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Axiomatic characterization of the interval function of a block graph

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ABSTRACT

In 1952 Sholander formulated an axiomatic characterization of the interval function of a tree with a partial proof. In 2011 Chvátal et al. gave a completion of this proof. In this paper we present a characterization of the interval function of a block graph using axioms on an arbitrary transit function R. From this we deduce two new characterizations of the interval function of a tree.

1. Introduction

One of the fundamental notions of metric graph theory is that of the *interval function* $I : V \times V \to 2^V$ of a connected graph $G = (V, E)$, where $I(u, v)$ is the set of vertices on shortest paths between u and v in G. The term interval function was coined in [16], which is the first extensive study of this function. The notion already existed long before. We do not know for sure, but one of the first occurrences might be the thesis of W.D. Duthie [11] of 1940 on "Segments in Ordered Sets", see also [12]. He characterized distributive lattices by postulates or 'axioms' on segments. This work was pursued by Sholander [26,27] in the early 1950s. Sholander studied median semilattices using segments. Median semilattices can also be studied as graphs: the Hasse diagram of a median semilattice is precisely a median graph (and vice versa). This was done for the first time by Avann in 1963 (who called the graph a 'unique ternary distance graph'), and later independently by Nebeský in 1971, and

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Mulder [15,19,16] in 1978–1980. Sholander also presented a set of axioms on segments that characterizes the segments (intervals) of a tree. But he gave a partial proof for this characterization. Only recently, in 2011, a completion of this proof was presented by Chvátal, Rautenbach and Schäfer [10].

Sholander also pursued another line of study in his papers [26,27], viz. that of betweenness in the language of ternary relations. This generalized results by Pitcher and Smiley [25]. Sholander used this notion of betweenness to characterize median betweenness, a structure that is equivalent to median semilattices and median graphs, see e.g. [20,16,21]. Amongst the results in [26] was a characterization of a tree betweenness. A new characterization of a tree betweenness was obtained recently by Burigana [2], with a short new proof by Chvátal et al. [10].

The focus of Sholander was on sets of axioms with as few axioms as possible. This was also the approach of later authors, see [20,10]. In this approach the axioms are necessarily of rather complex nature. In [16] and later work a different approach was taken: here the choice has been to find axioms that are as elementary as possible, and also such that they are applicable in the most general setting, not that of only very well-structured graphs or ordered sets (such as median graphs and median semilattices), see e.g. [22–24,6]. In [16] five simple and elementary properties of the interval function were given that are now known as the 'five classical' axioms for the interval function. In [18] the interval function of a connected graph is characterized by a set of axioms that includes these five classical, elementary axioms. The approach in [18] was as follows. First as much as possible was deduced using the five classical axioms only. Then the road blocks were determined that prevented any further consequence. From these road blocks two more axioms were inferred that, together with the five classical axioms, characterize the interval function of a connected graph. These two extra axioms were more complicated, but still minimal in the sense that weaker axioms would not do the trick.

In [14] a similar approach for betweenness was chosen: using axioms as simple as possible to study betweenness in a broad context. As opposed to the above idea of betweenness as a ternary relation, a betweenness in [14] was formulated in terms of a function $R : V \times V \to 2^V$. One advantage of this approach is that now it could be used in other contexts. In [17] a unifying approach for moving around in discrete structures such as graphs and partially ordered sets was presented: a *transit function* $R : V \times V \to 2^V$ satisfying three elementary axioms. It includes all of the above functions, but also other so-called path functions, like the *induced path function* J, see [6,7], where $J(u, v)$ consists of the vertices on induced paths between u and v. Recently in [4] characterizations of some graph classes were obtained using betweenness axioms on the interval function and on the induced path function.

In this paper we return to the interval function. Above we mentioned the Sholander characterization (with a completion of the proof by Chvátal et al.) by a set of axioms with as few axioms as possible. Here we choose the other approach (from [16,14,18]): try to find a set of axioms that are each as simple and elementary as possible. We present a characterization of the interval function of a block graph. All but one of the axioms are simple and elementary in the sense that these are the axioms for a betweenness from [14]. As corollaries we obtain two new characterizations in the case of trees. Our sets of axioms have one axiom in common with the Sholander set for trees. In our characterizations the remaining axioms form an actually weaker set than those in the Sholander characterization.

A betweenness sensu Sholander is a special type of a ternary relation $\mathcal{B}$ on V, where a triple (u, x, v) is in $\mathcal{B}$ means that x is *between* u and v. We can translate this into a function $R : V \times V \to 2^V$ by defining $R(u, v)$ to be the set of all x between u and v. The axioms on $\mathcal{B}$ then translate to axioms on R. With this translation in mind we study the tree betweenness of Burigana [2] and Chvátal et al. [10]. Below we obtain another characterization of the interval function of a tree that involves axioms that are actually weaker than the axioms of Sholander and those of Burigana/Chvátal et al. For more information on tree betweenness and other literature we refer the reader to their papers [26,2,10].

We investigate the independence of the axioms in our various characterizations. We present our results in the context of transit functions. Besides this we present a characterization of the interval function of a path and a star.

2. Axioms on transit functions

Throughout this paper V is a finite nonempty set. A *transit function* on V is a function $R : V \times V \to 2^V$, where 2^V is the power set of V, satisfying the following three axioms.

(t1) $u \in R(u, v)$, for all u, v in V.
(t2) $R(u, v) = R(v, u)$, for all u, v in V.
(t3) $R(u, u) = \{u\}$, for all u in V.

The third axiom could be deleted. It is usually added to exclude degenerate cases. For instance, the function $F(u, v) = V$, for all u, v in V, satisfies the first two axioms, but will not enlighten us about any aspect of an underlying structure. In the sequel we will see that in many relevant cases (t3) follows from other axioms. If $G = (V, E)$ is a graph with vertex set V, then we say that R is a transit function on G. The *underlying graph* G_R of a transit function R is the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that in general G and G_R need not be isomorphic graphs, see [17]. For one instance of this phenomenon we refer to Example 18.

