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S. Etikyala and R. I. Sujith

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# Change of criticality in a prototypical thermoacoustic system

S. Etikyala[a)] and R. I. Sujith[b)]
*Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai 600036, India*

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In this paper, we report on the existence of the phenomenon of change of criticality in a horizontal Rijke tube, a prototypical thermoacoustic system. In the experiments, the phenomenon is shown to occur as the criticality of the Hopf bifurcation changes with varying air flow rates in the system. The dynamics of a nonlinear system exhibiting Hopf bifurcation can be described using a Stuart-Landau equation (SLE) in the vicinity of the bifurcation point. The criticality of Hopf bifurcations can be determined by the Landau constant of the Stuart-Landau equation, which represents the effect of nonlinearities in the system. We propose an SLE to model the bifurcations seen in the horizontal Rijke tube. We identify a rescaled version of Strouhal number as the Landau constant, which determines the criticality of the bifurcation in the present study. *Published by AIP Publishing.*
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**The qualitative change in the dynamics of a system can happen either gradually or abruptly through bifurcations. Transitions into oscillations, in many natural systems, happen through Hopf bifurcation. The nature of the transitions, being abrupt or smooth, is described simply by the criticality of the bifurcation. The criticality of the Hopf bifurcations is decided by the stabilizing nature of dominant nonlinearities in the system. Variations in critical parameters in the system affect and alter the nature of nonlinearites from stabilizing to de-stabilizing, thus subjecting the criticality to change. In this paper, we investigate the change in criticality of the Hopf bifurcations leading to self-sustained oscillations in a thermoacoustic system. In a thermoacoustic system, oscillations are established through positive feedback between the heat release rate and sound waves. The oscillations are often found to be detrimental, sometimes even causing failure of systems such as gas turbine engines, rockets, etc. We show that varying the flow rates can alter the criticality of the bifurcations using a prototypical thermoacoustic system, a horizontal Rijke tube. Further, we identify the Landau constant in the Stuart-Landau equation (SLE) that describes the dynamics of this system, as the rescaled Stouhal number of the flow in the system to describe the observed change in criticality. Identification of Landau constant in a real-world system such as gas turbine engines encountering thermoacoustic instabilities, can be of practical importance in recognizing its dynamical behavior.**

## I. INTRODUCTION

Many natural systems are nonlinear in nature and exhibit transitions from one dynamical state to another through bifurcations. A bifurcation is a qualitative change in the dynamics of the system with a small change in the control parameter.[1] Some standard examples of bifurcations include saddle-node or fold, pitchfork and Hopf bifurcations.[1] Hopf bifurcations are used to describe the dynamics of an oscillatory system where the oscillations emerge as a result of change in system parameters.

Oscillations are often established through a Hopf bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues crosses the imaginary axis in the complex plane. As a result, a limit cycle takes birth from the fixed point in the dynamical system at the bifurcation point. This point at which bifurcation occurs, is known as the Hopf point. When the control parameter is changed in a dynamical system, at the Hopf point, the steady-state amplitude of the oscillations can change abruptly, in which case the bifurcation is subcritical Hopf.[1,2] In contrast, the case in which the amplitude of the oscillations changes continuously, the bifurcation is supercritical Hopf.[1,2] However, the criticality of a Hopf bifurcation, typically is decided by the stabilizing nature of the nonlinearities present in the system.[1,3]

Hysteresis is observed in the case of subcritical Hopf bifurcations, when the control parameter is decreased or reversed. The oscillations cease to happen at a point other than the Hopf point and the fixed point reappears through a saddle-node bifurcation.[1–6] This point at which saddle-node bifurcation occurs is known as the fold point.[1] Studying such bifurcations is very essential in estimating the behavior of the nonlinear system at hand and eventually understanding the criticality of its bifurcations.[1,7]

