quad (9)$$

where $G_0 = G(0)$, $G_1 = G'(0)$ and $G_2 = G''(0)/2$, by substituting Eq.(8) and Eq. (9) into Eq. (5) and expanding it by the Maclaurin series form as

$$H(s) = H_0 + H_1 s + H_2 s^2 + H_3 s^3 + Os^4 \text{ or} \quad (10)$$

$$\begin{aligned} H(s) &= I - (G_0 K_0)^{-1} s + (G_0 K_0)^{-1} (I + G_0 K_1 + G_1 K_0) \\ &(G_0 K_0)^{-1} s^2 + (G_0 K_0)^{-1} [G_0 K_2 + G_1 K_1 + G_2 K_0 - \\ &(I + G_0 K_1 + G_1 K_0)(G_0 K_0)^{-1} (I + G_0 K_1 + G_1 K_0)] s^3 + Os^4 \end{aligned} \quad (11)$$

where $H_0$ is full matrix, $H_0 = H(0)$, $H_1 = H'(0)$ and $H_2 = H''(0)/2$. *R(s)* also can be expressed in a Maclarurin series as

$$R(s) = R_0 + R_1 s + R_2 s^2 + R_3 s^3 + Os^4 \quad (12)$$

where $R_0 = I = R(0)$, $R_1 = R'(0)$ and $R_2 = R''(0)/2$, By comparing each element of *H(s)* and *R(s)* in Eq.(10) and Eq.(12) for the first three *s* terms ($s, s^2, s^3$), we can express $K_0$, $K_1$ and $K_2$ in terms of the process model parameters and the desired closed-loop response parameters. We can get the ideal multivariable controller $G_c(s)$ to achieve the desired closed-loop responses *R(s)* as

$$G_c = G_P^{-1} R [I - R]^{-1} \quad (13)$$

Now, we can design the decentralized multivariable controller, if the feedback multivariable controller $G_c$ is related to the IMC-controller *(Q)* as:

$$G_c = Q[I - G_p Q]^{-1} \quad (14)$$

where *Q(s)* is the IMC controller. The choice of $R_i(s)$ will determine $Q_i(s)$ as

$$Q_i(s) = \frac{(\alpha_i s+1)G_{p,i-}^{-1}(s)}{(\tau_{ic}s+1)^2} \tag{15}$$

where $\tau_{ic}$ is the only tuning parameter to be selected by the user to achieve the appropriate compromise between performance and robustness and to keep the value of the manipulated variable within the bounds.

The tuning parameter ($\tau_{ic}$) is selected such that the desired closed loop response should be achieved and the output of the controller initial variations should not be high. The tuning parameter ($\tau_c$) is selected usually half value of the open loop time constant. In general, this gives controller output within the limit. Substituting Eq.(15) in Eq.(14), and expanding by the Maclaurin series, we get

$$G_{c,i}(s) = \frac{1}{s}\left[ f_i(0) + f_i^{'}(0)s + \frac{f_i^{''}(0)}{2}s^2 + Os^3 \right] \tag{16}$$

By neglecting the higher order derivative terms, the ideal multivariable controller can be found. Finally $K_{ci}$, $\tau_{Ii}$ and $\tau_{Di}$ of the centralized PID controller can be obtained by

$$K_{c,i} = f_i^{'}(0)\,;\ \tau_{I,i} = \frac{f_i^{'}(0)}{f_i(0)}\,;\ \tau_{D,i} = \frac{f_i^{''}(0)}{2f_i^{'}(0)} \tag{17}$$