The notion of transit function was introduced in [17] as a unifying concept for many functions on graphs that have been studied so far, e.g. the (geodesic) interval function I, the induced path function J, see [6,7], the triangle-path function T, see [5,9], the all-paths function A, see [3]. It was also meant to create a framework for new problems and ideas. The four mentioned functions are all so-called *path transit functions*, because they are defined in terms of paths in G. See [17,8] for further information on path transit functions. In [17] the problem is proposed to characterize any transit function in terms of *transit axioms*, that is, axioms in terms of the function only, independent of the graph on which the function is defined. Nebeský [21] obtained a very interesting impossibility result: there does not exist a characterization of the induced path function J of a connected graph using transit axioms only.

Our focus in this paper is on the interval function. Let $G = (V, E)$ be a graph with distance function d, where $d(u, v)$ is the length of a shortest u, v-path or u, v-geodesic. Then the interval function I_G of G is defined by

$$I_G(u, v) = \{x \mid d(u, x) + d(x, v) = d(u, v)\},$$

that is, the set of vertices lying on shortest paths between u and v. When no confusion arises we usually write I instead of I_G.

The geodesic intervals $I(u, v)$ in G inherently have the structure of a betweenness (defined below), but arbitrary transit functions may not have these properties. The following *betweenness axioms* were introduced in [14] to capture basic aspects of the idea of betweenness. The first of these tells us that, if x is between u and v but distinct from v, then v is *not* between u and x. The second tells us that, if x is between u and v and y is between u and x, then y is between u and v.

$(b1)$ $x \in R(u, v), x \neq v \Rightarrow v \notin R(u, x)$, for all u, v, x in V.
$(b2)$ $x \in R(u, v) \Rightarrow R(u, x) \subseteq R(u, v)$, for all u, v, x in V.

A *betweenness* in the sense of [14] is a function $R : V \times V \to 2^V$ satisfying $(t1)$, $(t2)$ and these two betweenness axioms. Below we will see that this notion is weaker than the betweenness considered by Sholander [26,27], Burigana [2] and Chvátal et al. [10]. The idea behind the betweenness in the sense of [14] is that it is also applicable to transit functions different from the interval function. For instance, in [14,6,7] the case is studied for which graphs the induced path function J is a betweenness, that is, satisfies the axioms $(b1)$ and $(b2)$. Note that axioms $(t1)$ and $(b1)$ imply axiom $(t3)$. So a betweenness is a transit function.

In the first extensive study of the interval function [16] five simple properties of the interval function $I(u, v)$ of any connected graph were presented. In [18] these properties were coined as the five *classical axioms* on I. These five transit axioms are $(t1)$ and $(t2)$, the betweenness axiom $(b2)$, and the following two axioms.

$(c4)$ $x \in R(u, v) \Rightarrow R(u, x) \cap R(x, v) = \{x\}$, for all u, v, x in V.
$(c5)$ $x \in R(u, v), y \in R(u, x) \Rightarrow x \in R(y, v)$, for all u, v, x, y in V.

Obviously, axioms $(t1)$ and $(c4)$ imply $(t3)$. So any function satisfying the five classical axioms is a transit function. First we present two simple implications for the classical axioms and $(b1)$.

Proposition 1. *Axioms* $(t1)$, $(t2)$ *and* $(c4)$ *imply axiom* $(b1)$.

Proof. Let $R : V \times V \to 2^V$ be a function on V satisfying axioms $(t1)$, $(t2)$ and $(c4)$. Let x be in $R(u, v)$ with $x \neq v$. By $(c4)$, we have $R(u, x) \cap R(x, v) = \{x\}$. From $(t1)$ and $(t2)$, it follows that v lies in $R(x, v)$. Since $v \neq x$, we have that v is not in $R(u, x)$. □

From this proposition it follows that any transit function satisfying the five classical axioms is a betweenness in our sense.

Proposition 2. *Axioms* $(t1)$, $(t2)$, $(t3)$ *and* $(c5)$ *implies axiom* $(c4)$.

Proof. First we prove that $(t3)$ and $(c5)$ imply $(b1)$. Let x be a vertex in $R(u, v)$ distinct from v. If v were in $R(u, x)$, then by taking $y = v$ in $(c5)$, we would get x in $R(v, v)$. By $(t3)$ we have $R(v, v) = \{v\}$. But this contradicts x and v being distinct.

Next let x be in $R(u, v)$, and let y be any vertex in $R(u, x)$ distinct from x. From $(c5)$ it follows that x is in $R(y, v)$. But now $(t2)$ and $(b1)$ imply that y is not in $R(x, v)$. Thus we have $[R(u, x) - \{x\}] \cap R(x, v) = \emptyset$. Hence $R(u, x) \cap R(x, v) = \{x\}$ by $(t1)$, and we are done. □

Note that we used $(t3)$ in the above proof. If $(t3)$ is not assumed then it turns out that axioms $(c4)$ and $(c5)$ are independent. First we consider the degenerate function $R(u, v) = V$ on a graph with at least 3 vertices. This function trivially satisfies $(t1)$, $(t2)$ and $(c5)$, but clearly does not satisfy $(t3)$ or $(c4)$. Next we present two examples that show that the converse of neither Proposition 1 nor Proposition 2 is true. In particular, Example 4 shows that $(c4)$ does not imply $(c5)$. The *induced path interval* $J(u, v)$ is the set of all vertices on induced paths between u and v. This defines the *induced path function* $J : V \times V \to 2^V$.

Example 3 (*A Betweenness R That Does Not Satisfy* $(c4)$)**.** The *k-fan* F_k consists of a path P on k vertices and an additional vertex y adjacent to all vertices on the path. Take $k \geq 5$. We consider the induced path function J on F_k. It is straightforward to check that J is a betweenness on this fan (satisfies axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$). This also follows trivially from any of the main results in [7]. Let u and v be the end vertices of P, and let x be a vertex on P that is not adjacent to u or v. Then y belongs to both $J(u, x)$ and $J(x, v)$. Obviously, $J(u, v) = V$. So this choice of vertices u, v, x does not satisfy axiom $(c4)$. □

Example 4 (*A Betweenness R That Does Not Satisfy* $(c5)$)**.** Take the 4-fan as in Example 3. So P is a u, v-path of length 3, and y is adjacent to all four vertices of P. It is straightforward to check that the induced path function J of this 4-fan satisfies $(c4)$. But it does not satisfy $(c5)$: now take x to be the vertex on P adjacent to v. Then y is in $J(u, x)$, but x is not in $J(y, v) = \{y, v\}$. □

Both examples and propositions were already given in [13].

