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Computational complexity of minimum P_4 vertex cover problem for regular and $K_{1,4}$-free graphs

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ABSTRACT

In $\mathtt{VCP}_4$ problem, it is asked to find a set $S \subseteq V$ of minimum size such that $G[V \setminus S]$ contains no path on 4 vertices, in a given graph $G = (V, E)$. We prove that it is APX-complete for 3-regular graphs as well as 3-regular bipartite graphs. We show that a greedy based algorithm approximates $\mathtt{VCP}_4$ within a factor of 2 for regular graphs. We also show that $\mathtt{VCP}_4$ is APX-complete for $K_{1,4}$-free graphs and a local ratio based algorithm generates a solution which is within a factor of 3.

1. Introduction

A graph property Π is nontrivial hereditary if there exists an infinite family of graphs satisfying Π, an infinite family not satisfying Π and if G satisfies the property Π then every induced subgraph of G also satisfies the property Π. A large class of graph optimization problems can be expressed as, given a graph $G = (V, E)$ and a nontrivial property Π, find a minimum size/weight vertex set $S \subseteq V$ such that $G[V \setminus S]$ satisfies the property Π. In [13], Lewis and Yannakakis proved that if Π is a nontrivial hereditary property, then the vertex-deletion problem for Π is NP-hard and if Π can be tested in polynomial time, then such vertex-deletion problem for Π is NP-complete. Lund and Yannakakis [14] proved that such kind of minimum vertex deletion problems are APX-complete and they can be approximated within a constant factor. This approximation factor depends on a finite forbidden graph characterization of the property Π. A property Π has a finite forbidden graph characterization if there exists a finite set $\mathcal{H}$ of graphs such that G has property Π if and only if no element of $\mathcal{H}$ is an induced subgraph of G. A trivial greedy algorithm for this class of vertex deletion problems gives a solution with approximation factor k, where k is the number of vertices in a largest graph in $\mathcal{H}$.

One of the interesting nontrivial hereditary properties is "at most degree b", where b is a nonnegative integer. The associated vertex deletion problem is known as *Minimum bounded degree vertex deletion* (Min-BDD-b). In Min-BDD-b, given a graph $G = (V, E)$, find a minimum size vertex set S such that degree of each vertex in $G[V \setminus S]$ is at most b. Such a graph property has a finite forbidden set characterization as a result of which the associated vertex deletion problem can be approximated within a factor of $b + 2$. However, Fujito [8] improved this trivial approximation factor to $b + 1$ (for $b \geq 2$) which is based on local ratio approximation of [3]. In Minimum Vertex Cover (VC), given a graph $G = (V, E)$, we are asked to find a set $S \subseteq V$ of minimum size such that degree of each vertex in $G[V \setminus S]$ is 0. For $b = 0$, Min-BDD-b is same as VC.

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For $b = 1$, given a graph $G = (V, E)$, we are asked to find a set $S \subseteq V$ of minimum size such that degree of each vertex in $G[V \setminus S]$ is at most 1 (it is related to dissociation number of G [21]). Based on local ratio approximation method `Min-BDD-0` and `Min-BDD-1` are approximated within a factor of 2 due to Fujito [8].

In this paper, we consider the minimum vertex deletion problem associated with the nontrivial hereditary property "at most k-path", i.e. given an undirected graph G and a positive integer k, find a minimum size vertex set F such that $G[V \setminus F]$ contains no k-path (path on k vertices). We shall denote this problem as k-path vertex cover ($\mathtt{VCP}_k$). A subset F of the vertex set V is called a k-path vertex cover if every path on k vertices, not necessarily induced, in G has at least one vertex from F. The k-path vertex cover number, $\psi_k(G)$, of G is the cardinality of a minimum k-path vertex cover for G. The $\mathtt{VCP}_k$ problem which generalizes the well studied VC was introduced by Brešar et al. in [5] motivated by its relation to secure connection in wireless networks [15]. Tu and Zhou in [19] considered the vertex weighted version of the $\mathtt{VCP}_k$ motivated by its applications in traffic control along with the aforementioned. In general, $\mathtt{VCP}_k$ is known to be NP-complete, for $k \geq 2$, [5,19]. Brešar et al. [5] present a linear time algorithm for trees and investigate upper bounds on $\psi_k(G)$ in regular graphs. For any graph $G = (V, E)$ without isolated vertices, they show that $\psi_k(G) \leq n - \frac{k-1}{k} \sum_{u \in V} \frac{2}{1+d(u)}$, where $n = |V|$. In [4], Brešar et al. give lower bounds for $\psi_k(G)$ in regular graphs. They show that for any d-regular graph G, $\psi_k(G) \geq \frac{d-k+2}{2d-k+2} n$ and also give the exact values of $\psi_k(G)$ for certain graph classes.