The process model contains uncertainties in its parameters; and hence the stability and robustness analysis of the multi-loop control system becomes more important. In this paper, the inverse maximum singular values of the Input and output uncertainties (which includes parametric uncertainties also) are considered to evaluate the performance of the proposed controller. The maximum singular value uncertainty models are considered to demonstrate the stability and robustness of the proposed control system. First, for a process multiplicative input uncertainty, $G(s)[I+\Delta_I(s)]$, the closed loop system is stable if [15]:

$$\|\Delta_I(j\omega)\| < \frac{1}{\bar{\sigma}}\{[I + G_C(j\omega)G(j\omega)]^{-1}G_C(j\omega)G(j\omega)\} \tag{18}$$

where, $\bar{\sigma}$ is the maximum singular value of the closed loop system. For the process output uncertainty $[I+\Delta_O(s)]G(s)$, the stability, robustness of the closed loop system can be obtained [15]. For the multiplicative output uncertainty, the closed-loop system is stable if

$$\|\Delta_O(j\omega)\| < \frac{1}{\bar{\sigma}}\{[I + G(j\omega)G_C(j\omega)]^{-1}G(j\omega)G_C(j\omega)\} \tag{19}$$

where $\Delta_I(s)$ and $\Delta_O(s)$ are stable. The frequency plots obtained from the right-hand side part of Eq.(18) and Eq.(19) indicate the stability bounds of the closed-loop system. The area under the curve represents the stability of the system. More area under the curve indicates the high stability of the system. By using this plot, it is easy to compare the stability of the controllers. The control system which gives the maximum area under the curve is the most stable one.

### *3.1 Performance measure*

To analyse the performance of the control systems, the following performance indices are used:

$$IAE = \int_0^\infty |e(t)|dt \tag{20}$$

where, $e(s)=r(s)-u(s)$.

$$TV = \sum_{i=1}^{\infty} |u_i - u_{i-1}| \tag{21}$$

Total Variation (TV) is also a performance criterion for the closed-loop response, but in contrast to the ISE and IAE, the objective for TV is to measure the total variation in the controller output signal, *u*. To evaluate the manipulated input usage, we compute the TV of the controller output *u*, which is the sum of all its moves up and down. This provides a good measure of the smoothness of the control action[16].

## 4. CASE STUDIES

Two TITO systems have been used to test the adequacy of this proposed method. In the present case studies, we considered mild interaction systems, so that the decentralized controllers stabilize the unstable systems. If the system having significant interactions then closed loop response will give a large overshoot. To reduce this overshoot, dual loop control structure is required (Govindhakannan & Chidambaram, 1997). In the following sections, we will present the simulation results. The effectiveness of the proposed decentralized control system is compared with that of the reported method [5].

*Example 1*: Consider a MIMO plant :

$$G_P(s) = \begin{bmatrix} \frac{2.5e^{-1s}}{15s-1} & \frac{1e^{-1.5s}}{14s+1} \\ \frac{1e^{-1.5s}}{15s+1} & \frac{-4e^{-1s}}{20s-1} \end{bmatrix} \tag{22}$$

Notice that in this case the interactions are such that $\lambda_{ij} < 1$. The Relative Gain Array ($\Lambda$) calculated as

$$\Lambda = \begin{bmatrix} 0.9091 & 0.0909 \\ 0.0909 & 0.9091 \end{bmatrix} \tag{23}$$

The parameters of the resulted decentralized controllers and the reported decentralized PID controllers are given in Table 1. The tuning parameters are usually selected such that good output responses and less initial variations in u are obtained. In the present work $\tau_c$=4 is giving good performance and best results in terms of total IAE values. Fig 2 shows the closed-loop servo responses and interactions for a step change in $y_{r1}$. Fig 2 also shows the closed-loop servo responses and interactions for step change in $y_{r2}$. *Fig 3* shows the closed-loop regulatory responses for a step change given in $d_1$.

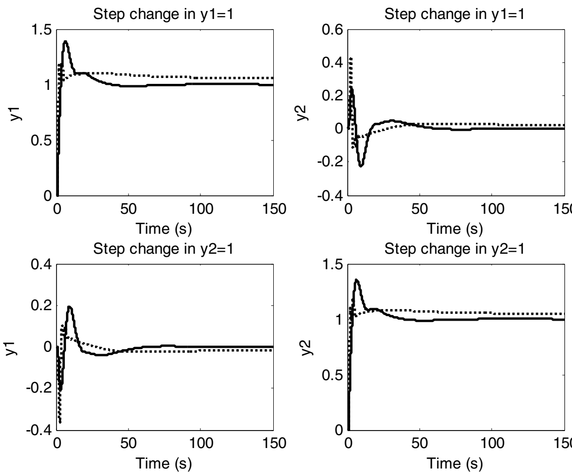


Fig 2: Servo responses for the Example 1. solid: proposed decentralized; dash: reported decentralized (GC method).

