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Abstract

This paper characterizes the aspects of exchangeability in the Shapley value. We show that, in the Shapley value, each player's prospects of joining a t-player game as the last member of the game is a moment sequence of the uniform distribution. We prove that, with finite exchangeability, the Shapley value is the only value in which the probability assignment is a unique mixture of independent and identically distributed probabilities. We also derive the Shapley value using the de Finetti's theorem.

Keywords: Shapley value; Exchangeability; de Finetti's Theorem

1 Introduction

n-person cooperative games with transferable utility (TU games) are characterized by the *characteristic function* $v(S)$. The set of all n-players is denoted as N. $v(S)$ is defined on all subsets of N with $v(\phi) = 0$

$$v(S) : 2^n \rightarrow \mathbb{R} \tag{1}$$

Also

$$v(R \cup S) \geq v(R) + v(S) \text{ if } R \cap S = \phi \tag{2}$$

This means that $v(S)$ is a *superadditive* set function. $v(S)$ measures the worth of a coalition S. The different *solution concepts* of cooperative game theory define the distribution of $v(S)$ among the members of S. One such solution concept for TU games is proposed by Shapley (1953) and is called the *Shapley value* $\Phi(v)$; where $\Phi(v) = (\phi_1(v), ..., \phi_i(v), ..., \phi_n(v))$. It is defined in terms of $v(S)$; and is a *unique* value satisfying the three axioms of *symmetry*, *efficiency*, and *additivity*.

The notion of the Shapley value is based on the evaluation of *prospects* of playing a game. Shapley (1953, p. 307) notes

> At the foundation of the theory of games is the assumption that the players of a game can evaluate, in their utility scales, every "prospect" that might arise as a result of play. In attempting to apply the theory to any field, one would normally expect to be permitted to

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include, in the class of "prospects", the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance.

Based on this principle, the Shapley value $\phi_i(v)$ (Eq. 3) is the *expected average payoff* for player i when the player i has the belief that the coalition he joins is *equally likely* to be any size $t-1$ $(0 \le t-1 \le n-1)$ and the coalitions of size t are *equally likely* (Harsanyi, 1977).

$$\phi_i(v) = \sum_{\substack{i \in T \\ T \subseteq N}} \frac{1}{n} \binom{n-1}{t-1}^{-1} [v(T) - v(T - \{i\})] \tag{3}$$

The probability assignment is analogous to drawing at random all $n-1$ balls without *replacement* from an urn containing $n-t$ balls marked '0' and $t-1$ balls marked '1'. The *composition* of the urn is unknown a priori and there are n such urns (composition of balls marked '1' varying from 0 to $n-1$). In the Shapley value, all these n urns are assigned equal probability $\frac{1}{n}$.

It means, in the Shapley value, each player *subjectively* assigns probabilities to the events which define their position in the game. "In some ways the most important concept of the subjectivistic theory is that of exchangeable events" (Kyburg and Smokler, 1980, p. 15). This paper explores the aspects of *exchangeability* in the Shapley value. In subsection 2.1, we discuss the aspects of exchangeability in the multilinear extensions. It is shown that the diagonal property in the multilinear extension implies exchangeability; and, this permits us to use the weaker condition of exchangeability (compared to the independence in the multilinear extensions) to derive the Shapley value. In subsection 2.2, we explore the aspects of exchangeability in the Shapley's original proof. We show that, in the Shapley value, each player's prospects of joining a t-player game as the last member of the game is a moment sequence of the uniform distribution. In subsection 2.3, we focus on exchangeability aspects in the semivalues. We prove that, with finite exchangeability, the Shapley value is the only value in which the probability assignment is a unique mixture of independent and identically distributed probabilities. With finite exchangeability, a semivalue cannot be a unique mixture of independent and identically distributed probabilities. Appendices, at the end of the paper, provide the theoretical background for the paper.

2 Exchangeability and the Shapley value

In this section, we discuss the aspects of exchangeability in the Shapley value.