In [18] a characterization of the interval function of a connected graph is given involving the five classical axioms, see the Introduction for more information on this.

Already as early as 1952 Sholander [26] gave a characterization of the interval function of a tree, although without a complete proof. In his paper intervals were still called segments. The completion of the proof was presented by Chvátal et al. in [10]. Sholander's axioms were the following three axioms.

(S) There exists an x such that $R(u, v) \cap R(v, w) = R(v, x)$, for all u, v, w in V.
(T) $R(u, v) \subseteq R(u, w) \Rightarrow R(u, v) \cap R(v, w) = \{v\}$, for all u, v, w in V.
(U) $R(u, x) \cap R(x, v) = \{x\} \Rightarrow R(u, x) \cup R(x, v) = R(u, v)$, for all u, v, x in V.

Sholander [26] proved that his axioms (S) and (T) imply the five classical axioms. So any function $R : V \times V \to 2^V$ satisfying (S) and (T) is a betweenness in our sense. Proposition 1 and Example 3 show that our concept of a betweenness is an essentially weaker concept than a Sholander function R satisfying axioms (S) and (T).

Sholander [26] also considered ternary relations $\mathcal{B}$ on a set V. We can translate such a ternary relation into a function $R : V \times V \to 2^V$ by defining $R(u, v)$ to be the set of all x for which (u, x, v) is in $\mathcal{B}$. With this translation in mind, Sholander introduced a *betweenness* as a function R satisfying axiom $(t3)$ and the following additional axiom.

(C) $x \in R(u, v), y \in R(x, z) \Rightarrow x \in R(v, y)$ or $x \in R(z, u)$, for all u, v, x, y, z in V.

It turns out that this axiom is quite strong: Sholander proved that axioms $(t3)$ and (C) imply axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and the following axiom.

(Υ) $y \in R(u, x), x \in R(y, v), y \neq x \Rightarrow x \in R(u, v)$, for all u, v, x, y in V.

This axiom was called axiom $(J0)$ in [4]. In Section 4 we will see that our four betweenness axioms do not imply axiom (Υ). So our concept of betweenness from [14] is also weaker than a Sholander betweenness. Another Sholander axiom has become known as *modularity*, see also [16,1,28].

(Mod) $R(u, v) \cap R(v, w) \cap R(w, u) \neq \emptyset$, for all u, v, w in V.

Sholander gave a characterization of a "tree betweenness". Phrased in our terminology it reads as follows: *a function $R : V \times V \to 2^V$ is the interval function of a tree if and only if it satisfies $(t3)$, (C) and (Mod).*

In [2] Burigana presented a slightly different axiom set that characterizes a tree betweenness in the Sholander tradition. His proof comprised more than four pages. Chvátal et al. [10] presented a one-page proof, and also deleted one axiom that followed from the remaining axioms. We rephrase these results in our functional terminology. The Burigana/Chvátal et al. approach is different than ours. They consider the function R^* that can be obtained from our function R by setting $R^*(u, v) = R(u, v) - \{u, v\}$. The underlying graph of R^* is then defined by: uv is an edge if $u \neq v$ and $R^*(u, v) = \emptyset$. Note that at first sight this looks like a converse of our axiom $(t1)$. But the effect is that, if we translate everything into our terminology, then $(t1)$ and $(t3)$ are presumed throughout. Also note that $x \in R^*(u, v)$ implies that u, x, v are all distinct. Taking into account this different approach the axiom set of Burigana consists of analogues of the axioms $(t1)$, $(t2)$, $(t3)$, $(b1)$, $(c5)$, (Υ) and (*Mod*). The analogues of $(t2)$, $(b1)$, $(c5)$ and (Υ) can be obtained in a simple way: just presume that all variables involved in the axiom are distinct. Axiom $(b1)$ is superfluous and does not occur in the result of Chvátal et al. [10]. A small step in the proof of Chvátal et al. is that axioms (Υ), $(c5)$ and $(t2)$ imply $(b2)$.

In Section 4 we present a characterization of the interval function of a tree where the axiom set involves weaker axioms than the Burigana/Chvátal et al. characterization: we replace axiom $(c5)$ by the two axioms $(c4)$ and $(b2)$ and we replace the modularity axiom by a much weaker axiom. Moreover $(t3)$ is now superfluous.

3. The interval function of a block graph

First we recall some definitions. A graph is separable if it contains a cut vertex, that is, a vertex, the removal of which increases the number of components. A *block* in a connected graph is a maximal non-separable subgraph. Hence a block is either a K_2 or a maximal 2-connected subgraph. A connected graph G is a *block graph* if every block in G is a complete graph. Loosely speaking it is a tree-like structure of cliques. Trivially complete graphs and trees are block graphs. In this section we characterize the interval function of a block graph.

In [7] the following lemma is proved. Unfortunately, the use of some of the axioms was not made explicit. Hence, and also for the sake of completeness, we give a full proof of the lemma here. Note that in [7] it was used to study the question for which graphs the induced path function J is a betweenness. So this lemma applies to more functions than just the interval function I.

Lemma 5. *Let R be a betweenness on V. Then the underlying graph G_R of R is connected.*

Proof. Let u and v be any two distinct vertices in G_R. We prove the existence of a u, v-path in G_R by induction on $|R(u, v)|$. Note that by $(t1)$ and $(t2)$ we have $u, v \in R(u, v)$, so $|R(u, v)| \geq 2$. If $|R(u, v)| = 2$, then $R(u, v) = \{u, v\}$. So by the definition of G_R there is an edge between u and v, which constitutes a u, v-path.