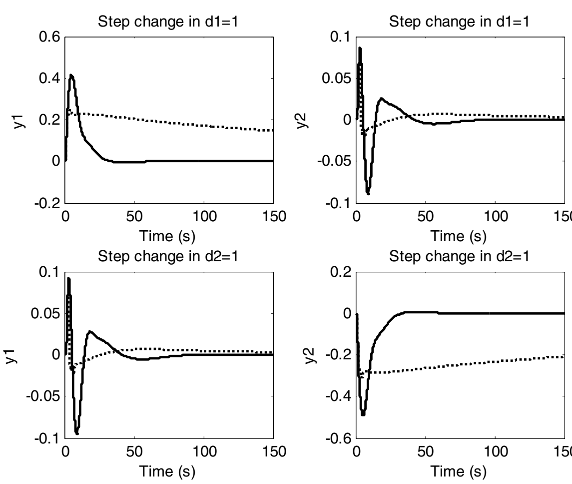


Fig 3: Regulatory responses for the Example1.solid: proposed decentralized; dash: reported decentralized (GC method).

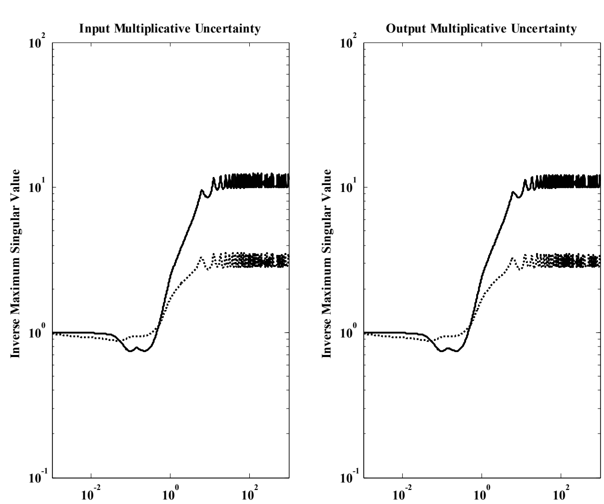


Fig 4: Stability regions of input and output uncertainties for Example 1. solid: proposed decentralized; dash: reported decentralized (GC method).

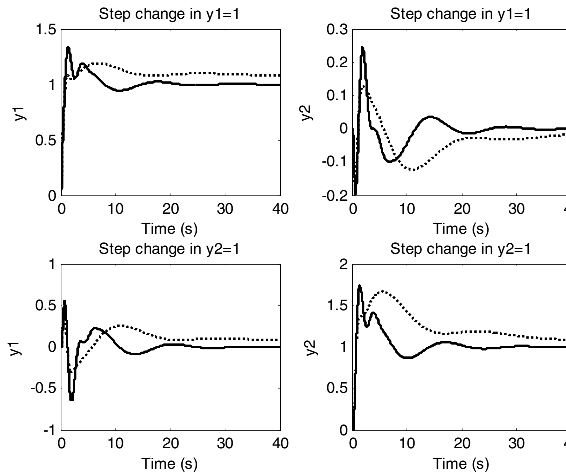


Fig 5: Servo responses for the Example 2. solid: proposed decentralized; dash: reported decentralized (GC method).

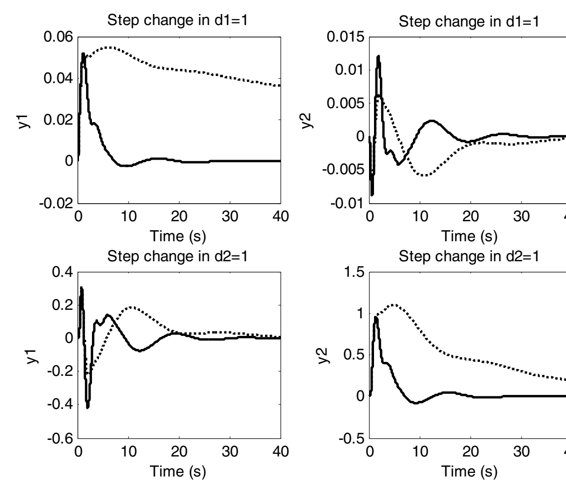


Fig 6: Regulatory responses (Example 2). solid: proposed decentralized; dash: reported decentralized (GC method).