2.1 In the Multilinear Extensions

Owen (1972) derives the Shapley value by extending the characteristic function v for a game with carrier N to the unit n-cube $[0,1]^N$ and called it the *multilinear extension* of v. It is denoted as f

with domain $[0,1]^N$.

$$f(x_1, ..., x_n) = \sum_{T \subseteq N} \left[\prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i) \right] v(T) \;\; \forall 0 \leq x_i \leq 1 \tag{4}$$

f coincides with v at the vertices of the unit n-cube. If x_i is the probability of an event E_i that a player i is joining a random coalition X, then the term in the bracket in Eq. 4 gives the probability of formation of coalition of size T (assuming E_i are independent); and f gives the expected value of the coalition X (Owen, 1972). For deriving the Shapley value, Owen (1972) considers the partial derivatives of f and proves that

$$\phi_i(v) = \int_0^1 \frac{\partial f(x, ..., x)}{\partial x_i} dx \tag{5}$$

It means that the Shapley value is obtained by integrating the partial derivatives of f along the main diagonal of the unit n-cube. This condition is known as the *diagonal property*.

Let us reconsider the probabilistic interpretation of f. The integration along the diagonal means that, at any point along the integration path, each player is equally committed to joining the coalition X. In other words, the events E_i are equiprobable. This leads us to a very important concept of *exchangeability*.

Definition 1 (Renyi (1970)) *The events $\{E_n\}$ $(n = 1, 2, ...)$ are called exchangeable if for any $k \geqslant 1$, the probability*

$$P(E_{n_1} \cap E_{n_2} \cap ... \cap E_{n_k}) = p_k$$

depends only on k, but does not depend on the choice of different integers $n_1 < n_2 < ... < n_k$.

Intuitively, it means that we assign equal probabilities to the joint occurrence of any k events chosen arbitrarily from the sequence $\{E_n\}$. From the above definition, if the events $E_1, ..., E_n$ are independent and equiprobable, they are also exchangeable; but, the converse is not true (Renyi, 1970).

Lemma 1 *In the multilinear extension of games, diagonal property implies exchangeability.*

From lemma 1, the Shapley value can be derived using the weaker condition of exchangeability (compared to the independence of events).

If x_i is the probability that a player i is joining a random coalition X; and we assume that player i considers E_j $(j \neq i)$ as exchangeable (instead of independent as in Owen (1972)). With respect to player i, the expected value of coalition X is

$$g(x, ..., x_i, ..., x) = \sum_{\substack{i \in T \\ T \subseteq N}} \left[\int_0^1 x^{t-1}(1-x)^{n-t} dF(x) \right] \cdot x_i \cdot v(T) + \sum_{\substack{i \notin T' \\ T' \subset N}} \left[\int_0^1 x^{t'}(1-x)^{n-t'-1} dF(x) \right] \cdot (1-x_i) \cdot v(T') \quad (6)$$

The term $\int_0^1 x^{t-1}(1-x)^{n-t} dF(x)$ in Eq. 6 is the probability of the simultaneous occurrence of $t-1$ and nonoccurrence of $n-t$ exchangeable events using the infinite form of the de Finetti's theorem (see Appendix B). Differentiating Eq. 6 with respect to x_i, we get

$$\frac{\partial g(x, ..., x_i, ..., x)}{\partial x_i} = \sum_{\substack{i \in T \\ T \subseteq N}} \left[\int_0^1 x^{t-1}(1-x)^{n-t} dF(x) \right] v(T) - \sum_{\substack{i \notin T' \\ T' \subset N}} \left[\int_0^1 x^{t'}(1-x)^{n-t'-1} dF(x) \right] v(T') \quad (7)$$

In Eq. 7, let $T' = T - \{i\}$. This implies $t' = t - 1$. Any pivotal player i can make T a winning coalition and $T - \{i\}$ a losing coalition. This gives

$$\frac{\partial g(x, ..., x_i, ..., x)}{\partial x_i} = \sum_{\substack{i \in T \\ T \subseteq N}} \left[\int_0^1 x^{t-1}(1-x)^{n-t} dF(x) \right] [v(T) - v(T - \{i\})] \quad (8)$$

Under complete ignorance about the probability distribution F of the unknown probability x, player i assumes uniform distribution ($dF(x) = dx$) using the Bayes's postulate (see Appendix B). Hence

$$\frac{\partial g^u(x, ..., x_i, ..., x)}{\partial x_i} = \sum_{\substack{i \in T \\ T \subseteq N}} \left[\int_0^1 x^{t-1}(1-x)^{n-t} dx \right] [v(T) - v(T - \{i\})] \quad (9)$$

From Owen (1972), the right hand side of Eq. 9 is $\int_0^1 \frac{\partial f(x, ..., x)}{\partial x_i} dx$ which is equal to $\phi_i(v)$ (Eq. 5). Hence, we prove the Shapley value using the exchangeability condition.