Assume that $|R(u, v)| > 2$. Then there is a vertex x in $R(u, v)$ distinct from u and v. By $(b1)$ we have $v \notin R(u, x)$. By $(b2)$ we have $R(u, x) \subseteq R(u, v)$. So $|R(u, x)| < |R(u, v)|$. By induction, there is a u, x-path. Similarly, by $(t2)$, $(b1)$ and $(b2)$, we have $|R(x, v)| < |R(u, v)|$. Hence by induction there is also an x, v-path. These two paths together form a u, v-walk, which contains a u, v-path, and we are done. □

Note that in the proof of Lemma 5 we need both betweenness axioms $(b1)$ and $(b2)$ to make the induction work. To characterize the interval function of a block graph we introduce the following axiom, which is weaker than Sholander's axiom (U).

(U^*) $R(u, x) \cap R(x, v) = \{x\} \Rightarrow R(u, v) \subseteq R(u, x) \cup R(x, v)$, for all u, v, x in V.

The 3-fan, also known as diamond or kite, is usually denoted by $K_4 - e$, since it can be obtained from K_4 by deleting one edge. For any path P, the vertex set of P is denoted by $V(P)$. Now we are ready to prove our main result.

Theorem 6. *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and (U^*) if and only if G_R is a block graph and $R = I_{G_R}$.*

Proof. First let R be the interval function of a block graph G. Clearly we have $G_R = G$. Moreover, R being an interval function, R satisfies axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$. Since G is a block graph, $R(u, v) = V(P)$, where P is the unique shortest u, v-path. Assume that $R(u, x) \cap R(x, v) = \{x\}$. Then there are two possibilities. First x is on P. In this case $R(u, x) \cup R(x, v) = R(u, v)$. Second x is adjacent to two consecutive vertices y and z on P. In this case $R(u, v) = V(P) = [R(u, x) \cup R(x, v)] - \{x\}$. So axiom (U^*) is satisfied.

Conversely assume that $R : V \times V \to 2^V$ is a betweenness satisfying axiom (U^*). Note that G_R is connected by Lemma 5. So, if d is the distance function of G_R, then $d(u, v)$ is finite for any two vertices u and v in V. By axioms $(t1)$ and $(t2)$ we have $R(u, v) = R(v, u)$ and $u, v \in R(u, v)$. Moreover a betweenness satisfies $(t3)$. We use these facts in the sequel without mention. We split the proof in a number of claims.

Claim 1. *If P is an induced u, v-path in G_R, then $R(u, v) \subseteq V(P)$.*

We use induction on the length $\ell(P)$ of P. If $\ell(P) = 0$, then $u = v$, and $R(u, u) = \{u\} = V(P)$. If $\ell(P) = 1$, then u and v are adjacent. So by definition $R(u, v) = \{u, v\} = V(P)$. Now assume that $\ell(P) \geq 2$. Let x be the neighbor of v on P, and let P' be the subpath of P between u and x. By induction, we have $R(u, x) \subseteq V(P')$. Hence v is not in $R(u, x)$. So $R(u, x) \cap R(x, v) = R(u, x) \cap \{x, v\} = \{x\}$. By axiom (U^*), we have $R(u, v) \subseteq R(u, x) \cup R(x, v) \subseteq V(P') \cup \{x, v\} = V(P)$.

Claim 2. *G_R does not contain an induced cycle of length at least* 4.

Assume the contrary, and let C be an induced cycle of length at least 4. Take two non-adjacent vertices u and v on C. Then we have two internally disjoint induced paths P and Q in C between u and v. By Claim 1 we have $R(u, v) \subseteq V(P)$ as well as $R(u, v) \subseteq V(Q)$. This implies that $R(u, v)$ cannot contain any internal vertex of P and also not any internal vertex of Q. So we have $R(u, v) = \{u, v\}$. But this is impossible, since u and v are not adjacent. This settles Claim 2.

Claim 3. *G_R does not contain an induced $K_4 - e$.*

Assume the contrary. Let u and v be the two non-adjacent vertices, and let x and y be the other two vertices. By Claim 1 we have $R(u, v) \subseteq \{u, x, v\}$ and $R(u, v) \subseteq \{u, y, v\}$. Since u and v are not adjacent, we have a contradiction.

Claim 4. *G_R is a block graph.*

By Claim 2 and 3 every block in G_R is a complete graph. Hence, G_R being connected, it is a block graph.

Claim 5. *$R = I_{G_R}$.*

Write $I = I_{G_R}$. Since G_R is a block graph, there is a unique shortest path between any two vertices in G_R. So by Claim 1 we have $R(u, v) \subseteq I(u, v)$. We prove that $R(u, v) = I(u, v)$ by induction on $d(u, v)$. First we have $R(u, u) = \{u\} = I(u, u)$. If $d(u, v) = 1$, then by definition we have $R(u, v) = \{u, v\} = I(u, v)$. If $d(u, v) = 2$ with x the common neighbor of u and v, then by Claim 1 we have $R(u, v) \subseteq \{u, x, v\}$. But we also have $R(u, v) \neq \{u, v\}$. So $R(u, v) = \{u, x, v\} = I(u, v)$. Now let $d(u, v) \geq 3$, and let P be the shortest u, v-path. Hence we have $R(u, v) \subseteq I(u, v) = V(P)$. Since u and v are not adjacent, there must be a vertex z on P distinct from u and v that is in $R(u, v)$. Assume that $R(u, v) \neq I(u, v) = V(P)$. Then there must be a vertex y on P that is not in $R(u, v)$. We may choose z and y to be adjacent on P. Without loss of generality y is between u and z on P. By axiom $(b2)$, we have $R(u, z) \subseteq R(u, v)$. So y does not belong to $R(u, z)$. Now, z being an internal vertex of the shortest u, v-path P, we have $d(u, z) < d(u, v)$. So, by induction, $R(u, z) = I(u, z)$. But, y being on the shortest path between u and z, we have that y is in $I(u, z)$. This yields a contradiction and settles Claim 5, by which the proof is complete. □

4. The interval function of a tree

In this section we present three new characterizations of the interval function of a tree. The first two are corollaries of Theorem 6. As an intermediate result, we characterize the interval function of a graph that is a tree or a complete graph. We consider the following new axiom. It is just in between our axiom (U^*) and Sholander's axiom (U).

