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Characterization of minimum cycle basis in weighted partial 2-trees[☆,☆☆]

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ABSTRACT

For a weighted outerplanar graph, the set of lex short cycles is known to be a minimum cycle basis (Liu and Lu, 2010). In our work, we show that the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees (graphs of treewidth at most two), which is a superclass of outerplanar graphs. In general graphs, a minimum cycle basis is a subset of the set of lex short cycles, where as the equality is known to hold only for the weighted outerplanar graphs.

1. Introduction

A cycle basis is a compact description of the set of all cycles of a graph and has various applications including the analysis of electrical networks [7]. Let $G = (V(G), E(G))$ be an edge weighted graph and let $m = |E(G)|$ and $n = |V(G)|$. A *cycle* is a connected graph in which the degree of every vertex is two. Let $E(G) = \{e_1, \dots, e_m\}$. An *incidence vector* $(x_1, \dots, x_m)$ is associated with every cycle C in G, where for each $1 \le i \le m$, x_i is 1 if $e_i \in E(C)$ and 0 otherwise. The *cycle space* of G is the vector space over $\mathbb{F}_2^m$ spanned by the incidence vectors of cycles in G. A *cycle basis* of G is a minimum set of cycles whose incidence vectors span the cycle space of G. The weight of a cycle C is the sum of the weights of the edges in C. A cycle basis $\mathcal{B}$ of G is a *minimum cycle basis* (MCB) if the sum of the weights of the cycles in $\mathcal{B}$ is minimum.

Motivation. For a weighted graph G, Horton has identified a set $\mathcal{H}$ of $O(mn)$ cycles and has shown that a minimum cycle basis of G is a subset of $\mathcal{H}$ [6]. Liu and Lu have shown that the set of *lex short cycles* (defined later) is a minimum cycle basis in weighted outerplanar graphs [9]. We generalize this result for partial 2-trees, which is a superclass of outerplanar graphs.

Our contribution. The following are the main results in this work.

Theorem 1.1. *Let G be a weighted partial 2-tree on n vertices and m edges. Then the number of lex short cycles in G is $m - n + 1$.*

Theorem 1.2. *For a weighted partial 2-tree G, the set of lex short cycles is a minimum cycle basis.*

Related work. The characterization of graphs using cycle basis was initiated by MacLane [10]. In particular, MacLane showed that a graph G is planar if and only if G contains a cycle basis B, such that each edge in G appears in at most two cycles of B.

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However, he referred to a cycle basis as a *complete independent set of cycles*. Formally, the concept of cycle space in graphs was introduced in [3] after four decades. Later, it was characterized that a planar 3-connected graph G is a Halin graph if and only if G has a planar basis B, such that each cycle in B has an external edge [13]. There after, it was shown that every 2-connected outerplanar graph has a unique MCB [8]. Subsequently, it was proven that Halin graphs that are not necklaces have a unique MCB [12].

The first polynomial time algorithm for finding an MCB was given by Horton [6]. Since then, many improvements have taken place on algorithms related to minimum cycle basis and its variants. A detailed survey of various algorithms, characterizations and the complexity status of cycle basis and its variants was compiled by Kavitha et al. [7]. The current best algorithm for MCB runs in $O(m^2n/\log n)$ time and is due to Amaldi et al. [1].