In a physical system, the nature of nonlinearities changes as one of the system parameters is varied, which can change the criticality of a Hopf bifurcation. Recent research on the existence of change of criticality includes studies that reported its occurrence in nonlinear systems, across different areas such as ecology,[8–10] quantum-dot optoelectronic devices,[11] and models related to dynamos in magneto-hydrodynamic turbulence,[12] phase transitions under the influence of external agents,[13] etc. Experimental studies include plate tectonic studies,[14] jamming transition from a fluid to a disordered solid state[15] and bluff body wakes in turbulent systems.[16,17]

a)Electronic mail: sreelekha.etikyala@gmail.com
b)Electronic mail: sujith@iitm.ac.in

In the present work, we study the change of criticality in a thermoacoustic system. A thermoacoustic system is a nonlinear dynamical system which undergoes large amplitude limit cycle oscillations as a result of the positive feedback established between the heat release rate fluctuations and sound waves.[18–21] These large amplitude oscillations constitute a self-sustained oscillatory state known as thermoacoustic instability, and are detrimental to the systems such as gas turbine engines and rockets, sometimes even causing a catastrophic failure.[4–6,19–25] The transition of a thermoacoustic system into its harmful oscillatory state happens through a Hopf bifurcation.[4] Depending on the experimental system and its parametric variation, the bifurcation can either be subcritical or supercritical.[4,20,24] In the case of practical engineering systems undergoing subcritical bifurcations, avoiding the bifurcation is difficult as it happens suddenly, and large amplitude oscillations are established immediately. When the parameter is reversed, the detrimental oscillations sustain till the fold point due to hysteresis.

For a prototypical thermoacoustic system known as Rijke tube, a model was developed by Balasubramanian and Sujith[19] to study the subcritical Hopf bifurcations. Further, Subramanian *et al.*[6] performed a weakly nonlinear analysis of a thermoacoustic system to derive the Stuart-Landau equation (SLE). The SLE is known to describe the weakly nonlinear dynamics in the vicinity of bifurcation points in systems displaying oscillations. The SLE represents the normal form of a homogenous system close to a Hopf bifurcation. The criticality of the bifurcation is then determined by the sign of the coefficient of the leading nonlinearities, known as a Landau constant. Many physical systems such as flows undergoing Taylor-Couette and shear flow instability, bluff body wakes and their instabilities as a result of bifurcation are studied using a SLE.[26–29] The SLE has also been used in modeling pertinent nonlinear phenomena such as frequency lock-in[30] and resonance in systems with flows.[31]

Experimentally, Hopf bifurcations of both criticalities have been reported to occur in the same combustor at different operating conditions by Lieuwen in 2002.[4] It was observed that supercritical bifurcations occurred for higher inlet velocity and subcritical for low inlet velocity.[4] Later in 2012, Illingworth *et al.*[5] have found the criticality of bifurcations changing with respect to variation in system control parameters (Peclet number) in their study on a ducted diffusion flame using numerical continuation. Although SLE generically models both the criticalities, Subramanian *et al.*[6] have shown SLE for subcritical bifurcations in a Rijke tube model.[19] However, there was not much attention paid so far to change in criticality of the Hopf bifurcations in thermoacoustic systems. The present work is aimed at investigating the change of criticality observed in a prototypical thermoacoustic system.

We investigate a Rijke tube, the prototypical thermoacoustic system with an electrical heater, experimentally. From the experimental data, we observe a change in criticality of the bifurcations at different air flow rates. A model is constructed using a SLE, with coefficients that depend on the system parameters, to capture the phenomena of crossover in the criticality. Further, we find that the Landau constant of the system is a linear function of Strouhal number of the flow, which is a non-dimensional ratio of the flow time-scale to the acoustic time-scale of the system.