(U') $R(u, x) \cap R(x, v) = \{x\}, R(u, v) \neq \{u, v\} \Rightarrow R(u, x) \cup R(x, v) = R(u, v)$, for all u, v, x in V.

It is straightforward to check that, if R is the interval function of a triangle K_3, then it satisfies (U'). So we can expect a broader class than just the trees. The graph consisting of a triangle and an extra vertex adjacent to exactly one vertex of the triangle is called a *paw*.

Theorem 7. *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and (U') if and only if G_R is a tree or a complete graph and $R = I_{G_R}$.*

Proof. First let R be the interval function of a graph G that is a tree or a complete graph. Clearly we have $G_R = G$. Moreover, R being an interval function, R satisfies axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$. If G is a tree, then $R(u, v) = V(P)$, where P is the unique u, v-path. So $R(u, x) \cap R(x, v) = \{x\}$ holds if and only if x is on P. Hence $R(u, x) \cup R(x, v) = R(u, v)$. Now let G be a complete graph. Then $R(u, v) = \{u, v\}$, for any two distinct vertices u and v. So axiom (U') is trivially satisfied.

Conversely assume that $R : V \times V \to 2^V$ is a betweenness satisfying axiom (U'). By Theorem 6 the underlying graph G_R is a block graph, and R is the interval function of G_R. Assume that G_R contains an induced paw S. Let u be the vertex of degree 1 in S, let w be the vertex of degree 3 in S, and let x and v be the vertices of degree 2 in S. Then we have $R(u, x) = \{u, w, x\}$ and $R(x, v) = \{x, v\}$ and $R(u, v) = \{u, w, v\}$. Clearly, the vertices u, x, v violate axiom (U'). So G_R does not contain an induced paw. This implies that G_R is either a tree or a complete graph with at least three vertices. □

Now we present two new characterizations of the interval function of a tree. Both involve axiom (U), and some of the five elementary classical axioms.

Theorem 8. *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$ and (U) if and only if G_R is a tree and $R = I_{G_R}$.*

Proof. If G_R is a tree and R is the interval function of G_R, then it is straightforward to check that R is a betweenness satisfying axiom (U).

For the converse note that axiom (U') is weaker than axiom (U). Hence by Theorem 7 the underlying graph G_R of R is either a tree or a complete graph and $R = I_{G_R}$. But (U) clearly forbids a triangle in G_R. So G_R is a tree. □

Note that our Example 3 shows that axioms $(t1)$, $(t2)$, $(b1)$ and $(b2)$ are weaker than axioms (S) and (T). So Theorem 8 is actually a new characterization of the interval function of a tree. For our second characterization we need another lemma. It turns out that we can replace the two betweenness axioms $(b1)$ and $(b2)$ by the single classical axiom $(c4)$.

Lemma 9. *Axioms $(c4)$ and (U) imply axiom $(b2)$.*

Proof. Let $R : V \times V \to 2^V$ be a function on V satisfying axioms $(c4)$ and (U). Take x in $R(u, v)$. By $(c4)$, we have $R(u, x) \cap R(x, v) = \{x\}$. Hence by (U) we have $R(u, x) \cup R(x, v) = R(u, v)$. Therefore $R(u, x) \subseteq R(u, v)$. □

Using Proposition 1 and Lemma 9 the following Theorem is an immediate corollary of Theorem 8.

Theorem 10. *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies axioms $(t1)$, $(t2)$, $(c4)$ and (U) if and only if G_R is a tree and $I_{G_R} = R$.*

From what we have so far we deduce another characterization of the interval function of a tree that involves some of the axioms and a condition on the underlying graph G_R. So it is not a fully axiomatic characterization.

Proposition 11. *Let $R : V \times V \to 2^V$ be a function satisfying the three axioms $(t1)$, $(t2)$ and (U). Then each component of G_R is a tree, and $R = I_H$ on each component H of G_R.*

Proof. We only give a sketch of the proof, because many of the arguments are similar to those in the proof of Theorem 6. The first step is to prove that, for any induced u, v-path P, we have $R(u, v) = V(P)$. This can be done by induction on the length of P similar as in Claim 1. As second step we prove that G_R does not contain an induced cycle of length at least 4, using the same arguments as in Claim 2. By (U) it is trivial that G_R does not contain a triangle. Hence each component is a tree. By the first step we have that R is the interval function on each component. □

Note that in the last step of this proof we did not need axiom $(b2)$. That R is the interval function on each component just follows from Step 1 and the fact that the component is a tree. In the proof of Claim 5 in Theorem 6 we really needed axiom $(b2)$. With this Proposition in hand, we can replace axioms $(b1)$ and $(b2)$ in Theorem 8 and axiom $(c4)$ in Theorem 10 by the condition that G_R is connected.

Theorem 12. *Let $R : V \times V \to 2^V$ be a function on V. Then G_R is connected and R satisfies axioms $(t1)$, $(t2)$ and (U) if and only if G_R is a tree and $I_{G_R} = R$.*

We have now presented three axiomatic characterizations that use axiom (U): Sholander's from 1952 with a full proof in [10] and our two above. Axiom (U) is rather strong, because in itself it almost forces that there be a unique path between any two vertices. To explore the reach of the axiom we consider the following example.

Example 13. Let $C = u_1 \to u_2 \to \cdots \to u_k \to u_1$ be a directed cycle on k vertices with $k \geq 3$. Write $V = \{u_1, u_2, \ldots, u_k\}$. We define the function $R : V \times V \to 2^V$ as follows. For vertices u and v, the set $R(u, v)$ is the set of vertices on the directed path from u to v in C. Then R satisfies axioms $(t1)$, $(t3)$, $(b1)$, $(b2)$, $(c4)$, $(c5)$ and (U) but not $(t2)$. It also satisfies the axiom $(t1')$, viz. v lies in $R(u, v)$. In the directed cycle there is a unique directed path between any two vertices. But clearly C is not a tree. □

An interesting point arises here: by replacing axiom $(t2)$ by $(t1')$ as in Example 13, we could develop results on such functions with a directed graph as underlying graph. What can be done in this case? We will not pursue this question here.