Graph preliminaries. In this paper, we consider only simple, finite, connected, undirected and weighted graphs. We refer [14] for standard graph theoretic terminologies. Let G be an edge weighted graph. Let $X \subseteq V(G)$ be a set of vertices. $G-X$ denotes the graph obtained after deleting the set of vertices in X from G. We use $G[X]$ to denote the subgraph induced by the vertices in X. A set $X \subseteq V(G)$ of vertices is a *vertex separator* if $G - X$ is disconnected. A *path* is a connected graph that has two vertices of degree one and the rest being two. We use $P(u, v)$ to denote a path joining the vertices u and v. If the end vertices of the path under consideration are clear from the context, then we simply use P to denote such a path. A *component* of G is a maximal connected subgraph. K_3 denotes a cycle on 3 vertices and K_2 denotes an edge. $K_{2,3}$ is a complete bipartite graph (V_1, V_2) such that $|V_1| = 2$, $|V_2| = 3$. A graph is *planar* if it can be drawn on the plane without any edge crossings. A planar graph is *outerplanar* if it can be drawn on the plane such that all of its vertices lie on the boundary of its exterior region. A 2-tree is defined inductively as follows: K_3 is a 2-tree; if G' is a 2-tree and $(x, y) \in E(G')$, then the graph with the vertex set $V(G') \cup \{u\}$ and edge set $E(G') \cup \{(u, x), (u, y)\}$ is a 2-tree. A graph is a *partial 2-tree* if it is a subgraph of a 2-tree. Alternatively, a graph of treewidth (defined in [11]) at most two is a *partial 2-tree*. An *H-subdivision* (or subdivision of H) is a graph obtained from a graph H by replacing edges with pairwise internally vertex disjoint paths.

2. On lex short cycles in weighted partial 2-trees

We first define lex shortest paths and lex short cycles and explore their structural properties in weighted partial 2-trees. Then by using these structural properties, a procedure is defined to decompose a weighted partial 2-tree. This procedure helps in computing the number of lex short cycles in weighted partial 2-trees and further proves the main result.

Let G be a weighted graph associated with a weight function $w : E(G) \to \mathbb{N}$. For a totally ordered set S, $\min(S)$ denotes the minimum element in S. Let $V(G)$ be a totally ordered set. The notion of lex shortest path and lex short cycle presented here is from [5]. A path $P(u, v)$ between two distinct vertices u and v is *lex shortest path* if for all the paths P' between u and v other than P, exactly one of the following three conditions hold: (1) $w(P') > w(P)$, (2) $w(P') = w(P)$ and $|E(P')| > |E(P)|$, and (3) $w(P') = w(P)$, $|E(P')| = |E(P)|$ and $\min(V(P') \setminus V(P)) > \min(V(P) \setminus V(P'))$, where $w(P) = \Sigma_{e \in E(P)} w(e)$. The lex shortest path between any two vertices u and v is unique. For a subgraph H of G, $lsp_H(x, y)$ denotes the lex shortest path between vertices x and y in H; $lsp(u, v)$ denotes the lex shortest path between x and y in G. A cycle C is *lex short* if for every two vertices u and v in C, $lsp(u, v)$ is in C. The set of lex short cycles of G is denoted by $LSC(G)$. For a subgraph H of G, the total order of $V(H)$ is the order induced by the total order of $V(G)$. For any connected graph, the cardinality of a cycle basis is $m - n + 1$. Thus by the following two lemmas, the set of lex short cycles in outerplanar graphs is a minimum cycle basis.

Lemma 2.1 ([5]). *A minimum cycle basis of a weighted graph G is a subset of $LSC(G)$.*

Lemma 2.2 ([9]). *For a simple weighted outerplanar graph G, $|LSC(G)| = m - n + 1$.*

We now present our lemmas and theorems that are required to prove the main results.

Lemma 2.3. *Let G be a graph and $\{u, v\}$ be a vertex separator in G. Let P be a path between u and v. Then there exist one component H in $G - \{u, v\}$, such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$.*

Proof. Since $\{u, v\}$ is a vertex separator, the number of components in $G - \{u, v\}$ is at least two. If there are no internal vertices in P, then none of the components in $G - \{u, v\}$ contain $V(P)$ and $E(P)$. Consider the other case where P has at least one internal vertex. Since the end vertices of P are u and v, and P is a path, the graph $P - \{u, v\}$ is a path in $G - \{u, v\}$. Since $\{u, v\}$ is a vertex separator and $P - \{u, v\}$ is a path, it follows that $P - \{u, v\}$ is in a single component in $G - \{u, v\}$. Then we have at least one component H in $G - \{u, v\}$, such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$. □

