Analysis of Trip and Stop Duration for Shopping Activities

Simultaneous Hazard Duration Model System

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A joint hazard-based model for the analysis of simultaneous (mutually interdependent) duration processes is proposed. The proposed model generalizes independent hazard-based models by accounting for correlations between simultaneous duration processes. Furthermore, the model also permits the use of flexible and variable hazard function parameters to capture realistic features observed empirically in activity duration data (e.g., bimodal peaks). To account for correlated processes (duration processes) that underlie observed stop and trip durations, the proposed model relies on an implicit component of error structure that combines a baseline hazard function (log–logistic distribution) with a mixing (lognormal) distribution. This model is estimated by the simulated maximum-likelihood technique and is used to analyze activity and trip duration for shopping activities. The results highlight the need to account for duration dependence effects in activity–travel durations. Furthermore, hazard-based models that disregard correlation across joint duration processes can provide biased estimates and inaccurate forecasts. Empirical results from San Francisco, California (1996), activity diary data imply that stop and trip durations for shopping activities are positively correlated. The hazard rate profile (shape and intensity) also varies significantly across individuals, suggesting the need for targeted demand management measures. At a substantive level, the results indicate the role of personal, household, and situational attributes on activity and trip duration decisions. These findings and models have important applications in the analysis of activity–travel dimensions of duration and timing and the evaluation of alternate travel demand management measures.

Because of the limitations of conventional planning models, recent attention in travel demand research is shifting toward the development of behavioral, dynamic, and policy-based models of activity and travel patterns (*1, 2*). In particular, investigations on the timing and duration dimensions in activity–travel patterns are being supported by the availability of richer data and more general analysis methodologies (*3–6*). This paper presents a joint hazard-based model of activity (also referred to as "stops") and trip durations to capture the interdependency between the two duration processes (duration processes are correlated processes that underlie observed stop and trip durations).

This investigation is motivated by the following considerations: (*a*) stop duration directly affects the number of cold starts and has important implications for congestion management and air quality programs (*6*); (*b*) trip duration affects several travel dimensions, including destination, mode, and route choices; and (*c*) activity durations also influence the congestion encountered on the network by altering the spatial and temporal distributions of flows on the network (*7*).

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Therefore, several investigators have modeled activity and trip durations of activity episodes using simultaneous regression-based models and hazard duration-based models (*8–10*) (a detailed review is provided below). While simultaneous equations capture correlations across multiple dimensions but not duration dependence effects, hazard duration models account for duration dependence but not correlations.

To address these limitations, this paper has two principal objectives. The first objective aims to specify and test flexible hazard-based duration models to analyze activity episode-level duration decisions. Two tasks are pursued under this objective. The first task proposes a model that jointly analyzes multiple and correlated duration dimensions by using a hazard duration-based framework. The second task aims to investigate the potential to relax the constant hazard profile assumptions by permitting shape (hazard profile) parameters to vary across individuals.

The second objective is to use the flexible hazard duration-based framework to investigate the following substantive issues that could affect the episode-level stop and trip durations of shopping activities. Are there systematic differences in hazard rate parameters (across individuals, e.g., across workers versus nonworkers, males versus females, or age segments)? What are the key covariates that affect activity and trip durations, and are there significant differences in their effects across different sociodemographic segments (e.g., across males versus females or age segments)? What is the effect of activity timing decisions (departure time) on the hazard rates of shopping activities (trip and stop durations)? Insights into these questions have important implications for activity–travel analysis and travel demand forecasts.

The proposed work builds on the large existing literature in activity duration analysis by proposing a more flexible and joint hazard duration model system of interrelated (simultaneous) duration dimensions. The increased flexibility arises from decomposition of the duration distribution implicitly into two components: independent log–logistic hazard distributions for activity and trip durations and correlated errors from a bivariate lognormal distribution (to account for correlation). Note that similar error components schemes have recently been used in discrete choice contexts to overcome restrictive assumptions in conventional models [for example, independence from irrelevant alternatives assumptions in the multinomial logit models (*6, 11*)]. The proposed model also enables the specification of more general hazard duration profiles (observed empirically) by allowing the profile shape and location parameter to vary systematically across individuals, while a few studies account for them by nonparametric methods (*6, 12*).