Next we consider the Burigana/Chvátal et al. characterization of the interval function of a tree: *a function* $R : V \times V \to 2^V$ *on* V *satisfies axioms* $(t1)$, $(t2)$, $(t3)$, $(c5)$, (Υ) *and* (Mod) *if and only if* G_R *is a tree and* $I_{G_R} = R$. A close inspection of the proof of Chvátal et al. shows that the modularity axiom (Mod) is used in just one small step of the proof. One might wonder "how necessary" this axiom is. It turns out that we can replace modularity by a much weaker axiom, see below. Moreover we can replace $(c5)$ by the weaker axioms $(c4)$ and $(b2)$. Note that we now get axiom $(t3)$ for free. Thus we get another new characterization of the interval function of a tree, which in this case is a stronger result than the Burigana/Chvátal et al. result.

(tf) $R(u, x) = \{u, x\}, R(x, v) = \{x, v\} \Rightarrow x \in R(u, v)$, for u, v, x in V.

The effect of this axiom is that triangles are forbidden: if ux and xv are edges and $u \neq v$, then $|(R(u, v))| \geq 3$, so u and v are not adjacent.

Theorem 14. *Let* $R : V \times V \to 2^V$ *be a function on* V*. Then* R *satisfies axioms* $(t1)$, $(t2)$, $(b2)$, $(c4)$, (tf) *and* (Υ) *if and only if* G_R *is a tree and* $R = I_{G_R}$.

Proof. First let R be the interval function of a tree. Then it is straightforward to check that R satisfies the six mentioned axioms.

Conversely assume that $R : V \times V \to 2^V$ is a function satisfying the six axioms. By Proposition 1 we also have that R satisfies $(b1)$. Hence by Lemma 5 the underlying graph G_R of R is connected. Again we prove a number of claims.

Claim 1. *If* P *is an induced* u, v*-path, then* $V(P) \subseteq R(u, v)$.

The proof is by induction on the length $\ell(P)$ of P. For $\ell(P) \leq 1$ the claim is obvious. So let $\ell(P) = 2$ and let x be the common neighbor of u and v on P. Axiom (tf) implies that x lies in $R(u, v)$.

Now let $\ell(P) \geq 3$. Let x be a vertex on P distinct from u and v that is not a neighbor of u and let Q be the subpath of P between u and x. Then $\ell(Q) < \ell(P)$. So by induction we have $V(Q) \subseteq R(u, x)$. Take any vertex y on Q distinct from u and x, and let Q' be the subpath between y and v. We have $\ell(Q') < \ell(P)$, so by induction we have $V(Q') \subseteq R(y, v)$. Then y is in $R(u, x)$ and x is in $R(y, v)$. So by axiom (Υ) we have x in $R(u, v)$. Thus we have shown for every vertex on P that it is in $R(u, v)$, except for the neighbor z of u on P. To prove that z also belongs to $R(u, v)$, we just reverse the roles of u and v in the previous argument and rename z as x.

Claim 2. *G_R does not contain an induced cycle.*

As observed above, axiom (tf) implies that G_R is triangle-free. Assume that G_R contains an induced cycle C of length at least 4. Let u be a vertex of C, and let x and w be the neighbors of u on C. Finally let v be the neighbor of x on C distinct from u. Then these four vertices are distinct, and v is on an induced path on C between x and w. So by Claim 1 we have x in $R(u, v)$ and v in $R(x, w)$. Hence by axiom (Υ) we have v in $R(u, w)$. But this is impossible, uw being an edge.

Thus we have proved that G_R is a connected cycle-free graph, hence a tree. What is left is to prove that $I_{G_R} = R$. Write $I = I_{G_R}$. Take two vertices u and v and let P be the path in G_R between u and v. By Claim 1 we have $I(u, v) = V(P) \subseteq R(u, v)$. Take any vertex z not on P. First assume that v is on the u, z-path Q. By Claim 1 we have $v \in V(Q) \subseteq R(u, z)$. So by $(b1)$ it follows that z is not in $R(u, v)$. A similar argument holds when u is on the v, z-path in G_R. Second let z be such that there exists an internal vertex x on P that is also on the u, z-path as well as the z, v-path in G_R. Now we have x in $R(u, z)$ as well as in $R(z, v)$. So if z were in $R(u, v)$, then we would get a conflict with $(c4)$. Hence in all cases it turns out that z is not in $R(u, v)$. So indeed $I = R$. $\square$

Note that the function R in Example 13 also trivially satisfies axioms (Υ) and (tf).

We make one observation here. Loosely speaking axioms (U) and (Υ) on their own almost force the underlying graph to be cycle-free. In a manner of speaking they are heavy duty axioms. So we pose the question: can we replace axiom (U), respectively (Υ), in the above theorems by a set of weaker axioms, each one of which does not yet 'force' the underlying graph to be cycle-free?

5. Independence of axioms

In Section 2 we observed some implications among the axioms. For instance axioms $(t1)$ and $(b1)$ imply $(t3)$, and also axioms $(t1)$ and $(c4)$ imply $(t3)$. In Proposition 1 and Lemma 9 we deduced two other implications. In this section we try to establish independence of the axioms in our results.

Our first example shows trivially that axiom $(t1)$ is independent.

Example 15 (*R Does Not Satisfy* $(t1)$)**.**

Let V be a set with $|V| \geq 2$, and let z be a fixed vertex in V. We define the function R by $R(u, v) = \{z\}$, for all u, v in V. Clearly R does not satisfy $(t1)$. But R satisfies axioms $(t2)$, $(b1)$, $(b2)$, (U), $(c4)$, (Υ) and (tf) trivially. $\square$

Example 13 shows that axiom $(t2)$ is independent of the other axioms in Theorems 8 and 14. Hence it is also independent of the other axioms in Theorems 6 and 7.