Lemma 2.4. *Let G be a partial 2-tree that is not outerplanar. Then there exists a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision in G, such that $G - \{u, v\}$ contains at least three components.*

Proof. A graph is outerplanar if and only if it contains no subgraph that is a subdivision of K_4 or $K_{2,3}$ [2]. Since a partial 2-tree does not contain a subdivision of K_4, a partial 2-tree is outerplanar if and only if it does not contain a subdivision of $K_{2,3}$. Consider a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision H in G. Since H is a $k_{2,3}$-subdivision, we have three internally vertex disjoint paths P_x, P_y and P_z between u and v that contains x, y and z respectively. Assume that $G-\{u, v\}$ has at most two components. In $G-\{u, v\}$, without loss of generality, let x and y be in a single connected component. Let (x', y') be a closest pair of vertices (with respect to the distance between them) such that $x' \in V(P_x)$ and $y' \in V(P_y)$. Let $P(x', y')$ be a shortest path between x' and y'.

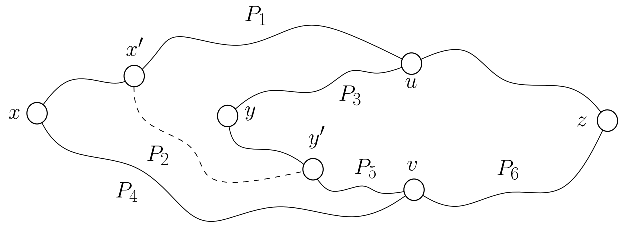

Fig. 1. A K_4-subdivision on the vertex set $\{u, v, x', y'\}$.

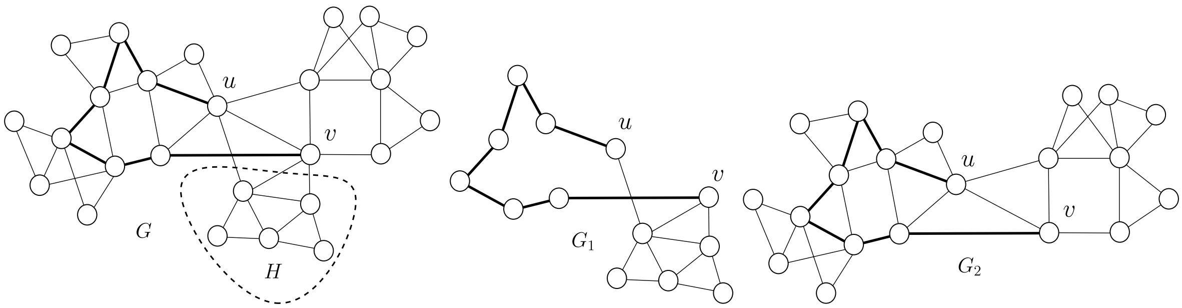

Fig. 2. For a weighted partial 2-tree G, $lsp(u, v)$ is shown in thick edges. Also $H \subset G$ is shown. G_1 and G_2 are the decomposed graphs of G.

Then $P(x', y')$ is edge disjoint from H. For instance, in Fig. 1, the path $P(x', y')$ is shown by dashed edges. For every two distinct vertices in $\{u, v, x', y'\}$, there is an internally vertex disjoint path in $H \cup P(x', y')$. Thus, there is a K_4-subdivision on the vertex set $\{u, v, x', y'\}$ in G, which is shown in Fig. 1. This is a contradiction as partial 2-trees do not contain any K_4-subdivision as a subgraph. Therefore, there exist two vertices u and v in G, such that $G - \{u, v\}$ contains at least three components. □

Observation 1. *Let G' be a weighted subgraph of a weighted graph G. Let P be a path and C be a cycle contained both in G and G'.*