## II. EXPERIMENTAL SETUP

In the present study, a horizontal Rijke tube with an electrically heated wire mesh, acting as a heat source, is used to perform the experiments. The tube is 1 m long with a square cross-section. The cross-sectional area of the duct is 9.2 cm × 9.2 cm. The mean flow is established with the help of a compressor. The flow rate of air provided by the compressor is measured and also regulated with the help of a mass flow controller (MFC) which is located downstream of the compressor. The MFC can measure the flow rates with an uncertainty of $\pm 0.01$ standard litre per minute (SLPM). The outlet of the compressor is connected to one end of the Rijke tube, via a rectangular chamber (120 cm × 45 cm × 45 cm), referred to as a decoupler. The decoupler acts as a silencer and eliminates the sound produced due to interactions between the compressor and the duct inlet. A programmable DC power supply (TDK-Lambda, GEN8–400, 0–8 V, 0–400 A) is used to power the mesh type heater. The mesh type heater used for the experiments presented in this study is similar to the one used by Matveev[20] and Gopalakrishnan and Sujith.[32] The uncertainty associated with heater power measurement is 0.4 W. In order to avoid electrical contact with the walls of the tube and also to prevent heat loss to the walls of the tube, a ceramic housing is provided around the mesh. The heater location is fixed throughout the study, at 25 cm from the end connected to the decoupler.

A piezoelectric transducer (PCB 103B02) mounted 30 cm from the open end is used to measure the acoustic pressure. The sensitivity of this transducer is 217.5 mV/kPa; the uncertainty in the measurements is 0.2 Pa. Data is acquired with the help of a National Instruments make PCI 6221 data acquisition card. The data was acquired at a sampling frequency of 10 kHz for 3 s. To ensure uniform conditions, the initial temperature was maintained at $19 \pm 3\,^{\circ}$C. The experiments were conducted only when the cold decay rate was 18.5 ($\pm 5\%$) $s^{-1}$ for the fundamental frequency. The decay rate resulting out of the acoustic damping of the duct is maintained within bounds to ensure repeatability.[25] We provided a sound in the duct, using a loud speaker (Ahuja AU60) mounted at 62.5 cm from the inlet, at the first eigenmode frequency (157 Hz) for a short duration of time. Once the loud speaker is switched off, the acoustic pressure decays down. The cold decay rate is determined by performing the Hilbert transform of the pressure signal and by calculating the logarithmic decay of its amplitude.[21]

In all the experiments, the heater power is varied as the bifurcation parameter till the system attains its oscillatory state from the non-oscillatory steady state. When the heater power is changed, other parameters of the system such as the flow rate and the heater location are maintained constant during a single experiment. Experiments are conducted by varying the heater power for a range of flow rates ($\dot{v}$) starting with 105 SLPM to 45 SLPM in steps of 5 SLPM.

We preheat the system for about 20 min before starting the experiments to minimize temperature variations as the heater power is changed. Systems with subcritical bifurcations, when operated in non-oscillatory state can go into their oscillatory states at values other than Hopf point if the control parameter is changed abruptly or discontinuously.[24] The heater power is changed in a quasi-steady manner to avoid this phenomenon. Here, the system is allowed to stay at a particular value for an adequate amount of time, 120 s for the system configuration used in this study, and reach the steady state asymptotically. The asymptotic state is confirmed by observing the steadiness in the temperature reached by the system and a duration of 120 s was found to be ideal for Rijke tube, used in this study (Figure 1).

## III. RESULTS

### A. Subcritical Hopf bifurcation

We begin our experimental study with a flow rate, where we obtain a subcritical Hopf bifurcation that enables us to study the hysteresis behavior. We have conducted experiments on the Rijke tube for an air flow rate ($\dot{v}$) of 100 SLPM indicating the Strouhal number being 0.22. Strouhal number is a non-dimensional ratio of the flow time-scale to the acoustic time-scale of the system. The Strouhal number can be calculated from the system parameters as $St = \frac{d_w A/\dot{v}}{2L/c_0}$, where, $c_0$ is the speed of sound at ambient temperature, $d_w$ is diameter of the heater wire, and $A$ and $L$ are the area and length of the duct respectively. A detailed account on the calculation of Strouhal number can be found in Gopalakrishnan and Sujith.[32]

As we gradually increase the heater power, the system goes from a non-oscillatory steady state to a self-excited oscillatory state through a Hopf bifurcation. Corresponding to every value of the heater power, we acquire the time series of acoustic pressure and plot the root mean square (RMS) value of the signal. This results in a bifurcation diagram as shown in Figure 2.