Take any connected graph G that is not a block graph. Its interval function I is a betweenness, and also satisfies axiom $(c4)$. Hence by Theorems 6–8 respectively, axioms (U^*), (U') and (U) are trivially independent of the other axioms in our theorems. If we take G to be triangle-free, then I also satisfies axiom (tf). But G not being a block graph, hence not a tree, its interval function I cannot satisfy (Υ). So axiom (Υ) is independent of the other axioms in Theorem 14.

The following example shows that axiom $(b1)$ is independent of the other axioms in Theorems 6–8. It also shows that $(c4)$ is independent of the other axioms in Theorem 14.

Example 16 *(R Does Not Satisfy $(b1)$ or $(c4)$)*. Let V be a set with $|V| \geq 3$. Define the function $R : V \times V \to 2^V$ by $R(u, v) = V$, for all distinct $u, v \in V(G)$, and $R(u, u) = \{u\}$, for all u in V. Then R trivially satisfies $(t1)$, $(t2)$, $(b2)$, (Υ) and (tf). Now, for any x distinct from u and v, we have $R(u, x) \cap R(x, v) = V \neq \{x\}$. So in this case (U) is trivially satisfied. If $x = u$, then $R(u, u) \cup R(u, v) = \{u\} \cup R(u, v) = R(u, v)$. So again (U) is satisfied. Similarly, if $x = v$, axiom (U) is satisfied. Now take distinct u, v. Then $R(u, v) = V$. So it contains a vertex x distinct from u and v. Hence $R(u, x) = V = R(u, v)$, so that axiom $(b1)$ is not satisfied. Trivially also $(c4)$ is not satisfied. □

Next we show the last independence of the axiom set in our main Theorem 6 on block graphs.

Example 17 *(R Does Not Satisfy $(b2)$)*.
Let G be the graph consisting of the path $P = u_0u_1 \ldots u_{2k}$ and an isolated vertex z, with $k \geq 1$. So G is not connected, and P is a path of even length at least 2. We define the transit function R on G as follows. It has G as underlying graph, and on P it is just the interval function I_P of P. So far only the intervals between z and any vertex on P are yet undetermined. We define $R(u_i, z) = R(z, u_i) = \{u_i, u_{i+1}, \ldots, u_{i+k}, z\}$. Here we assume that the indices are taken modulo $n = 2k + 1$, that is $u_{2k+1} = u_0$, and so forth. Loosely speaking, $R(u_i, z)$ consists of z, u_i and the k vertices of P following u_i (modulo n). We call these vertices the *k followers* of u_i on P.

Clearly R satisfies $(t1)$ and $(t2)$. On P it satisfies $(b1)$, $(b2)$ and (U) as well. Now we check the cases where intervals of the type $R(u, z)$ or $R(z, v)$, with u and v on P, are involved. Take x in $R(u, z)$ distinct from z. Then $R(u, x) = I_P(u, x)$, so that it does not contain z. Take x in $R(z, u)$ distinct from u, say $u = u_i$ and $x = u_{i+\ell}$ with $\ell \leq k$. Then u_i is not among the k followers of x on P. So u is not in $R(z, x)$. So R satisfies $(b1)$ overall. Clearly, $u_{(i+1)+k} = u_{i+(k+1)}$ lies in $R(u_{i+1}, z)$ but not in $R(u_i, z)$. Hence $R(u_{i+1}, z)$ is not contained in $R(u_i, z)$. Therefore R does not satisfy $(b2)$.

Finally we show that R satisfies (U^*). Consider $R(u, x) \cap R(x, v)$. We only have to check the cases that z is among u, x, v. First suppose that $z = x$. Then $R(u, z)$ and $R(z, v)$ both contain $k + 1$ vertices on P. Hence they have at least one common vertex on P. So $|R(u, z) \cap R(z, v)| \geq 2$, and (U^*) is trivially satisfied. Now suppose that $z = u$, say. Let $x = u_i$ and $v = u_j$. In order that we have $R(z, x) \cap R(x, v) = \{x\}$, we need v to be on the part of P between u_0 and u_i such that v does not belong to the k followers of x on P. But now $R(z, x) \cup R(x, v)$ contains z and the vertices of P between u_j and u_{i+k}. So it contains $R(z, v)$, and again (U^*) is satisfied. □

Note that we have thus established independence of most of the axioms in our theorems. For one important case we do not have an answer yet. Is axiom $(b2)$ independent of axioms $(t1)$, $(t2)$, $(b1)$ and (U)? We do not know whether these four axioms imply $(b2)$ or not. Similarly, what if we replace (U) by (U')? Also we do not know whether $(b2)$ is independent of the other axioms in Theorem 14. This remains an open problem. Here we present some partial answers on this question of independence. Proposition 11 tells us that, if $(t1)$, $(t2)$ and (U) are satisfied, then each component of G_R is a tree. What we cannot prove is that G_R is connected. For this we seem to need both $(b1)$ and $(b2)$ or other axioms that would do the trick. So any example that would show that $(b2)$ is independent of the other four axioms in Theorem 8 must have a disconnected underlying graph. We present two examples that show some independencies.

Example 18 *(R Satisfies $(t1)$, $(t2)$ And (U) But Neither $(b1)$ Nor $(b2)$)*. Let C be an odd cycle of length $2k + 1$ with $k \geq 2$, and let V be the vertex set of C. We define $R(u, u) = \{u\}$, for all u in V. For distinct u and v, we define $R(u, v)$ to be the set of all vertices on the longest u, v-path in C. Note that G_R is the edgeless graph. Clearly R satisfies $(t1)$ and $(t2)$. We have $R(u, u) \cup R(u, v) = R(u, v)$. So in this case (U) is trivially satisfied. Now take distinct u, x, v. Let y be the neighbor of x in $R(u, x)$. To avoid that y is also in $R(x, v)$, we must have that u is on the longest x, v-path. Therefore $R(u, x) \cap R(x, v) \neq \{x\}$, so that again (U) is trivially satisfied.