(a) *If P in G is lex shortest, then P in G' is lex shortest.*
(b) *If $C \in LSC(G)$, then $C \in LSC(G')$.*

Proof. Let u and v be the end vertices of P. Let $\mathcal{P}$ and $\mathcal{P}'$ be the sets of paths between u and v in G and G', respectively. The path P is the minimum element in $\mathcal{P}$ according to the three conditions of the definition of lex shortest path. Also $\mathcal{P}' \subseteq \mathcal{P}$ and $P \in \mathcal{P}'$. Thereby P is the minimum element in $\mathcal{P}'$. Thus the first part of this observation holds true.

Recall that C is a lex short cycle if for every $x, y \in V(C)$, $lsp(x, y)$ is contained in C. Since C is in G' and for every $x, y \in C$, $lsp(x, y)$ is same as $lsp_{G'}(x, y)$, C is a lex short cycle in G'. □

By the principle of optimality, we have the following observation.

Observation 2. *Any subpath of a lex shortest path is a lex shortest path.*

Lemma 2.5. *The intersection of two lex shortest paths is either empty or a lex shortest path.*

Proof. Consider two lex shortest paths $lsp(x, y)$ and $lsp(u, v)$ in a graph G. Let $G' = (V(G'), E(G'))$, where $V(G') = V(lsp(x, y)) \cap V(lsp(u, v))$, $E(G') = E(lsp(x, y)) \cap E(lsp(u, v))$. Suppose $V(G') \neq \emptyset$ and G' is not a path, then we have at least two maximal paths $P(a, b)$ and $P(a', b')$ which are common to both $lsp(x, y)$ and $lsp(u, v)$, where $b \neq a'$. Consequently, the path $P_1(b, a')$ contained in $lsp(x, y)$ and the path $P_2(b, a')$ contained in $lsp(u, v)$ are different. By Observation 2, both P_1 and P_2 are lex shortest paths between b and a'; a contradiction to the fact that for any two vertices in a graph, there is a unique lex shortest path. □

Decomposition. We decompose a weighted partial 2-tree G that is not outerplanar into two weighted partial 2-trees G_1 and G_2 in such a way that $LSC(G)$ is equal to the disjoint union of $LSC(G_1)$ and $LSC(G_2)$. From Lemma 2.4, there exist two vertices $u, v \in V(G)$, such that $G - \{u, v\}$ is disconnected and has at least three components. Let P be the lex shortest path between u and v in G. By Lemma 2.3, there exists a component H in $G - \{u, v\}$, such that $V(P) \cap V(H) = E(P) \cap E(H) = \emptyset$. The operation $decomp(G, u, v)$ decomposes G into G_1 and G_2, where $V(G_1) = V(H) \cup V(P)$, $E(G_1) = E(H) \cup E(P) \cup \{(x, y) \in E(G) \mid x \in V(H) \text{ and } y \in \{u, v\}\}$, $G_2 = G[V(G) \setminus V(H)]$. An example is shown in Fig. 2 to illustrate the operation $decomp(G, u, v)$. We use the following notation for the rest of the paper. G is a weighted partial 2-tree that is not outerplanar. $\{u, v\}$ is a vertex separator that disconnects G into at least three components. H is a component in $G - \{u, v\}$, such that the sets $V(lsp(u, v)) \cap V(H)$ and $E(lsp(u, v)) \cap E(H)$ are empty. G_1 and G_2 are the graphs obtained from the operation $decomp(G, u, v)$. From the definition of $decomp(G, u, v)$, we have the following two observations.