The onset of the self-sustained oscillations occurs at a value of heater power, $K_H = 614.3$ W, to be called as a Hopf point. The bifurcation at this point leads to a sudden appearance of limit cycle oscillations with a frequency of 155 Hz. Till Hopf point, the system is in its non-oscillatory state. The RMS values of the inherent noise levels in the system are measured and found to be in the range of 4–10 Pa. This results in a non-zero value for RMS of the measured pressure till the bifurcation point as shown in Figure 2. There is a sudden increase in the RMS value of acoustic pressure by 80 Pa at the Hopf point, as seen from the figure. The difference is at least 15 times the average of noise levels measured from the experiments. Thus, a subcritical Hopf bifurcation is found at 100 SLPM. These observations are consistent with the previous findings.[20,22,24,32]

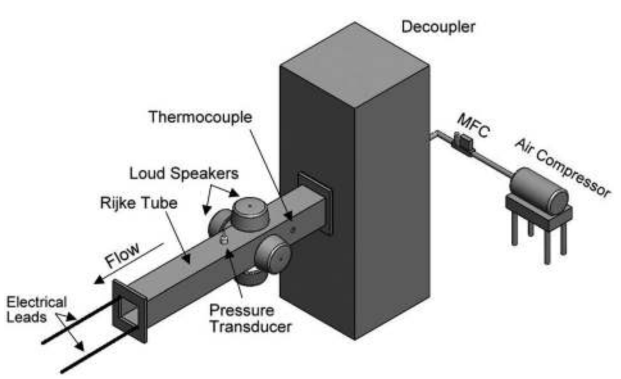


FIG. 1. Schematic diagram of the horizontal Rijke tube setup used in the present study. A pressure transducer (PCB 103B02) is used to measure the pressure fluctuations in the system. A thermocouple is used to monitor the temperature of the duct. Loud speakers are used to provide sound that is used to obtain the decay rates, related to damping of the system. A mass flow controller (MFC) is used to maintain and control the flow rates. For this study, the heater is fixed at 25 cm into the tube from the decoupler.

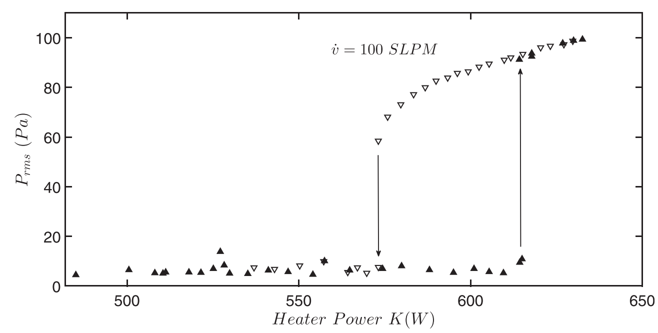


FIG. 2. Bifurcation diagram at a Strouhal number of 0.22 corresponding to an air flow rate of 100 SLPM. The figure shows variation in RMS values of acoustic pressure, $P_{rms}$ when the heater power is changed. The sudden appearance of high amplitude oscillations in the Rijke tube shows that the transition is subcritical. The forward and backward paths are indicated by filled upward (▲) and hollow downward facing triangles (∇) respectively.

Once established, the limit cycle oscillations persist for values higher than the value corresponding to the Hopf point. The experimental results are presented till a heater power of 630 W in Figure 2. We observe a marginal increase in the amplitudes of oscillations, maintaining the same frequency, which is close to the duct acoustic mode. We have studied the behavior of the system on the reverse path, by plotting the RMS values of the time series of acoustic pressure acquired while decreasing the heater power. For a flow rate of 100 SLPM, the presence of hysteresis is evident from the figure.

Upon decrement in heater power, the system returns to its non-oscillatory steady state through a saddle-node bifurcation, at the fold point $K_F = 573.1$ W, with a value different from that of the Hopf point. Hysteresis observed in this subcritical bifurcation can be seen in Figure 2 by noting the change in heater power value where there is an abrupt change in the amplitude of the oscillations with respect to the direction of variation in heater power. This enables the system to exist in either of the two stable states in the hysteresis zone depending on whether the heater power is increased or decreased. This property of the system is known as bistability. In Rijke tube, hysteresis has been reported and explained by Matveev[20] and Mariappan and Sujith.[22] They have also established that the hysteresis is more prominent for higher flow rates. This brings us to question naturally about what happens to the hysteresis properties of the prototypical thermoacoustic system at low air flow rates.