To see that $(b1)$ is not satisfied, take three vertices u, x, v such that xu and uv are edges. Then $R(u, v) = V = R(u, x)$. For $(b2)$ not being satisfied take u and v to be two vertices at distance 2 with y as common neighbor. Let x be the neighbor of u distinct from y. Now $R(u, v) = V - \{y\}$ and x lies in $R(u, v)$, but we have $R(u, x) = V$. □

Example 19 *(R Satisfies $(t1)$, $(t2)$ And $(b1)$ But Neither (U) Nor $(b2)$)*.
Let S be the paw with vertex set $V = \{u, v, w, x\}$ with u the vertex of degree 1 and x the vertex of degree 3. Let R be the transit function with S as underlying graph defined as follows for the two non-adjacent pairs: $R(u, v) = \{u, v, w\}$ and $R(u, w) = \{u, w, x\}$. Now w is in $R(u, v)$, but $R(u, w)$ is not a subset of $R(u, v)$. So $(b2)$ is not satisfied. Moreover $R(u, v) \cap R(v, x) = \{v\}$, but $R(u, v) \cup R(v, x) = V \neq \{u, x\} = R(u, x)$. So (U^*) does not hold, and in particular (U) does not hold. It is easy to verify that $(b1)$ holds. □

6. The interval function of special classes of trees

Two vertices that have maximum distance in a connected graph $G = (V, E)$ are called *diametrical*. Of course diametrical pairs of vertices exist in any connected graph. In some graphs they play a special role. It might be that there is a pair u, v in G

such that $I(u, v) = V$. Such a pair is necessarily diametrical. Graphs having such a pair are abundant. Some special instances are such different graphs as paths, hypercubes and even cycles. The hypercubes and the even cycles even have the property that any vertex is in such a pair. The following axiom catches the existence of such a pair.

(D) There exist p and q in V such that $R(p, q) = V$.

If we combine this axiom with the ones that make G_R a block graph, then obviously we get a path. So we have the following result.

Theorem 20. *Let $R : V \times V \to 2^V$ be a function on V. Then R satisfies axioms $(t1)$, $(t2)$, $(b1)$, $(b2)$, (U^*) and (D) if and only if G_R is a path and $I_{G_R} = R$.*

Here we would like to suggest the following question: is there a characterization of the interval function of a hypercube involving (almost) only elementary axioms amongst which axiom (D)?

There is another subclass of the trees that admits a characterization involving an extra axiom, viz. the stars $K_{1,n}$ with $n > 1$. It turns out that with this extra axiom (St) we can weaken axiom (U). This weaker axiom $(p2)$ is on the other hand stronger than axiom (tf).

(St) $|R(u_1, u_2) \cap R(v_1, v_2)| = 1$, for all distinct u_1, u_2, v_1, v_2 in V.
$(p2)$ $R(u, x) = \{u, x\}, R(x, v) = \{x, v\} \Rightarrow R(u, v) = R(u, x) \cup R(x, v)$, for all u, v, x in V.

Note that axiom $(p2)$ in itself does not guarantee that $R(u, v) = V(P)$, for any shortest u, v-path P in G_R. For instance let G be a graph without induced C_4 or $K_4 - e$. Then the interval function of G satisfies $(p2)$. Trivially $(p2)$ forbids triangles.

Theorem 21. *Let $R : V \times V \to 2^V$ be a function on V. Then R is a betweenness satisfying axioms $(p2)$ and (St) if and only if G_R is a K_1, a K_2 or a star and $R = I_{G_R}$.*

Proof. It is straightforward to check that the interval function of a star is a betweenness satisfying the two extra axioms.

Assume that R is a betweenness satisfying the extra axioms (St) and $(p2)$. If $|V| = 1$ or 2, then $G_R = K_1$ or K_2, respectively, and we are done. So let $|V| \geq 3$. By Lemma 5 we know that G_R is connected. Moreover, G_R is triangle-free due to axiom $(p2)$. Let u and v be non-adjacent vertices having a common neighbor x. Then by $(p2)$ we have $R(u, v) = \{u, x, v\}$. Now it follows that G_R does not contain a C_4 or $K_4 - e$ as an induced subgraph. For, assume the contrary, and let u and v be non-adjacent vertices in this subgraph and x and y be their common neighbors in the subgraph. Then we would have $R(u, v) = \{u, x, v\} = \{u, y, v\}$, a contradiction.

Also G_R does not contain a path on four vertices as induced subgraph. Assume to the contrary that $uxvy$ is such a path. Then we have $R(u, v) = \{u, x, v\}$ and $R(x, y) = \{x, v, y\}$. But this contradicts axiom (St). Hence G_R also does not contain an induced cycle of length at least 5. From all this we deduce that the blocks of G_R are complete graphs, so that it is a block graph. If there were two distinct cut vertices, then we would get an induced path of length at least 3. Since this is impossible, G_R contains at most one cut vertex. Finally, G_R being triangle-free, each block of G_R is a K_2. Hence G_R is a star. □

7. Concluding remarks

We obtained a characterization of the interval function of a block graph. As a consequence, we obtained a characterization of the interval function of a tree that used weaker and more elementary axioms than Sholander's classical result on "tree segments" of 1952. We also presented a characterization of the interval function of stars, in which the heavy duty axiom (U) is replaced by two simpler axioms. Another characterization of the interval function of a tree improved upon the characterizations of Burigana and Chvátal et al. Moreover we presented a number of examples that showed various independencies of axiom sets. But it is still open whether axiom $(b2)$ is independent of the other axioms in Theorem 8, viz. axioms $(t1)$, $(t2)$, $(b1)$ and (U), or whether it is independent form the other axioms in Theorem 14.

Along the way we mentioned a few interesting open problems. For instance, can we avoid the heavy duty axioms (U) and (Υ) in the case of trees? Is there a characterization of the interval function of the hypercube that involves axiom (D)? What can we do when we replace axiom $(t2)$ with the dual $(t1')$ of $(t1)$, with $(t1')$ being the axiom: $v \in R(u, v)$, for u, v in V? As Example 13 shows, we then move into the area of directed graphs. These are just a few of the many open problems that are still abundant in this area.

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