When $k = 2$, $\mathtt{VCP}_2$ is equivalent to VC which is APX-complete even when restricted to cubic graphs [1]. The best known approximation factor for VC in general graphs is $2 - \Theta(\frac{1}{\sqrt{\log n}})$ given by Karakostas in [10]. Also, Khot and Regev in [11] prove that Vertex Cover cannot be approximated within any constant factor better than 2 based on unique game conjecture.

When $k = 3$, the $\mathtt{VCP}_3$ is the dual to the Dissociation number problem. Dissociation number of a given graph $G = (V, E)$ is the number of vertices in a maximum induced subgraph of G of degree at most 1. The problem of finding the dissociation number of a graph was introduced by Yannakakis in [21] and proved to be NP-hard even in bipartite graphs and planar graphs. It is not hard to observe that $\mathtt{VCP}_3$ is identical to `Min-BDD`-b with $b = 1$. Therefore, $\mathtt{VCP}_3$ is APX-complete and Fujito's local ratio based algorithm [8] approximates $\mathtt{VCP}_3$ within a factor of 2. In [19], the authors presented a primal–dual factor 2 approximation algorithm for $\mathtt{VCP}_3$, which is based on a new integer linear program formulation. Tu and Yang [18] give a 1.57-approximation to $\mathtt{VCP}_3$ in cubic graphs.

When $k \geq 4$, $\mathtt{VCP}_k$ is not the same as bounding the degree by $k - 2$. The approximability of the generalized problem of the two above mentioned node deletion problems in the direction of bounding the degree has been studied extensively in [6–8,16]. The method of formulating $\mathtt{VCP}_3$ as an integer linear program formulation as in [19] does not get extended to have an integer linear program formulation for $\mathtt{VCP}_k$, for $k > 3$. Based on the lower bound results for `Min-BDD`-b [16], we expect that $\mathtt{VCP}_k$ may not be approximable within a constant factor for arbitrary k.

We define $\mathtt{VCP}_4$ as: Given a graph $G = (V, E)$, find a set $S \subseteq V$ of minimum size such that $G[V \setminus S]$ contains no path on 4 vertices. In Section 3, we show that $\mathtt{VCP}_4$ is NP-complete for cubic planar graphs and APX-complete for cubic graphs, cubic bipartite graphs and $K_{1,4}$-free graphs. In Section 4, we give a greedy 2-approximation algorithm to $\mathtt{VCP}_4$ in regular graphs. In Section 5, we use the local ratio technique used by Fujito [8] to get a 3-approximation in $K_{1,4}$-free graphs.

2. Notations and definitions

In this paper we follow the notations about graphs from West [20]. Let $G = (V, E)$ and $S \subseteq V$. We shall refer to S as a *P_4-VC* for G if $G[V \setminus S]$ contains no path on 4 vertices. If G has no path on 4 vertices then we refer to G as a P_4-free graph, though the path mentioned here is not necessarily induced.

Following [2], we next recall some basic concepts regarding approximation algorithms for NP-optimization problems. The class NPO is the set of all NP-optimization problems.

For an instance x of a problem $\pi \in$ NPO, $m^*(x)$ denotes the measure (or weight) of an optimal solution of x, i.e. $m^*(x) = goal_{y \in sol(x)} m(x, y)$ where $sol(x)$ denotes the finite set of feasible solutions of x and $m(x, y)$ denotes the nonnegative measure of the solution y of the instance x of π. Given an instance x of $\pi \in$ NPO and $y \in sol(x)$, the performance ratio of y with respect to x is defined by

$$R_\pi(x, y) = \max\left\{\frac{m(x, y)}{m^*(x)}, \frac{m^*(x)}{m(x, y)}\right\}.$$

A polynomial-time algorithm A for a problem $\pi \in$ NPO is an *ϵ-approximate* algorithm for π, if $R_\pi(x, A(x)) \leq \epsilon$, for some $\epsilon \geq 1$ and for any instance x of π, where $A(x)$ is the solution for x given by A. The class APX is the set of all $\pi \in$ NPO which have ϵ-approximate algorithm for some constant $\epsilon \geq 1$. In order to introduce the notion of completeness for the class APX, several approximation preserving reductions are introduced. Among them is L-reduction [17] which is most commonly used and is defined as follows:

π_1 is said to be *L-reducible* to π_2 [17], in symbols $\pi_1 \leq_L \pi_2$, if there exist a function f from instances of π_1 to instances of π_2 and two positive constants α, β such that:

1. $m^*_{\pi_2}(f(x)) \leq \alpha \cdot m^*_{\pi_1}(x)$.
2. For any $x \in I_{\pi_1}$ and for any $y \in sol_{\pi_2}(f(x))$ we can in polynomial time find a solution $y' \in sol_{\pi_1}(x)$ such that

$$|m^*_{\pi_1}(x) - m_{\pi_1}(x, y')| \leq \beta \cdot |m^*_{\pi_2}(f(x)) - m_{\pi_2}(f(x), y)|.$$

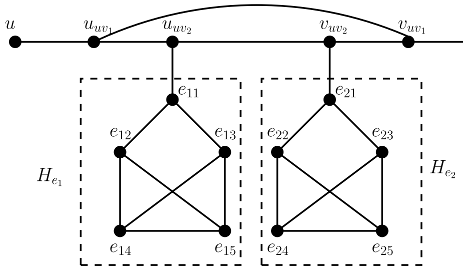

Fig. 1. An edge gadget in H corresponding to an edge $e = (u, v)$ in G.