Fig 3 also shows the closed-loop regulatory responses for a step change given in $d_2$. Fig 2 and Fig 3 show that the proposed decentralized PID controllers give improved performance when compared to that of the reported decentralized [5] PID controller design method. For Govindhakanan and Chidambaram (GC) method, the detuning parameter F = 2 which gives the best performance is selected. Table 2 shows that the sum of the IAE and TV values for the main responses and the interactions are lesser for the proposed centralised control scheme.

As stated earlier, the frequency plot obtained for the right hand side part of Equation (18) and Equation (19) indicates the stability bounds of the closed loop system. Fig 4 shows the stability bounds for the Example 1. In this Figure, the region below the curve represents the stability region and above the curve represents instability region. From Fig 4, present control schemes is robust than the decentralised scheme proposed by [5]. As stated earlier, the performance of the proposed decentralized control system is better than that of the literature reported decentralized [5] control systems design method.

*Example 2:* Consider the example given below by:

$$G_P(s) = \begin{bmatrix} \frac{0.3960e^{-0.2s}}{4.572s-1} & \frac{1.7255e^{-0.4s}}{1.807s+1} \\ \frac{-0.0585e^{-0.2s}}{2.174s+1} & \frac{1.9713e^{-0.4s}}{1.801s-1} \end{bmatrix} \quad (24)$$

Notice that in this case the interactions are such that $\lambda_{ij} < 1$. The Relative Gain Array [13] ($\Lambda$) can be calculated as

$$\Lambda = \begin{bmatrix} 0.8855 & 0.1145 \\ 0.1145 & 0.8855 \end{bmatrix} \quad (25)$$

The parameters of the resulted decentralized, reported decentralized PID controllers are given in Table 1. The tuning parameters are selected such that good output response and less initial variation in u are obtained. In the present work $\tau_c = 1$ is giving good performances. Fig 5 shows the closed-loop servo responses and interactions for step change in $y_{r1}$. Fig 5 also shows the closed-loop servo responses and interactions for step change in $y_{r2}$. Fig 6 shows the closed-loop regulatory responses for a step change given in $d_1$. Fig 6 also shows the closed-loop regulatory responses for a step change given in $d_2$. Fig 5 and Fig 6 show that the proposed decentralized PID controllers give improved performance when compared to that of the reported decentralized [5] PID controller design method. For GC method, the detuning parameter F = 1 which gives the best performance is selected. Table 3 shows that the sum of the IAE and TV values for the main responses and the interactions are lesser for the proposed centralised control scheme. The performance of the proposed decentralized control system is better than that of the reported decentralized [5] control system design method.

## 5. CONCLUSIONS

The proposed decentralized control system, derived analytically using Maclariun Series, gives improved responses and decreased interactions when compared to that of the reported decentralized [5] PID control system design method. Performance and robustness are analysed in the presence of the process input multiplicative uncertainty and output multiplicative uncertainty. The proposed decentralized control system gives similar robust performance, compared to with that of the reported method. Proposed control system shows an improvement in the main responses about 71.8% and reduces the interaction by 70.5%. Similar improvement is obtained for the regulatory problems also.