### B. Supercritical Hopf bifurcation

The results of the experiments conducted at a lower flow rate of air ($\dot{v} = 60$ SLPM) corresponding to a Strouhal number of 0.36 are presented in Figure 3. The initiation of limit cycle oscillations happens at a heater power value of $K_H = 335.2$ W. There are two interesting aspects to note about the Hopf bifurcation at $K = K_H$ in Figure 3. First, the limit cycle oscillations constitute an RMS value of about 7 Pa which is only 2 Pa more than the average noise levels measured from these set of experiments. Second, the fold point corresponding to the loss of oscillations in the reverse path is found to be at a heater power value of $K_F = 335.8$ W. From a visual inspection of the bifurcation diagram, there is no hysteresis. This is again a strong indication of the supercritical nature of the bifurcation. We remark that the step size chosen for varying the heater power is small enough to confirm the absence of hysteresis zone.

In systems undergoing supercritical bifurcations, the amplitude of pressure oscillations increases and decreases continuously in the same manner. Thus amplitudes obtained while decreasing, retrace the path followed by the amplitudes obtained while increasing ensuring no hysteresis. The amplitudes vary as square root of the control parameter, as expected from the solutions of a supercritical bifurcation.[1] This is illustrated in the inset of Figure 3 showing the parabolic fit for $P_{rms}$ in both directions, post the Hopf point. The fit accounts for the noise floor. Thus, we observe a continuous change in $P_{rms}$ values in supercritical as opposed to the case of a subcritical bifurcation.

### C. The crossover of bifurcations

In order to understand the change in criticality observed from the experiments at two different flow rates so far, we have conducted experiments over a range of flow rates of air. The results corresponding to the bifurcations undergone by the system at various flow rates are plotted using a surface in Figure 4.

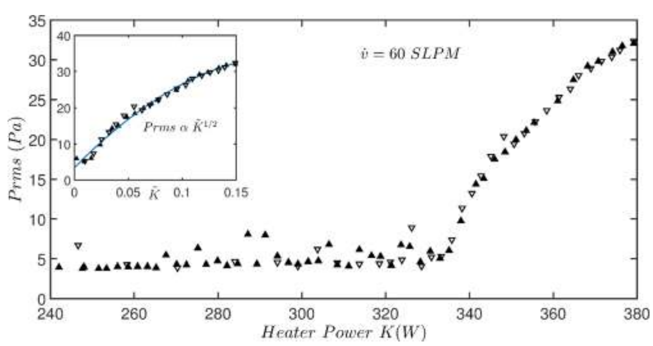


FIG. 3. Bifurcation diagram at a Strouhal number of 0.36 corresponding to the flow rate of 60 SLPM. The figure shows variation in RMS values of acoustic pressure with respect to heater power. The gradual birth of stable limit cycle oscillations in Rijke tube shows that the transition is supercritical. In the inset, a parabolic fit is shown inherent to capture the continuous increase in the RMS values of the pressure data. The intercept on the axis of $P_{rms}$ corresponds to the noise levels in the experimental setup. There is no hysteresis zone as the increase and decrease of $P_{rms}$ with heater power follow the same trend. Note that the heater power is rescaled to $\tilde{K} = (K - K_H)/K_H$ in the inset.

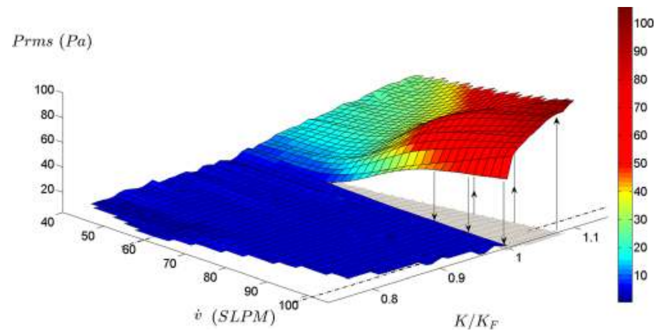


FIG. 4. Results from the experiments conducted over a range of air flow rates and their corresponding bifurcation. The bistable region is shaded gray. The magnitudes of the RMS are color coded respectively. The surface represents both the criticalities of the system such as (a) supercritical at 60 SLPM (Fig. 3) and (b) subcritical at 100 SLPM (Fig. 2). Dotted lines are used to indicate the sections (a) and (b). The hysteresis appears gradually beyond a certain flow rate of 65 SLPM. From 70 SLPM and higher rates, the width of the bistable region increases for the subcritical bifurcations.