Empirical activity–trip data from a 1996 San Francisco, California, Bay Area survey are used in this study (*13*). The model formulation and estimation procedures are also discussed. The proposed model is applied empirically to estimate the joint system of hazard-based models of trip and stop durations for shopping activities. The performance of the proposed model is compared with those of simultaneous regression-based models, independent hazard-based models, and conventional proportional hazard assumptions (fixed alpha parameters) through suitable statistical tests. Finally, concluding remarks and opportunities for future work are proposed.

BACKGROUND AND LITERATURE REVIEW

Several investigators have modeled stop and trip durations of activity episodes using two broad approaches (*2, 10, 14*). In the first and more widely used approach, regression-based models or variants (simultaneous or structural equations) are used to analyze the relationship between activity duration and underlying explanatory factors. Recent applications include simultaneous discrete or continuous models of various activity and timing duration data (*14–16*). Bhat found that in-home and out-of-home stay durations for workers are correlated and are influenced by individual and household attributes such as age, gender, and number of children (*9*). Using data from Washington, D.C., Kuppam and Pendyala observed longer durations of in-home and maintenance activities for females than for males (*16*).

Although this regression-based approach enables the joint analysis of activity and trip durations, it does not adequately account for duration dependence observed empirically. Duration dependence refers to the effect by which "the time at which an activity will be completed (terminated) depends on the time already spent in the activity" (*12*). For instance, a person who has already spent 2 h during an activity is perhaps more likely to terminate the current activity than one who has only spent 10 min. Disregard of duration dependence can lead to a poor model fit, inaccurate forecasts, and ineffective demand management strategies. Another shortcoming of the application of regression models to duration data is that they cannot satisfactorily account for censored data. To overcome the latter restriction, Kasturirangan and Pendyala develop a tobit-based model of episode-level duration to account for the fact that activity episodes cannot have a negative duration (*17*).

The second major approach relies on the use of hazard duration models to address some of these shortcomings. A hazard function essentially describes the probability that an event (activity) will terminate between times t and $t + dt$ (where dt is the time differential) given that it has lasted until time t. This framework enables the analysis of duration dependence through the specification of a hazard function. A key advantage of the hazard-based framework is that it can naturally accommodate censored durations. Because of these advantages, hazard-based duration models have been applied in several fields, including statistics, biometrics, and reliability engineering (*18–21*).

The appealing features of hazard models have also been recognized in the analysis of activity and travel duration data. Niemeier and Morita found that shopping duration is reduced as the number of stops increases (*8*), whereas Kim and Mannering noted that commute time adversely affects shopping duration (*22*). Bhat used a multiple-durations model to analyze the durations of recreational and shopping activities during the evening work-to-home commute (*9*). That study found that work duration and departure before the evening peak reduced the durations of shopping stops. Note that the multiple durations in that model refer to several possible exit states, but recreational and shopping trips were modeled as being mutually independent. Mannering et al. used a hazard-based model to analyze the stability and transferability of home-stay durations (*5*).

Although the hazard-based framework is quite general, current applications have imposed restrictive assumptions because of analytical tractability and interpretability considerations. For instance, although trip and activity duration decisions are likely to be correlated, they have typically been modeled by using independent hazard models. Bhat observed that that there are few empirical simultaneous hazard models of duration decisions (*6*). Among the few transportation applications, Yee and Niemeier (*23*) proposed an approximate sandwich estimator [based on the work of Lin and Wei (*24*)] to account for the correlation between repeated duration decisions (of the same dimension) across different panel waves. Bhat has sought to account for the correlation between the duration decision and the discrete activity type choice (*9*).