$$\phi_i(v) = \frac{\partial g^u(x, ..., x_i, ..., x)}{\partial x_i} = \sum_{\substack{i \in T \\ T \subseteq N}} \left[\int_0^1 x^{t-1}(1-x)^{n-t} dx \right] [v(T) - v(T - \{i\})] \quad (10)$$

Dubey et al. (1981) define the right hand side of Eq. 8 as a *semivalue* generated by distribution

F. In the semivalues (except the Shapley value), the efficiency axiom is relaxed. They consider the sampling from an infinite set of players; and prove that the probability assignment in a semivalue is a unique mixture of independent and identically distributed probabilities. However, with a finite set of players, the probabilities in a semivalue (except the Shapley value) as a mixture of independent and identically distributed probabilities can be assigned in uncountably many ways. We discuss these issues in section 2.3.

2.2 In the Shapley's Proof

In previous section, we have derived the Shapley value using the exchangeability condition. In this section, we explore the concept of exchangeability in the Shapley's original proof (see Appendix A). Let us look at the term in the braces in Eq. 11

$$\phi_i(v) = \sum_{\substack{i \in T \\ T \subseteq N}} \left\{ \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s} \right\} [v(T) - v(T - \{i\})] \tag{11}$$

and compare it to the right hand side of the generalization of the Poincaré identity in Eq. 12

$$P(E_{n_1} \cap ... \cap E_{n_k} \cap \bar{E}_{m_1} \cap ... \cap \bar{E}_{m_l}) = \sum_{j=0}^{l} (-1)^j \binom{l}{j} p_{k+j} \tag{12}$$

This shows that $\sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s}$ is actually the probability of exactly t success in n trials with $p_{t+j} = \frac{1}{t+j}$ (here $t + j = s$). Since the probability of t success in n trials depends only on t and is independent of order, these events are *exchangeable*.

Lemma 2 *In the Shapley value, a player assigns equal probability to all the coalitions of size t ($1 \leq t \leq n$). In other words, the probability assignment to a coalition of size t is independent of the identity of the players.*

Any player i, who is already in the coalition of size t, assigns probability one to his presence in the coalition and considers all other $t-1$ places as exchangeable and are independent of player i's presence in the coalition. In other words

$$p_t = p_{t-1} \cdot 1 = \frac{1}{t} \tag{13}$$

This gives $p_0 = 1$. Using the difference operator Δ, the expression in braces in Eq. 11 can be written as

$$\sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s} = (-1)^{n-t} \Delta^{n-t} p_{t+n-t} \tag{14}$$

The expression on left hand side of Eq. 14 indicates probability and is nonnegative, it means $(-1)^{n-t} \Delta^{n-t} p_{t+n-t} \geq 0$. The sequences $\{p_t\}$ following this condition are called absolute monotonic sequences (see Appendix B). Using Eq. 13, we can reindex the sequence as $\{p_{t-1}\}$ with $p_0 = 1$. From the Hausdorff moment property, an arbitrary absolute monotonic sequence $\{p_{t-1}\}$ with $p_0 = 1$ can be represented as a moment sequence of a unique probability distribution. It means

$$p_{t-1} = \int_0^1 x^{t-1} dF(x) \tag{15}$$

With $p_{t-1} = \dfrac{1}{t}$, the unique F can only be *uniformly distributed*.

Theorem 1 *In the Shapley value, each player's "prospects" of joining a t-player game ($1 \leq t \leq n$) as the last member of the game is a moment sequence of the uniform distribution.*

We have used the infinite form of the de Finetti's theorem. This is applicable as the sampling is done from a very large set of players, defined in Shapley (1953), as the universe of players. However, the finite and infinite exchangeability agrees in the case of uniform distribution. In other words, the Bayes-Laplace process (BL_∞) and the Prevost-L' Huilier process (PL_n) are stochastically identical (see Appendix B).