A problem $\pi \in$ NPO is APX-hard if, for any $\pi' \in$ APX, $\pi' \leq_L \pi$, and problem π is APX-complete if π is APX-hard and $\pi \in$ APX.

3. APX-hardness proofs for cubic graphs

We have already mentioned that the property $\Pi = P_4$-free is a nontrivial hereditary property. Also, one can prove that this property has a finite forbidden graph characterization. i.e. a graph G contains no P_4 if and only if G does not contain any one of the graphs ⊠, ⍂, ⊔, ⍁ and ⊓ as an induced subgraph. Because of this characterization and each forbidden graph has 4 vertices, $\mathtt{VCP}_4$ is approximable within a factor of 4. From this observation it follows that $\mathtt{VCP}_4$ is in APX. Next we show that $\mathtt{VCP}_4$ is APX-hard for cubic graphs using the following known results.

Claim 1. *If $G = (V, E)$ is a k-regular graph with $k \geq 1$ and S_{opt} is a minimum vertex cover in G, then $|S_{opt}| \geq \frac{|V|}{2}$.*

Proof. Suppose $|S_{opt}| < \frac{|V|}{2}$. Since G is k-regular, $|E| \leq k|S_{opt}| < k\frac{|V|}{2}$. This contradicts the fact that $|E| = k\frac{|V|}{2}$. □

Lemma 1 ([1]). *For cubic graphs, $\mathtt{VC}$ is APX-complete.*

Theorem 1. *For cubic graphs, $\mathtt{VCP}_4$ is APX-complete.*

Proof. We exhibit an L-reduction from $\mathtt{VC}$ for cubic graphs to $\mathtt{VCP}_4$ for cubic graphs. For a given 3-regular graph $G = (V, E)$, an instance of $\mathtt{VC}$, we construct a cubic graph $H = (V', E')$, an instance of $\mathtt{VCP}_4$, by replacing each edge $(u, v) \in E$ by a gadget as given in Fig. 1. From this construction, it follows that $|V'| = |V| + \frac{3|V|}{2}14 = 22|V|$ and $|E'| = 33|V|$.

Now, let $S \subseteq V$ be a vertex cover for G. Consider $S' = S \cup \{u_{uv_1} : (u, v) \in E, u \notin S, v \in S\} \cup \{u_{uv_1} : (u, v) \in E, \{u, v\} \subseteq S\} \cup \{e_{11}, e_{12}, e_{21}, e_{22} : e \in E\}$. From the construction of S', it should be noted that for any edge $(u, v) \in E$ with both u and v in S, exactly one of the vertices u_{uv_1} and v_{uv_1} is chosen arbitrarily in S'. We claim that S' is a P_4-VC for H. For $i \in \{1, 2\}$, let H_{e_i} be the subgraph of H induced by $\{e_{i1}, e_{i2}, e_{i3}, e_{i4}, e_{i5}\}$ for $e \in E$. We must delete 2 vertices from H_{e_i} to make it P_4-free. Deleting e_{i1} and e_{i2} kills all P_4s that contain vertices of H_{e_i}. Let $(u, v) \in E$. Since S is a vertex cover for G, $|S \cap \{u, v\}| \geq 1$. Without loss of generality, $u \in S$. Hence for the corresponding edge gadget, we have in S', u and v_{uv_1}. If in addition $v \in S$, then $v \in S'$ and this implies that there is no path on 4 vertices in the remaining graph involving vertices from this edge gadget. Suppose $v \notin S'$. Then $v_{pv_1}, v_{qv_1} \in S'$ where v_{pv_1} and v_{qv_1} are the only other neighbours of v in H. Therefore, $H[V' \setminus S']$ does not contain any path on 4 vertices involving vertices from this edge gadget. Thus S' is a P_4-VC for H. Let $|V| = n$. Then $|E| = \frac{3n}{2}$, since G is a cubic graph. For each edge-gadget there are 5 vertices from H in S' in addition to vertices from V. Therefore, we have that $|S'| = |S| + 5|E| = |S| + \frac{15n}{2}$. Let S_{opt} be a minimum vertex cover for G, then by Claim 1, $|S_{opt}| \geq \frac{n}{2}$. Let S'_{opt} be a minimum P_4-VC for H, then $|S'_{opt}| \leq |S_{opt}| + \frac{15n}{2} \leq |S_{opt}| + 15|S_{opt}| \leq 16|S_{opt}|$.