Observation 3. *For $x, y \in V(lsp(u, v))$, $lsp(x, y)$ is in each G_i, $i \in \{1, 2\}$.*

Proof. The proof follows from the construction of G_1 and G_2, and Observation 2. □

Observation 4. *For $i \in \{1, 2\}$, for $x \in V(G_i)$ and $y \in V(G_i) \setminus V(lsp(u, v))$, every path $P(x, y)$ in G, such that the internal vertices of P are in $V(G_i) \setminus \{u, v\}$, is present in G_i.*

Proof. Assume that $P(x, y)$ is not in G_i. In the path P from x to y, let (a, b) be the first edge, such that $(a, b) \notin E(P)$. Clearly, $b \notin V(G_i)$. It follows that a is an intermediate vertex in P and $a \in \{u, v\}$; a contradiction to the premise that no intermediate vertex in P belong to $\{u, v\}$. Hence, the observation. □

Lemma 2.6. *For $i \in \{1, 2\}$ and for any two vertices $x, y \in V(G_i)$, $lsp(x, y)$ is in G_i.*

Proof. If no vertex is common in $lsp(x, y)$ and $lsp(u, v)$, then from Observation 4, $lsp(x, y)$ is in G_i. If at least one vertex is common in $lsp(x, y)$ and $lsp(u, v)$, then due to Lemma 2.5, $lsp(x, y) \cap lsp(u, v)$ is $lsp(a, b)$ for some $a, b \in V(lsp(u, v)) \cap V(lsp(x, y))$. The $lsp(x, y)$ can be viewed as a union of three paths $P(x, a)$, $P(a, b)$ and $P(b, y)$. From Observation 3, $P(a, b)$ is contained in G_i. If $x = a$, then trivially $P(x, a)$ appears in G_i. Also, if $y = b$, then clearly $P(b, y)$ appears in G_i. If $x \neq a$, then $x \notin V(lsp(u, v))$. Similarly, if $y \neq b$, then $y \notin V(lsp(u, v))$. From Observation 4, it follows that both $P(x, a)$ and $P(b, y)$ appear in G_i. These observations imply that $lsp(x, y)$ is in G_i. □

Corollary 2.7. *For $i \in \{1, 2\}$, if there is a cycle C in $LSC(G_i)$, then C is in $LSC(G)$.*

Proof. From Lemma 2.6 and Observation 1(a), for every $x, y \in V(G_i)$, $lsp_{G_i}(x, y)$ and $lsp(x, y)$ are same. Since $C \in LSC(G_i)$, for every $x, y \in V(C)$, $lsp_{G_i}(x, y)$ is contained in C. Consequently, for every $x, y \in V(C)$, $lsp(x, y)$ is contained in C. Hence $C \in LSC(G)$. □

Theorem 2.8. $LSC(G) = LSC(G_1) \uplus LSC(G_2)$.

Proof. Since $E(G_1) \cap E(G_2)$ is $E(lsp(u, v))$, $LSC(G_1)$ and $LSC(G_2)$ are disjoint. We now prove that $LSC(G) \subseteq LSC(G_1) \uplus LSC(G_2)$. Let $C \in LSC(G)$. If C contains at most one vertex from $\{u, v\}$, then C is contained either in G_1 or G_2, because $\{u, v\}$ is a vertex separator. Consider the other case when C contains both u and v. Since $C \in LSC(G)$, C contains $lsp(u, v)$. Note that $lsp(u, v)$ is contained both in G_1 and G_2. Observe that $E(C) \setminus E(lsp(u, v))$ belongs to $E(G_i)$ for some $i \in \{1, 2\}$, because $E(G_1) \cap E(G_2) = E(lsp(u, v))$. Hence, C is either in G_1 or G_2. In both of the cases, by Observation 1(b), C is either in $LSC(G_1)$ or $LSC(G_2)$. Therefore, $LSC(G) \subseteq LSC(G_1) \uplus LSC(G_2)$. From Corollary 2.7, $LSC(G_1) \uplus LSC(G_2) \subseteq LSC(G)$. Hence, $LSC(G) = LSC(G_1) \uplus LSC(G_2)$. □

Lemma 2.9. *The number of $K_{2,3}$-subdivisions in each of G_1 and G_2 is less than the number of $K_{2,3}$-subdivisions in G.*