Theorem 2 *The Shapley value is an expected utility representation where the probability assignment is an uniformly distributed mixture of hypergeometric processes. In other words, in the Shapley value, probabilities are assigned using the Prevost–L' Huilier process (PL_n).*

2.3 In the Semivalues

Let $\mathcal{P}_{n-1}$ is a $2^{n-1} - 1$ dimensional simplex representing all probabilities on 2^{n-1} possible sequences which define the position of player i in the game. Any element of $\mathcal{P}_{n-1}$ is represented as $\mathbf{p} = (p_1, p_2, \ldots, p_{2^{n-1}})$. Let $S(t-1, n-1)$ be the set of sequences with exactly $t-1$ successes. The number of elements in $S(t-1, n-1)$ is $\binom{n-1}{t-1}$. Let $\mathcal{E}_{n-1}$ be the subset of exchangeable probabilities in $\mathcal{P}_{n-1}$. $\mathcal{E}_{n-1}$ has n extreme points $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, where $\mathbf{e}_{t-1}$ ($0 \leq t-1 \leq n-1$) is the probability putting mass $\binom{n-1}{t-1}^{-1}$ at each of the coordinates indicating the sequences in $S(t-1, n-1)$ and mass 0 on the other coordinates. Each point in $\mathcal{E}_{n-1}$ has a unique representation as a mixture of the n extreme points. Figure 1 shows $\mathcal{E}_2$ for a 3-player game. This is the geometric representation of the finite form of the de Finetti's theorem. The above discussion on geometric representation is related to the article by Diaconis (1977). The probability assignment in the Shapley value represents the barycenter of $\mathcal{E}_{n-1}$.

The points in the shaded portion indicate the subclass of mixture of independent and identically distributed probabilities $(x^2, x(1-x), (1-x)x, (1-x)^2)$ in $\mathcal{E}_2$. For finite exchangeable sequences, the points in the shaded portion can be represented as a mixture of independent and identically distributed probabilities in uncountably many ways (Diaconis, 1977). The points in the shaded portion are the probability assignments of the semivalues; and hence we can deduce that, with a finite set of players, a semivalue cannot be a unique mixture of independent

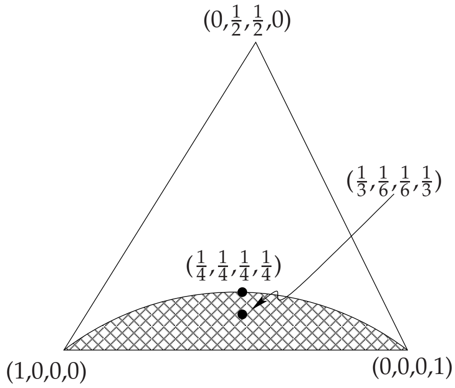

Figure 1: $\mathcal{E}_2$

and identically distributed probabilities. The probability assignment in the Shapley value (represented as point $(\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{3})$ in Figure 1) is a unique mixture of independent and identically distributed probabilities, even in the finite case, is due to uniform distribution. The proof is as follows.

Assume $F(x)$ is the mixing distribution for the point $(\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{3})$; hence, we get

$$\int_0^1 x^2 dF(x) = \frac{1}{3} \qquad \int_0^1 x(1-x)dF(x) = \frac{1}{6}$$
$$\int_0^1 (1-x)xdF(p) = \frac{1}{6} \qquad \int_0^1 (1-x)^2 dF(p) = \frac{1}{3}$$

Solving these four equations, we get

$$\int_0^1 xdF(x) = \frac{1}{2} \qquad \int_0^1 x^2 dF(x) = \frac{1}{3}$$

From these two equations indicating the first and second moments of $F(x)$, it can be observed that one distribution which satisfies these equations is the uniform distribution $(F(x) = x)$. However, for the uniqueness of the probability assignment in the Shapley value, we require that no other frequency distribution for x has the same property. This problem was first observed in a scholium of the Bayes's fundamental paper on posterior probability (Bayes, 1763); and, Murray (1930) in his notes on the Bayes's scholium proves that only uniform distribution $(F(x) = x)$ satisfies the equations $\int_0^1 xdF(x) = \frac{1}{2}$ and $\int_0^1 x^2 dF(x) = \frac{1}{3}$. This can be extended to any n-player game.

Theorem 3 *With a finite set of players, the Shapley value is the only value which can be expressed as a unique mixture of independent and identically distributed probabilities. The unique mixing distribution is the uniform distribution.*

Also, the Shapley value is the only semivalue satisfying the efficiency axiom. In Figure 1, point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is the probability assignment of a semivalue known as the *Banzhaf value* (Banzhaf, 1965; Coleman, 1971). In the Banzhaf value, all the possible 2^{n-1} sequences are equally likely.