Conversely, let S' be any P_4-VC for H. We construct another P_4-VC, say S'' for H such that $\{u, v\} \cap S' \neq \varnothing$ for every $(u, v) \in E$ and $|S''| \leq |S'|$. For an edge-gadget corresponding to $e \in E$, S' contains at least 2 vertices from each of the two sets $\{e_{i1}, e_{i2}, e_{i3}, e_{i4}, e_{i5}\}$, for $i = 1, 2$. We can assume that they are fixed. We do not need any other vertices from these two sets. If S' has u_{uv_1} then $S' \cap \{u_{uv_1}, v_{uv_1}, v_{uv_2}, v\} \neq \varnothing$. It is sufficient to replace $S' \cap \{u_{uv_1}, v_{uv_1}, v_{uv_2}, v\}$ by $\{v\}$ in S''. If $S' \cap \{u_{uv_1}, v_{uv_1}\} = \varnothing$ then $\{u, v\} \cap S' \neq \varnothing$ and $\{u_{uv_2}, v_{uv_2}\} \cap S' \neq \varnothing$. If $v \in S'$, then we can substitute v_{uv_2} or u_{uv_2} by u_{uv_2} in S''.

Thus, we have constructed S'' such that $|S''| \leq |S'|$. Also, by construction of S'', $\{u, v\} \cap S'' \neq \varnothing$ for all edges in E. Hence, $S'' \cap V$ gives a vertex cover for G. Also, $|S| = |S''| - 15n/2$. Let S' be any P_4-VC set for H and S be the corresponding vertex cover for G and S'_{opt} be a minimum P_4-VC set for H and S_{opt} be the corresponding minimum vertex cover for G. Then,

$$|S'| - |S'_{opt}| \geq |S| + \frac{15n}{2} - |S_{opt}| - \frac{15n}{2}$$
$$|S'| - |S'_{opt}| \geq |S| - |S_{opt}|.$$

This gives an L-reduction from $\mathtt{VC}$ for cubic graphs to $\mathtt{VCP}_4$ for cubic graphs with $\alpha = 16$ and $\beta = 1$. □

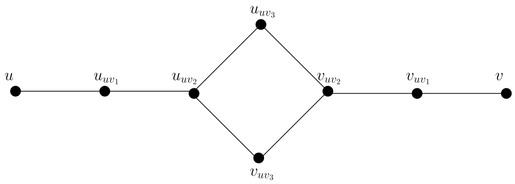

Fig. 2. An edge gadget in H corresponding to an edge (u, v) in G.

VC restricted to cubic planar graphs is known to be NP-hard [9] and the construction of the graph H from a cubic graph G, in the above reduction, preserves planarity property. Hence we have the following corollary.

Corollary 1. *For cubic planar graphs,* $\mathtt{VCP}_4$ *is* NP*-complete.*

Next we shall proceed towards proving that $\mathtt{VCP}_4$ is APX-hard for $K_{1,4}$-free graphs. In order to prove it, it is enough to prove that $\mathtt{VCP}_4$ is APX-complete for cubic bipartite graphs. Before that we need to observe that $\mathtt{VCP}_4$ is approximable within a factor 2 when restricted to bipartite graphs which follows from a result of [12].

Lemma 2 ([12]). *For bipartite graphs,* $\mathtt{VCP}_4$ *is approximable within a factor of* 2.

Theorem 2. *For bipartite graphs with degree at most* 3 , $\mathtt{VCP}_4$ *is* APX*-complete.*

Proof. Let $G = (V, E)$ be a cubic graph. We construct the graph $H = (V', E')$ as follows. We define V' as, $V' = V \cup \{u_{uv_1}, u_{uv_2}, u_{uv_3}, v_{uv_1}, v_{uv_2}, v_{uv_3} : (u, v) \in E\}$. We transform each edge (u, v) into a graph on 8 vertices as given in Fig. 2. H is a bipartite graph of degree at most 3 where $X = V \cup \{u_{uv_2}, v_{uv_2} : (u, v) \in E\}$ and $Y = \{u_{uv_1}, u_{uv_3}, v_{uv_1}, v_{uv_3} : (u, v) \in E\}$ give a bipartition of V'. Let $S \subseteq V$ be a vertex cover for G. Let $S' = S \cup \{v_{uv_2} : u \in S, v \notin S, (u, v) \in E\} \cup \{u_{uv_2} : u, v \in S, (u, v) \in E\}$. It should be noted that we choose exactly one of u_{uv_2} and v_{uv_2} into S' whenever both endpoints u and v of an edge are in S. Then $H[V' \setminus S']$ has no path on 4 vertices. Also, S' has exactly $|E| + |S|$ many vertices.