Many existing duration models are based on the assumption of proportional hazard functions. In this approach, it is assumed that hazard rates vary systematically across individuals through a multiplicative factor (applied to the baseline hazard) that depends on the level of the covariates. Consequently, all individuals have the same hazard profile, although the intensity of hazard function can vary (*25, 26*). Because empirical data provide evidence of varying hazard shapes across individuals, Bhat (*6*) and Kharoufeh and Goulias (*12*) recognized the need for more flexible models of heterogeneity (differences in hazard responses across respondents). While these two groups of researchers have proposed nonparametric models for heterogeneity, this study proposes a flexible parametric model for this purpose.

DATA DESCRIPTION AND PRELIMINARY ANALYSIS

Survey Data Description

Data from the 1996 San Francisco Bay Area Household Activity Survey are used in this study (*13*). These survey data contain records of 203,000 activities and 64,000 trips made by more than 1,200 households. The activity diary collected data on both the activity and travel patterns of all household members, including activity type, start and end times, location of activity, and travel time to activity location. The data were extensively cleaned to select only those records with consistent activity patterns and plausible activity and trip durations. The calibrations sample consisted of 1,234 shopping activity records, whereas the prediction data set consisted of 1,236 records.

The sociodemographic characteristics of the sample respondents are as follows. The sample was nearly evenly split among females (50.4%) and males (49.6%). Nearly 26% of the respondents were younger than 25 years, about 18% were between 25 and 36 years old, 29% were between 36 and 50 years old, 14% were between 50 and 65 years old, and 13% were older than 65 years of age. Nearly 60% of the respondents were employed in full-time or part-time jobs. While 23% of the activity episodes were work or school related, discretionary activities consisted of 10% shopping stops, 6% visiting stops, 8% recreational stops, 10% maintenance activities, and 20% meal trips. The mean (standard deviation) trip and stop durations for shopping trips were 13.75 (11.47) and 43.34 (38) min, respectively.

Empirical and Theoretical Analyses of Interdependency Between Activity and Trip Durations

This section examines whether correlations between trip and activity durations are significant enough to warrant the development of new models. Correlation coefficients between activity and trip durations are estimated for various activity types. The results indicate mild to moderate correlations between activity and trip durations [shop, –0.15; work (for workers), 0.142; visit, 0.139; serve passenger (for workers), –0.33; serve passenger (for nonworkers), –0.16].

Hausman's test was conducted to confirm the presence of endogeneity. The results reject the hypothesis that activity and trip durations for shopping processes are mutually exogenous at a 99% confidence level (the t-statistic is 8.44, which is greater than the t_{crit} value of 2.33, where t_{crit} is the critical t-statistic). These results illustrate the need to account for correlations between stop and trip durations. If the two duration processes are correlated, independent hazard models can lead to biased coefficient estimates and likelihood, as shown below.

Consider a simple hazard model with two correlated duration processes, s and t. Let the hazard functions for the two dimensions (h_s and h_t) be parameterized into a decomposable form, as follows:

$$h_s(u) = h_{s0}(u)e^{z_1} \quad \text{and} \quad h_t(v) = h_{t0}(v)e^{z_2}$$

where

- h_{s0} and h_{t0} = baseline hazard functions, which are assumed to be independent;
- u and v = time variates; and
- z_1 and z_2 = error terms, which are intended to capture the correlation between duration processes s and t.

The error terms (z_1 and z_2) are assumed to be bivariate normally distributed with a mean of 0, a variance of 1, and a correlation of ρ. Then, the joint conditional hazard function can be written as

$$h_{s,t}(u, v|z_1, z_2) = h_{s0}(u)e^{z_1}h_{t0}(v)e^{z_2}$$

The unconditional hazard function can be written as

$$h_{s,t}(u, v) = \int_{z_1}\int_{z_2} h_{s,t}(u, v|z_1, z_2)f(z_1, z_2)dz_1dz_2$$

and the integrated hazard function ($I_{s,t}$) takes the form of

$$I_{s,t}(u, v) = \int_u\int_v\int_{z_1}\int_{z_2} h_{s0}(u)h_{t0}(v)e^{z_1}e^{z_2}f(z_1, z_2)dz_1dz_2$$

where $f(z_1, z_2)$ represents the standard bivariate normal density function, given by

$$f(z_1, z_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\, e^{-\frac{[z_1^2+z_2^2-2\rho z_1 z_2]}{2}}$$