3 Remarks

The paper shows that the uniqueness of probability assignment in the Shapley value is due to exchangeability and the uniform distribution. In other words, exchangeability in the Shapley value leads to an absolute monotone sequence which can be represented as a moment sequence of a uniquely determined uniform distribution. In both finite and infinite exchangeable cases, the uniform distribution assigns unique probabilities to the coalitions in the Shapley value.

Shapley (1953) considers a universe of players U and the carrier of a game N is a subset of U. Shapley's assumption on the universe of players is necessary for proving the uniqueness of the probability assignment in the semivalues as a mixture of independent and identically distributed probabilities. However, from the exchangeability viewpoint, the assumption is redundant for the Shapley value due to the uniform distribution.

Appendix A Proof of the Shapley value

Shapley (1953) considers a very large set of players U called the *universe* of players. A *carrier* of a game v is any set $N \subseteq U$ with

$$v(S) = v(S \cap N) \quad \forall S \subseteq U \tag{16}$$

The players outside any carrier are *dummy* players. Any set containing the carrier is itself a carrier of a game. Shapley (1953) restricts the attention to games with *finite* carriers. Shapley (1953) considers three axioms:

1. Symmetry: $\phi_i(v) = \cdots \phi_n(v)$ in a symmetric game[1]
2. Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$
3. Additivity: $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$

and proves that $\phi_i(v)$ in Eq. 3 is the *unique* value satisfying axioms 1 to 3. The proof of the result is based on the fact that any n-person TU game v can be obtained as a *sum game* of a finite number of elementary games[2] v^S—one elementary game v^S for every nonempty coalition S. This means

[1]In a symmetric game, the characteristic function remains unchanged when we interchange any pair of players i and j (Harsanyi, 1977).

[2]An elementary game is one in which every coalition either wins or loses (Owen, 1995). An elementary game is always symmetric (Harsanyi, 1977).

$2^n - 1$ elementary games for v. Shapley (1953) assumes

$$v^S(R) = \begin{cases} \sum_{T\subseteq S}(-1)^{s-t}v(T) & \text{if } S \subseteq R \\ 0 & \text{if } S \not\subseteq R \end{cases} \tag{17}$$

Since elementary games are symmetric, it means each player i will obtain $\frac{v^S}{s}$ in each elementary game v^S. Using the additivity axiom

$$\phi_i(v) = \sum_{S\subseteq N} \frac{v^S}{s} = \sum_{\substack{i\in S\\ S\subseteq N}} \frac{v^S}{s} \tag{18}$$

Using Eq. 17,

$$\phi_i(v) = \sum_{\substack{i\in S\\ S\subseteq N}} \frac{1}{s} \sum_{T\subseteq S} (-1)^{s-t} v(T) \tag{19}$$

On simplifying Eq. 19

$$\phi_i(v) = \sum_{\substack{i\in T\\ T\subseteq N}} \left\{ \sum_{s=t}^{n} (-1)^{s-t} \binom{n-t}{s-t} \frac{1}{s} \right\} [v(T) - v(T - \{i\})] \tag{20}$$

Further simplification leads to the expression in Eq. 3.

Appendix B Exchangeability

B.1 Infinite Exchangeability

The following discussion on infinite exchangeability is based on Renyi (1970). A sequence $\{E_n\}$ ($n = 1, 2, \ldots$) is said to be exchangeable if the probability of the joint occurrence of any k distinct events, chosen arbitrarily from $\{E_n\}$, is same.

If $\{E_n\}$ is an exchangeable sequence of events, then the probability of simultaneous occurrence of k distinct events and nonoccurrence of l distinct events (different from k events) is

$$P(E_{n_1} \cap \ldots \cap E_{n_k} \cap \bar{E}_{m_1} \cap \ldots \cap \bar{E}_{m_l}) = \sum_{j=0}^{l} (-1)^j \binom{l}{j} p_{k+j} \tag{21}$$