Conversely, let $S' \subseteq V'$ be any P_4-*VC* for H. Let $A = \{u_{uv_2}, v_{uv_2} : (u, v) \in E\}$. We construct an $S'' \subseteq V'$ such that $|S''| \le |S'|$, $S'' \subseteq V \cup A$ and $|S'' \cap \{u_{uv_2}, v_{uv_2}\}| = 1$, for every $(u, v) \in E$. Initially, let $S'' = S'$. We need to delete at least 2 vertices from each edge gadget to make it P_4-free. For an edge $(u, v) \in E$, if the corresponding edge gadget in H has 3 vertices in S', then we replace these 3 vertices by u, v, u_{uv_2} in S''. If there are only 2 vertices from the edge gadget and if $u \in S'$ then at least one of $\{u_{uv_1}, u_{uv_2}, u_{uv_3}, v_{uv_2}\}$, say x, is in S'. We can then replace x by v_{uv_2} in S''. Suppose both u and v are not in S' and $u_{uv_2} \in S'$. Then at least one of $\{u_{uv_3}, v_{uv_1}, v_{uv_2}, v_{uv_3}\}$ is in S' and we can replace this vertex by v in S''. The only remaining possibility is that u_{uv_3} and v_{uv_3} are the 2 vertices in S'. Then we can replace them by u and v_{uv_2}. In any case, we do not increase the size of S' or include any new P_4. In addition, $S'' \cap \{u, v\} \ne \varnothing$ for every edge $(u, v) \in E$. Hence S'' is a P_4-*VC* for H and $S = S'' \cap V$ is a vertex cover for G. Also, $|S| = |S''| - |E|$. Since G is a cubic graph, $|E| = \frac{3|V|}{2}$. Thus $|S''| = |S| + \frac{3|V|}{2}$. We can thus assume that any $\mathtt{VCP}_4$ for H has the properties of S''. Let S'_{opt} be a minimum $\mathtt{VCP}_4$ for H and S_{opt} be the corresponding minimum *VC* for G. Then $|S'_{opt}| = |S_{opt}| + \frac{3|V|}{2} \le |S_{opt}| + 3|S_{opt}| = 4|S_{opt}|$. Let S' and S be respectively a $\mathtt{VCP}_4$ for H and the corresponding *VC* for G. Then $|S'_{opt}| - |S_{opt}| \le |S'| - |S|$. Hence this reduction proves to be an L-reduction from VC in cubic graphs to $\mathtt{VCP}_4$ in bipartite graphs of degree at most 3. □

Theorem 3. *For cubic bipartite graphs,* $\mathtt{VCP}_4$ *is* APX*-complete.*

Proof. Let $G = (V, E)$ be a cubic graph. We construct the graph $H = (V', E')$ from G as in the previous theorem. This graph H has vertices of degree either 2 or 3. To adjust the vertices of degree 2 we make two copies $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ of H. For each $x \in V'$ let x^1 and x^2 be the corresponding copies in H_1 and H_2, respectively. We take the union of the graphs H_1 and H_2, and for any pair of vertices x^1, x^2 which are of degree 2 we add the graph structure H_x (the subgraph induced by the vertex set $\{x_1, x_2, \ldots, x_{10}\}$) between them as given in Fig. 3. We shall refer to H_x as the vertex gadget corresponding to x. We shall denote the entire graph by $H_0 = (V_0, E_0)$. Since G is a cubic graph, we have that H_0 is cubic. We claim that it is also bipartite. In the proof of Theorem 2, we have observed that H is a bipartite graph with vertex bipartition sets as $X = V \cup \{u_{uv_2}, v_{uv_2} : (u, v) \in E\}$ and $Y = \{u_{uv_1}, u_{uv_3}, v_{uv_1}, v_{uv_3} : (u, v) \in E\}$. It should be noted that a vertex $x \in V'$ has a vertex gadget in H_0 if and only if $x \in Y$. Let $X_0 = \{x^1 : x \in X\} \cup \{x^2 : x \in Y\} \cup \{x_1, x_3, x_5, x_7, x_9 : x \in Y\}$ and $Y_0 = \{x^2 : x \in X\} \cup \{x^1 : x \in Y\} \cup \{x_2, x_4, x_6, x_8, x_{10} : x \in Y\}$. Then (X_0, Y_0) gives a vertex bipartition of H_0. Therefore, H_0 is a cubic bipartite graph.

Claim 2. *Any* P_4-*VC of* H_x *contains at least* 4 *vertices.*

Proof. If we delete any two vertices from H_x then the remaining graph can contain either a C_6, or a C_8, or a C_4 with 3 vertex disjoint paths attached to three distinct vertices of the C_4. In all these three cases, it is required to remove at least two vertices to make each one them P_4-free. □

Let $S \subseteq V$ be a vertex cover for G. We construct a P_4-*VC* for H_0 from S. By Claim 2, it is required to delete at least 4 vertices from each vertex gadget such that it becomes P_4-free. If we choose them to be x_1, x_4, x_7 and x_9, for each $x \in Y$, then the