The change of criticality is seen when the system undergoes a supercritical bifurcation for flow rates upto 65 SLPM and subcritical starting at 70 SLPM and higher. The hysteresis zone, acting as a visual signature of subcritical bifurcation from the figure, is very small at 70 SLPM. The hysteresis zone grows gradually gaining area as we increase the air flow rate and is represented using grey area in the figure. Hysteresis becomes significant at higher flow rates as expected from earlier studies.[24]

The increment in the hysteresis zone with varying flow rates can be quantified by tracking the width of the bistable zone. There is a power law dependence observed for the hysteresis width $\chi = (K_H - K_F)/K_H$ from the experiments of subcritical bifurcations. The variation of $\chi$ with the non-dimensional Strouhal number ($St$) is plotted in Figure 5. The power law interpreted from the experimental data is found to have an exponent of $-5.86$. The exponent found here is in agreement with that found in earlier experimental studies[32] with heater power and heater location as control parameters indicating its robustness.

A typical thermoacoustic system such as the Rijke tube has many degrees of freedom. However, a common feature

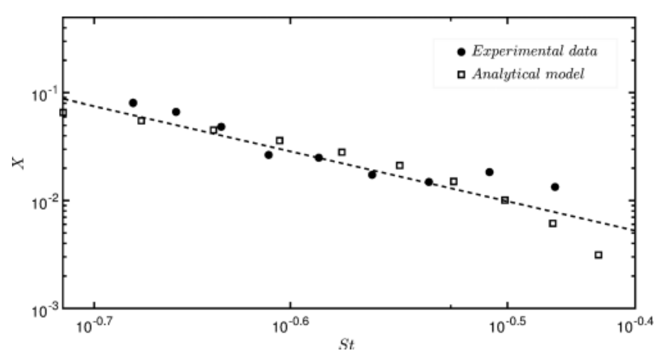


FIG. 5. Bistable region in the bifurcations is characterized by its width. Variation of the non-dimensional hysteresis width is plotted against the Strouhal number in a log-log scale. We observe that the Rijke tube undergoes a supercritical transition beyond $St = 0.36$. Results from analytical analysis are also presented using hollow squares. The change in hysteresis width is shown with respect to rescaled coefficient $b$, that is analogic to the Strouhal number. The dotted line is plotted as an indication of the power law.

in all such systems is that the large-scale modes dominate near a transition, where an amplitude equation such as an SLE would suffice to capture the system dynamics.[7] In Section IV, we propose an SLE for our thermoacoustic system that captures the aforementioned features.

## IV. STUART-LANDAU EQUATION FOR RIJKE TUBE

Any oscillatory system undergoing Hopf bifurcation can be studied using a Stuart-Landau equation (SLE) near the onset of the oscillations.[7,27–29,33] SLE can generically describe the dynamics of systems exhibiting oscillations in terms of a differential equation with system specific coefficients. For a complex amplitude $W = R\exp(i\varphi)$ of the oscillations of any general oscillator, the Stuart-Landau equation is given by[7]

$$\dot{W} = fW - g|W|^2W, \tag{1}$$

where $f$ and $g$ are functions of system parameters. In the case of weak nonlinearities, after some algebra, we can reduce Equation (1) to the following equations for real amplitude $r$ and the phase $\theta$ of the oscillations with coefficients $a$ and $b$ depending on $f$ and $g$

$$\dot{r} = ar + br^3, \tag{2}$$

$$\dot{\theta} = constant. \tag{3}$$

Equation (2), exhibits both supercritical and subcritical bifurcations depending on the sign of the coefficient $b$, known as the Landau constant. In other words, Landau constant represents the stabilizing effect of the leading nonlinearities in the system. We address the change in criticality of the bifurcation by studying both types of Hopf bifurcations with the SLE.