Proof. Recall that there is a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision in G, and G_1 and G_2 are obtained from $decomp(G, u, v)$. Without loss of generality, assume that $x \in V(H)$. Then at most one vertex from $\{y, z\}$ is in G_1. Further, observe that x is not in G_2. Therefore, no $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision exists in G_1 and G_2. Thus the lemma holds. □

Proof of Theorem 1.1. We use induction on the number of $K_{2,3}$-subdivisions in G. If the number of $K_{2,3}$-subdivisions in G is zero, then G is outerplanar, since G is a partial 2-tree. From Lemma 2.2, $|LSC(G)| = m - n + 1$ when G is outerplanar. Hence base case is true. If G is not an outerplanar graph, then there exists a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision in G. From Lemma 2.4, $G - \{u, v\}$ is disconnected and contains at least three components. Let P be the $lsp(u, v)$ in G and $k = |V(P)|$. We apply $decomp(G, u, v)$ to obtain G_1 and G_2 from G. For $i \in \{1, 2\}$, we use m_i and n_i to indicate $|E(G_i)|$ and $|V(G_i)|$, respectively. As per Lemma 2.9, the number of $K_{2,3}$-subdivisions in each G_i is less than the number of $K_{2,3}$-subdivisions in G. Then by induction hypothesis, for $i \in \{i, 2\}$, $|LSC(G_i)| = m_i - n_i + 1$. As P is present in G_1 and G_2, it follows that $n_1 + n_2 = n + k$ and $m_1 + m_2 = m + k - 1$. From Theorem 2.8, $LSC(G) = LSC(G_1) \uplus LSC(G_2)$. Hence $|LSC(G)| = |LSC(G_1)| + |LSC(G_2)| = m_1 - n_1 + 1 + m_2 - n_2 + 1 = m - n + 1$. Therefore, $|LSC(G)| = m - n + 1$. □

Proof of Theorem 1.2. For a simple weighted graph G, from Lemma 2.1, a minimum cycle basis of G is contained in $LSC(G)$. The cardinality of any cycle basis in a graph is known to be $m - n + 1$. For a weighted partial 2-tree G, by Theorem 1.1, we have $|LSC(G)| = m - n + 1$. Therefore, the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees. □

Concluding remarks. We have shown that the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees. This result is already known only for outerplanar graphs, which are a subclass of partial 2-trees. Further this cannot be extended even for partial 3-trees. We now present a family of partial 3-trees for which the set of lex short cycles is not a cycle basis, whose construction is as follows: Let $G_n = K_1 + C_{n-1}$ be a wheel graph on n vertices, where $n \geq 4$. A wheel graph on 9 vertices is depicted in Fig. 3. Note that G_n is planar. For every edge $e \in E(G_n)$, assign $w(e) = a$ if e is in external face, otherwise $w(e) = b$, where $a, b \in \mathbb{N}$ and $b \gg a$. Since every face in G_n is a lex short cycle, $|LSC(G_n)| = m - n + 2$ by Euler's formula. Therefore, $LSC(G_n)$ cannot be a cycle basis. Recently, a minimum cycle basis $\mathcal{B}$ in weighted partial 2-trees on n vertices is computed in $O(n) + \text{size}(\mathcal{B})$, where $\text{size}(\mathcal{B})$ is the number of edges in $\mathcal{B}$ counted according to their multiplicity [4]. Characterizing the graph class in which the set of lex short cycles forms a minimum cycle basis is open.

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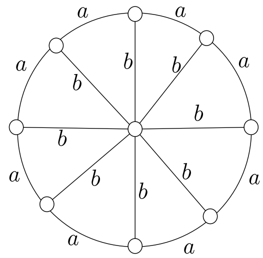

Fig. 3. For the wheel graph shown, if $b \gg a$, then the set of all triangles and the exterior face are lex short cycles.

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