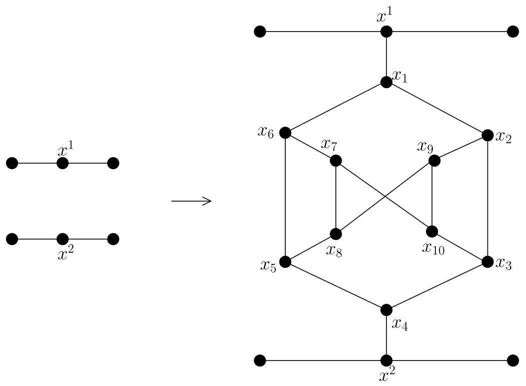

Fig. 3. Transformation of two degree 2 vertices x^1 and x^2 in $H_1 \cup H_2$ to degree 3 in H_0.

problem reduces to finding a P_4-*VC* in each of H_1 and H_2. Let $S' \subseteq V'$ be a P_4-*VC* for H constructed from S as in Theorem 2. Let $S_0 = \{x^1, x^2 : x \in S'\} \cup \{x_1, x_4, x_7, x_9 : x \in S' \cap Y\}$. Then S_0 is a P_4-*VC* for H_0.

Conversely, let $S_0 \subseteq V_0$ be a P_4-*VC* for H_0. For each vertex gadget H_x, we can replace $S_0 \cup V(H_x)$ by $\{x_1, x_4, x_7, x_9 : x \in S' \cap Y\}$ and this does not increase the size of S_0. $S_1 = S_0 \cap V_1$ is a P_4-*VC* for H_1 and $S_2 = S_0 \cap V_2$ is a P_4-*VC* for H_2. We choose the smallest of the two sets, say S_1, and replace S_2 by vertices in V_2 corresponding to S_1. We can then normalize S_1 to a P_4-*VC* of our requirement as in Theorem 2 and find a vertex cover for G accordingly.

Let S'_{opt} be a minimum $\mathtt{VCP}_4$ for H_0 and S_{opt} be the corresponding minimum *VC* for G. Then $|S'_{opt}| \leq |S_{opt}| + |E| + 4|Y| = |S_{opt}| + 17|E| = |S_{opt}| + \frac{51|V|}{2} \leq |S_{opt}| + 51|S_{opt}| = 52|S_{opt}|$. Let S' and S be respectively a $\mathtt{VCP}_4$ for H_0 and the corresponding *VC* for G. Then $|S'_{opt}| - |S_{opt}| \leq |S'| - |S|$. Hence this reduction proves to be an L-reduction from VC in cubic graphs to $\mathtt{VCP}_4$ in cubic bipartite graphs. □

Corollary 2. *For $K_{1,4}$-free graphs,* $\mathtt{VCP}_4$ *is* APX-*complete.*

Proof. The class of $K_{1,4}$-free graphs contains cubic bipartite graphs. Since VCP_4 is APX-complete on cubic bipartite graphs (by Theorem 3), VCP_4 is APX-hard on $K_{1,4}$-free graphs. Since P_4-free graphs has a finite forbidden subgraph characterization and each forbidden subgraph has 4 vertices, VCP_4 can be approximated within a factor of 4 on any graph. Therefore, VCP_4 is APX-complete on $K_{1,4}$-free graphs. □

4. Approximation algorithms

In the introduction of Section 3, we have mentioned that $\mathtt{VCP}_4$ is approximable within a factor of 4. However, in this section we will improve this approximation factor to 2 when the input graphs are restricted to regular graphs. We use a greedy approach to find an approximate solution to $\mathtt{VCP}_4$ in regular graphs.

Theorem 4. *Algorithm* 1 *gives a* 2-*approximate solution to* $\mathtt{VCP}_4$ *for k-regular graphs.*

Proof. By construction, each $F_i,\ i = k$ to 2, is an independent set. For $i = 2$ to $k-1$, each vertex in F_i can have at most one neighbour in F_{i+1}. Therefore, each subgraph $G[S_j],\ j = 1$ to $\lfloor \frac{k}{2} \rfloor + 1$, can have only stars and isolated vertices and hence is P_4-free. A maximum sized set, say S_j, has at least $\frac{n}{\lfloor \frac{k}{2} \rfloor + 1}$ many vertices. We consider the remaining set of vertices to be S and hence $|S|$ is at most $n - \frac{n}{\lfloor \frac{k}{2} \rfloor + 1}$ which is $\frac{\lfloor \frac{k}{2} \rfloor n}{\lfloor \frac{k}{2} \rfloor + 1}$.

A lower bound for the optimum solution in k-regular graphs is $\frac{k-2}{2k-2} n$ given by Brešar et al. [5]. Therefore, we have that

$$\frac{|S|}{|S_{opt}|} \leq \frac{\left(\lfloor \frac{k}{2} \rfloor\right) / \left(\lfloor \frac{k}{2} \rfloor + 1\right)}{(k-2)/(2k-2)}.$$

It can be easily seen that this fraction is always less than 2 for $k \geq 3$. Hence we have a 2-approximation for $\mathtt{VCP}_4$ in regular graphs. □

5. Approximation algorithm for $K_{1,4}$-free graphs

Let $G = (V, E)$ be such that G does not have $K_{1,4}$ as an induced subgraph and let $w : V \to Q^+$ be a weight function defined on V. We shall denote the set of vertices which have a non-zero weight under w as $V(w)$. We give a recursive algorithm using local ratio technique to find an approximate solution in G with the given weight function w.