Clearly, Equations (2) and (3) represent a supercritical Hopf bifurcation for $b < 0$ and subcritical Hopf bifurcation for $b > 0$ at $a = 0$. When $a > 0$ and $b > 0$, the model has unstable limit cycles as solutions representing a subcritical bifurcation. To analytically obtain the values of stable limit cycles, a stabilizing higher order nonlinearity, a fifth order term with a negative coefficient, is added to the model as

$$\dot{r} = ar + br^3 + cr^5. \tag{4}$$

A subcritical transition is observed when $b > 0$ for $c < 0$ in Equation (4). The $cr^5$ term stabilizes the system for large $r$[7]. The system undergoes a subcritical bifurcation at $a = a_H = 0$ and a saddle-node (fold) bifurcation at $a = a_F = b^2/(4c)$. A hysteresis zone results due to the aforementioned bifurcations, as we observe in experiments.

The solutions of $\dot{r} = 0$ at the fold point, where $a = a_F$ can be analytically evaluated as $r = \sqrt{-b/2c}$. The subcritical bifurcation occurs at $a = a_H$, the width of the hysteresis region indicated here depends on $b$ as

$$\chi = |a_H - a_F| = b^2/(4c), \tag{5}$$

where $a_H$ and $a_F$ are the values of $a$ at Hopf and fold points respectively. We observe from experiments, as shown in Figure 5, that the change in criticality occurs as we vary flow rates, in turn, the Strouhal number, $St$. The Landau constant ($b$), the coefficient of cubic nonlinearity in Equation (4), is chosen to be directly proportional to $St$ as

$$b \propto St_c - St, \tag{6}$$

where $St_c$ corresponds to the Strouhal number of the system at which the crossover of criticality occurs. In other words, this number corresponds to the case when the hysteresis width becomes completely negligible and is close to be zero. From the experiments, we found $St_c = 0.36$ for this configuration of the Rijke tube.

Substituting Equation (6) into Equation (4), we establish a SLE for the horizontal Rijke tube. The proportionality constant is chosen to be 60 for computation of $b$ in Equation (6), the coefficient $c$ to be $-5$ and $a$ is calculated from $a = -br^2 - cr^4$, to capture the experimental results in a qualitative sense. Solutions of SLE also capture the behavior of hysteresis width, which is summarized in Figure 5 for both experiments and the model.

The above one-dimensional model thus captures the salient features of supercritical and subcritical transitions exhibited by the prototypical thermoacoustic system. Here, the criticality of the bifurcation is essentially decided by the governing nonlinearities in the form of a Landau constant. From the relationship between Landau constant and the Strouhal number established in the model, we can conclude that Strouhal number itself is the Landau constant deciding the nature of criticality of the bifurcation. The study gains significance as the nature of criticality becomes important to be determined in developing strategies to prevent the onset of instabilities in systems such as gas turbine engines.

## V. CONCLUSIONS

The criticality of a bifurcation changes in a nonlinear dynamical system with the variation in the influence of its dominant parameters. We report on the existence of change of criticality in a prototypical thermoacoustic system from experiments. The behavior of the thermoacoustic system is qualitatively described using a SLE that best describes the bifurcation behavior of the system governed by its nonlinearities. We show that change in criticality of the Hopf bifurcations occurs as the Strouhal number is changed in our experiments. Landau constant, which decides the criticality of the bifurcation is modeled to be a linear function of Strouhal number.

An analysis of the variation of hysteresis width over the range of flow rates has shown that the width follows a power law with the Strouhal number associated with the oscillations observed in the Rijke tube. The study shows that the criticality of bifurcations in an oscillatory system such as Rijke tube is decided by the Strouhal number. The presented model is simple and used specifically to capture the qualitative features of the prototypical system. Thus, the work establishes scope for identification of Landau constant in real-world systems encountering thermoacoustic instabilities, which can be of practical importance in recognizing the dynamical behavior of the system.

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