This expression is a generalized form of Poincaré's identity. The right hand side of Eq. 21 can be expressed as

$$\sum_{j=0}^{l} (-1)^j \binom{l}{j} p_{k+j} = (-1)^l \Delta^l p_{k+l} \tag{22}$$

where Δ is the difference operator. The expression on left hand side of Eq. 21 is nonnegative, and hence $(-1)^l \Delta^l p_{k+l} \geq 0$. The sequences $\{p_k\}$ following this condition are called *absolute monotonic sequences*. It was proved by F Hausdorff that an arbitrary absolute monotonic sequence $\{p_k\}$ with $p_0 = 1$ can be represented as a moment sequence of a unique probability distribution. It means

$$p_k = \int_0^1 x^k dF(x) \tag{23}$$

Taking successive differences, we get

$$(-1)^l \Delta^l p_{k+l} = \int_0^1 x^k (1-x)^l dF(x) \tag{24}$$

If the exchangeable events can take only values 0 and 1 then using Eqs. 21 and 24, we get

$$P(E_{n_1} \cap ... \cap E_{n_k} \cap \bar{E}_{m_1} \cap ... \cap \bar{E}_{m_l}) = \int_0^1 x^k (1-x)^l dF(x) \tag{25}$$

This is the infinite form of the *de Finetti's theorem* (de Finetti, 1937). With infinite exchangeability, the probability of simultaneous occurrence of k distinct events and nonoccurrence of l distinct events (different from k events) is a unique mixture of a independent and identically distributed probabilities. If α_i is the indicator for event E_i and $S_{k+l} = \alpha_1 + ... + \alpha_{k+l}$, then from Eq. 25

$$P(S_{k+l} = k) = \int_0^1 \binom{k+l}{k} x^k (1-x)^l dF(x) \tag{26}$$

This can be stated as "For an infinite sequence of exchangeable events $\{E_n\}$, the probability of k successes in $k+l$ trials is a *unique* mixture of binomial distributions". This is analogous to probability of drawing with replacement k balls marked '1' in $k+l$ trials from an urn of unknown composition.

In the above discussion, the de Finetti theorem is considered for an infinite exchangeable sequence $\{E_n\}$, and it does not hold exactly if the sequence is finite. However, Diaconis (1977) proves that if a finite sequence of $k+l$ exchangeable events can be extended to an exchangeable sequence of length $n > k+l$, then the probability of k successes in $k+l$ trials can be approximated as the expression in Eq. 26.

B.2 Finite Exchangeability

Diaconis (1977) assertion on de Finetti's theorem is not applicable if the sequence cannot be extended at all. For example, in the case of sampling without replacement from an urn with finite balls. This leads us to *finite exchangeability*. The following discussion is based on Zabell (2005).

Let us consider $E_1, ..., E_n$ as a finite exchangeable sequence of 0, 1 events. If out of these

n events, k are '1's, then the notion of exchangeability assigns equal probability to all $\binom{n}{k}$ sequences of k '1's and $n-k$ '0's. This corresponds to drawing at random all n balls out of an urn containing k '1's and $n-k$ '0's. It means it is a *hypergeometric distribution*. There are $n+1$ such hypergeometric processes for $k=0$ to $k=n$.

"Every exchangeable probability assignment on sequences of length n is a unique mixture of $n+1$ hypergeometric processes". This is the finite form of de Finetti's Theorem.

B.3 Flat Priors

The finite and infinite exchangeability agrees in the case of flat priors (Diaconis, 1977). In a sequence of n trials, in case of complete ignorance, one's probability assignment for k heads should satisfy

$$P(S_n = k) = \frac{1}{n+1} \quad \text{(Bayes's Postulate)} \tag{27}$$

That is the number of heads can take any of the $n+1$ values (0, 1, ..., n), and under complete ignorance, all $n+1$ values are equally likely. Using Eq. 26 for $k=n$, we get

$$\int_0^1 x^n dF(x) = \frac{1}{n+1} \tag{28}$$

Hence, using the infinite form of the de Finetti's theorem, the unique F is the uniform distribution (flat priors over [0,1]).

Zabell (2005) defines the *Prevost–L' Huilier process* PL_n for finite exchangeable sequences in which each hypergeometric process is equiprobable with probability $\frac{1}{n+1}$. Zabell (2005) also considers the Bayes-Laplace process BL_∞ which is generated by picking p uniformly from the unit interval [0,1] and infinitely tossing a p-coin. Using the infinite form of de Finetti's theorem (Eq. 26), the probability of k heads in n trials is

$$P(S_n = k) = \int_0^1 \binom{n}{k} p^k (1-p)^{n-k} dp = \frac{1}{n+1} \tag{29}$$

This is same as for the Prevost–L' Huilier process PL_n. It means both processes are stochastically *identical* (Zabell, 2005).

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