Algorithm 1: Greedy Algorithm for $\mathtt{VCP}_4$ in k-regular graphs

Input: k-regular graph $G = (V, E)$;
Output: $S \subseteq V$ such that $G[V \setminus S]$ has no paths on 4 vertices;
$S = \emptyset$;
$F = \emptyset$;
$G' = G$;
for $i = k$ down to 3 **do**
  while there is a vertex of degree i in G' **do**
    Choose a vertex v with $d(v) = i$;
    $F_i = F_i \cup \{v\}$; $G' = G' \setminus \{v\}$;
    Remove all components in G' which are isomorphic to $K_{1,i}$ and call the resulting graph as G';
  end
  Remove all components in G' which are isomorphic to $K_{1,i-1}$ and call the resulting graph as G';
end
Choose a minimum P_4-VC, F_2, for G' such that
- F_2 is an independent set in G' and
- $d_{G'}(v) = 2, \forall v \in F_2$;

$F_1 = V \setminus \cup_{i=2}^{k} F_i$;
(Clubbing two sets at a time)
if k is odd **then**
  $S_1 = F_1$;
  $S_i = F_{2i-2} \cup F_{2i-1}$ for $i = 2$ to $\lfloor \frac{k}{2} \rfloor + 1$;
end
else
  $S_1 = F_1$; $S_2 = F_2$;
  $S_i = F_{2i-3} \cup F_{2i-2}$ for $i = 3$ to $\frac{k}{2} + 1$;
end
Choose j such that $|S_j| \geq |S_i|$, for $i = 1$ to $\lfloor \frac{k}{2} \rfloor + 1$;
$S = V \setminus S_j$;
Return S;

In the above algorithm, each iteration i finds the minimum ratio α_i and the weight function w_i slices the weight of each vertex (in the remaining graph) by a factor of α_i and w' gives the weight that is remaining. The minimum ratio is achieved at least at one vertex and hence the weight of this vertex is reduced to zero in the corresponding iteration. At each iteration i, the corresponding vertices whose weights are reduced to zero are taken into the set S_i and only those vertices which are left with a non-zero weight are considered in the next iteration. This assures that this recursive algorithm terminates in a finite number of steps. For each i, S_i is a solution to the graph $G[V(w^i)]$ and S_1 is the solution to the given graph G. Also, the solutions are minimal at each stage since the cleaning up step is also done recursively. Let $w_1, w_2, \ldots, w_t$ be the different weight functions obtained from the algorithm. Then $\{w_1, w_2, \ldots, w_t\}$ gives a decomposition of the given weight function w. That is, $\sum_{i=1}^{t} w_i(v) = w(v)$ for every $v \in V$. Therefore, we have that $w(S_1) = \sum_{v \in S_1} w(v) = \sum_{v \in S_1} \sum_{i=1}^{t} w_i(v) = \sum_{i=1}^{t} w_i(S_1)$.

We use the following result by Fujito [8] to get the performance ratio of Algorithm 2 which depends on a weight scheme defined on V, $\bar{w}$, and is for the node deletion problem for any graph property Π. Substituting 'no path on 4 vertices' by any graph property Π and $d_{G'}(v)$ by $\bar{w}_{G'}$ in the algorithm, we get the local ratio algorithm for $\mathtt{VCP}_4$.

Theorem 5 ([8]). *The Local Ratio Algorithm computes a solution of the node deletion problem for the graph property Π whose performance ratio is bounded by max $\frac{\bar{w}_G(S)}{\bar{w}_G(S^*)}$ where maximum is taken over any minimal and optimal solutions S and S^* respectively, in any graph G under weight $\bar{w}_G$.*

The weight scheme $\bar{w}_G$ mentioned here is computable in polynomial time given any graph G and is such that if S is any solution to G, then so is $S \cap V(\bar{w}_G)$. With $\bar{w}_G(v) = d_G(v)$, we get the same bound on the performance ratio of Algorithm 2. Now we have to only show a bound for $\frac{\bar{w}_G(S)}{\bar{w}_G(S^*)}$ where S and S^* are any minimal and optimal solutions respectively.