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**Table 1: Controller parameters for case studies.**

| | Example 1 | | Example 2 | |
|---|---|---|---|---|
| | Proposed Method | GC method | Proposed Method | GC method |
| $k_c$ | $\begin{bmatrix} 2.53149 & 0 \\ 0 & -2.05 \end{bmatrix}$ | $\begin{bmatrix} 4.64 & 0 \\ 0 & -3.65 \end{bmatrix}$ | $\begin{bmatrix} 20.01142 & 0 \\ 0 & 1.5380 \end{bmatrix}$ | $\begin{bmatrix} 20.6010 & 0 \\ 0 & 1.32447 \end{bmatrix}$ |
| $k_I$ | $\begin{bmatrix} 0.23177 & 0 \\ 0 & -0.1958 \end{bmatrix}$ | $\begin{bmatrix} 0.0134 & 0 \\ 0 & -0.0078 \end{bmatrix}$ | $\begin{bmatrix} 7.8326 & 0 \\ 0 & 0.4095 \end{bmatrix}$ | $\begin{bmatrix} 0.1892 & 0 \\ 0 & 0.0393 \end{bmatrix}$ |
| $k_D$ | $\begin{bmatrix} 0.48162 & 0 \\ 0 & -0.3805 \end{bmatrix}$ | $\begin{bmatrix} 2.1344 & 0 \\ 0 & -1.6790 \end{bmatrix}$ | $\begin{bmatrix} 0.61912 & 0 \\ 0 & 0.1731 \end{bmatrix}$ | $\begin{bmatrix} 1.8953 & 0 \\ 0 & 0.2474 \end{bmatrix}$ |

GC – Govindakanna and Chidambaram [5]; detuning parameter F=2 (example-1); F=1 for (example-2). For the present method $\tau_{1c}$= 4, $\tau_{2c}$= 4 (example-1); $\tau_{1c}$= 1, $\tau_{2c}$= 1(example-2)

**Table 2: Performance comparisons for Example 1.**

| Step change in | method | IAE | | | TV | | |
|---|---|---|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $\sum(y_1+y_2)$ | TV in $u_1$ | TV in $u_2$ | Sum of TV's |
| $y_{r1}$ | P-M | 6.342 | 3.349 | 9.691 | 55.2090 | 3.0669 | 58.2759 |
| | R-M | 24.900 | 9.010 | 33.910 | 125.6222 | 7.1099 | 132.7321 |
| $y_{r2}$ | P-M | 2.902 | 5.912 | 8.814 | 3.2682 | 43.4455 | 46.7137 |
| | R-M | 7.055 | 20.790 | 27.845 | 7.6159 | 97.8492 | 105.4651 |
| $d_1$ | P-M | 4.531 | 1.232 | 5.763 | 1.8292 | 0.8435 | 2.6727 |
| | R-M | 59.660 | 3.508 | 63.168 | 1.3675 | 0.6041 | 1.9716 |
| $d_2$ | P-M | 1.309 | 5.339 | 6.648 | 1.1085 | 1.7565 | 2.8650 |
| | R-M | 3.580 | 77.930 | 81.510 | 0.8134 | 1.2820 | 2.0954 |

P-M: Present method; R-M : reported method (GC method)

**Table 3: Performance comparisons for Example 2.**

| Step change in | Method | IAE | | | TV | | |
|---|---|---|---|---|---|---|---|
| | | $y_1$ | $y_2$ | $\sum(y_1+y_2)$ | TV in $u_1$ | TV in $u_2$ | Sum of TV's |
| $y_{r1}$ | P-M | 1.7810 | 1.131 | 2.912 | 102.6645 | 2.0066 | 104.6711 |
| | R-M | 8.2680 | 2.462 | 10.730 | 236.0659 | 2.1989 | 238.2648 |
| $y_{r2}$ | P-M | 2.6570 | 3.571 | 6.228 | 68.9714 | 24.2244 | 93.1958 |
| | R-M | 8.5070 | 12.300 | 20.807 | 76.0041 | 33.6093 | 109.6134 |
| $d_1$ | P-M | 0.1504 | 0.0510 | 0.2014 | 2.1476 | 0.0926 | 2.2402 |
| | R-M | 3.3980 | 0.1081 | 3.5061 | 1.4516 | 0.0607 | 1.5123 |
| $d_2$ | P-M | 1.7010 | 3.1670 | 4.868 | 41.3664 | 3.2434 | 44.6098 |
| | R-M | 3.2850 | 25.1500 | 28.435 | 29.7606 | 2.3894 | 32.15 |

P-M: Present method; R-M : reported method (GC method)