Theorem 6. *Algorithm 2 gives a 3-approximate solution to $\mathtt{VCP}_4$ for $K_{1,4}$-free graphs.*

Proof. Let S and S^* be any minimal and optimal solutions respectively. For convenience, we shall denote $\bar{w}$ as w during the analysis. Let $\delta(X, Y)$ be the set of edges between X and Y. It is not difficult to show the following two equations.

$$\begin{aligned} w(S) &= w(S \cap S^*) + w(S \setminus S^*) \\ &= 2\delta(S \setminus S^*, S \cap S^*) + w(S \cap S^*) - \delta(S \setminus S^*, S \cap S^*) + w(S \setminus S^*) - \delta(S \setminus S^*, S \cap S^*). \end{aligned}$$

Algorithm 2: Approximation Algorithm for $\mathtt{VCP}_4$ in $K_{1,4}$-free graphs using Local Ratio

Input: Graph $G = (V, E)$ which does not have $K_{1,4}$ as an induced subgraph and a weight function w defined on V;
Output: $S \subseteq V$ such that $G[V \setminus S]$ has no path on 4 vertices;
$i = 0$;
$S_i = \emptyset$;
LocalRatio(w){
$i = i + 1$;
$w^i = w$;
if G has no path on 4 vertices **then**
  Return $\emptyset$;
end
else
  $G' = G[V(w^i)]$;
  $\alpha_i = \min\{\frac{w^i(v)}{d_{G'}(v)} : v \in V(w^i)\}$;
  $w_i(v) = \alpha_i d_{G'}(v) \quad \forall v \in V(w^i)$;
  $w_i(v) = 0 \quad \forall v \notin V(w^i)$;
  $w'(v) = w^i(v) - w_i(v) \quad \forall v \in V(w^i)$;
  $w'(v) = 0 \quad \forall v \notin V(w^i)$;
  S' =LocalRatio(w');
  $S_i = S' \cup (V(w^i) - V(w'))$ /*Takes all zero weighted vertices into solution*/ ;
  for every $v \in V(w^i) - V(w')$ **do**
    If $S_i \setminus \{v\}$ is a solution in G' then $S_i = S_i \setminus \{v\}$;
  end
  Return S_i;
end
}

$$w(S^*) = \delta(S \setminus S^*, S \cap S^*) + w(S \cap S^*) - \delta(S \setminus S^*, S \cap S^*) + w(S^* \setminus S).$$

Therefore,

$$\frac{w(S)}{w(S^*)} \leq \max\left\{2, \frac{w(S \setminus S^*) - \delta(S \setminus S^*, S \cap S^*)}{w(S^* \setminus S)}\right\}.$$

Now we show that

$$w(S \setminus S^*) - \delta(S \setminus S^*, S \cap S^*) \leq 3w(S^* \setminus S).$$

We can assume that there are no cross edges between $S \setminus S^*$ and $S \cap S^*$ and hence it is enough to prove that $w(S \setminus S^*) \leq 3w(S^* \setminus S)$. Let $v \in S \setminus S^*$. Suppose there is a cross edge e from v to a vertex in $S^* \setminus S$. Then e gives a contribution of 3 to the RHS and only 1 to the LHS. Suppose v has at most 2 neighbours in $V \setminus S^*$ then we are done. Also, v cannot have 4 neighbours in $V \setminus S^*$. For, if so then there is an edge between two of its neighbours since G is $K_{1,4}$-free. This will give rise to a path on 4 vertices in $G[V \setminus S^*]$ contradicting the fact that S^* is a solution to G. Therefore, v can have at most 3 neighbours in $V \setminus S^*$. If there are 3 neighbours then no two of them are adjacent and there is an edge between one of these vertices and a vertex in $S^* \setminus S$, which has a contribution of 3 on the RHS. Also, these 3 vertices are adjacent to only v in $S \setminus S^*$ (otherwise there will be a path on 4 vertices in $G[V \setminus S^*]$). Therefore, this second cross edge is unique to v and we are done.

Assume there is no cross edge from v to $S^* \setminus S$. If v has at least 2 neighbours in $V \setminus S^*$ then none of these neighbours are adjacent to any other vertex in $S \setminus S^*$. v can have at most 3 neighbours in $V \setminus S^*$ and if it has only one then there can be at most 3 such vertices of $S \setminus S^*$ which are adjacent to the same vertex in $V \setminus S^*$ (using $K_{1,4}$-free property of G). Since S is a minimal solution to G, there is at least one path on 4 vertices in $G[V \setminus S] \cup \{v\}$ which gives a cross edge from $V \setminus S^*$ to $S^* \setminus S$. This proves the inequality and hence we have the 3-approximation. □

6. Conclusion

$\mathtt{VCP}_4$ is approximable within a factor of 2 in regular graphs and within a factor of 3 in $K_{1,4}$-free graphs. Also, it is approximable within a factor of 2 in bipartite graphs which is better than the trivial 4-approximation in graphs. This motivates us to expect that it can be approximable within a factor less than 4 in general graphs. Also, Min-BDD-b has a $2 + \log b$ factor approximation due to Okun and Barak [16]. A similar approximation (which is better than the trivial k factor approximation) for $\mathtt{VCP}_k$, for any k, is still unknown. Moreover, in Section 3, we exhibit a reduction from VC to $\mathtt{VCP}_4$ in cubic planar graphs to show NP-hardness of the problem. The question of whether $\mathtt{VCP}_4$ is APX-hard in cubic planar graphs remains open.

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