In contrast, if the two decisions are mutually independent, the correlation ρ is equal to 0 and the joint density simplifies as follows:

$$f(z_1, z_2) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{[z_1^2+z_2^2]}{2}}$$

If the two decision processes are positively (negatively) correlated, then the independent joint density underestimates (overestimates) the true density. Consequently, the hazard function, the integrated hazard function, and the likelihood estimates are all biased. The extent of this bias in the unconditional hazard function was ascertained by conducting 750 Monte Carlo draws of correlated normal errors (Figure 1). The results indicate that this bias increases with increasing correlation, further confirming the need to account for correlations between interrelated duration processes.

Heterogeneity in Hazard Duration Profiles

Figures 2*a* and 2*b* plot the duration distribution across different activity types. Both distributions exhibit duration dependence, as the likelihood of event termination tends to increase with increasing duration. Furthermore, the trip duration distribution varies across activity types (Figures 2*a* and 2*b*). For instance, trip durations for work or shopping activities display a monotonically decreasing trend, whereas the density of travel times increases for service activities initially and declines later. Therefore, flexible duration models such as log–logistic models that can capture possible nonmonotonic trends are more suitable than the Weibull distribution (which leads to monotonic hazard profiles). The plot of activity durations, on the other hand, reveals not only nonmonotonic profiles but also the presence of multiple local maxima. These variations in profiles across individuals are partly due to systematic differences in hazard functions and are referred to as "heterogeneity."

JOINT HAZARD-BASED MODEL OF ACTIVITY AND TRIP DURATIONS

Model Formulation

Consider a sequence of two duration episodes: S (stop and activity duration) and T (trip duration).

It is assumed that the baseline hazards of duration processes S and T, given by $h_{s0}(u)$ and $h_{t0}(v)$, respectively, are independently distributed according to log–logistic distributions:

$$h_{s0}(u) = \left[\frac{\alpha_s\lambda_{s0}(\lambda_{s0}u)^{\alpha_s-1}}{1+(\lambda_{s0}u)^{\alpha_s}}\right] \tag{1a}$$

$$h_{t0}(v) = \left[\frac{\alpha_t\lambda_{t0}(\lambda_{t0}v)^{\alpha_t-1}}{1+(\lambda_{t0}v)^{\alpha_t}}\right] \tag{1b}$$

where α is the shape parameter and λ is an intensity parameter.

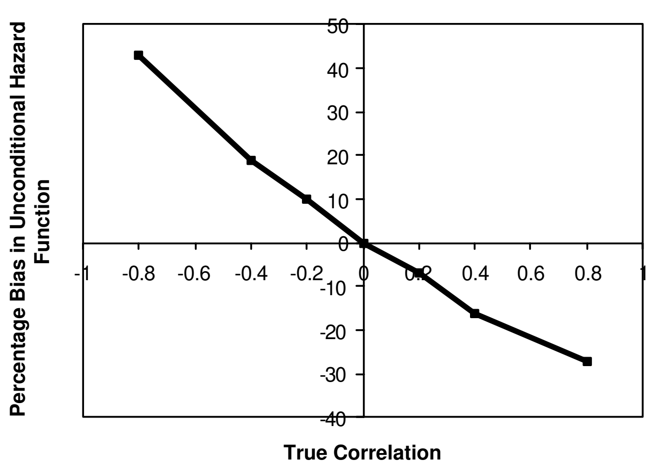

FIGURE 1 Bias in unconditional hazard function under independence assumption.

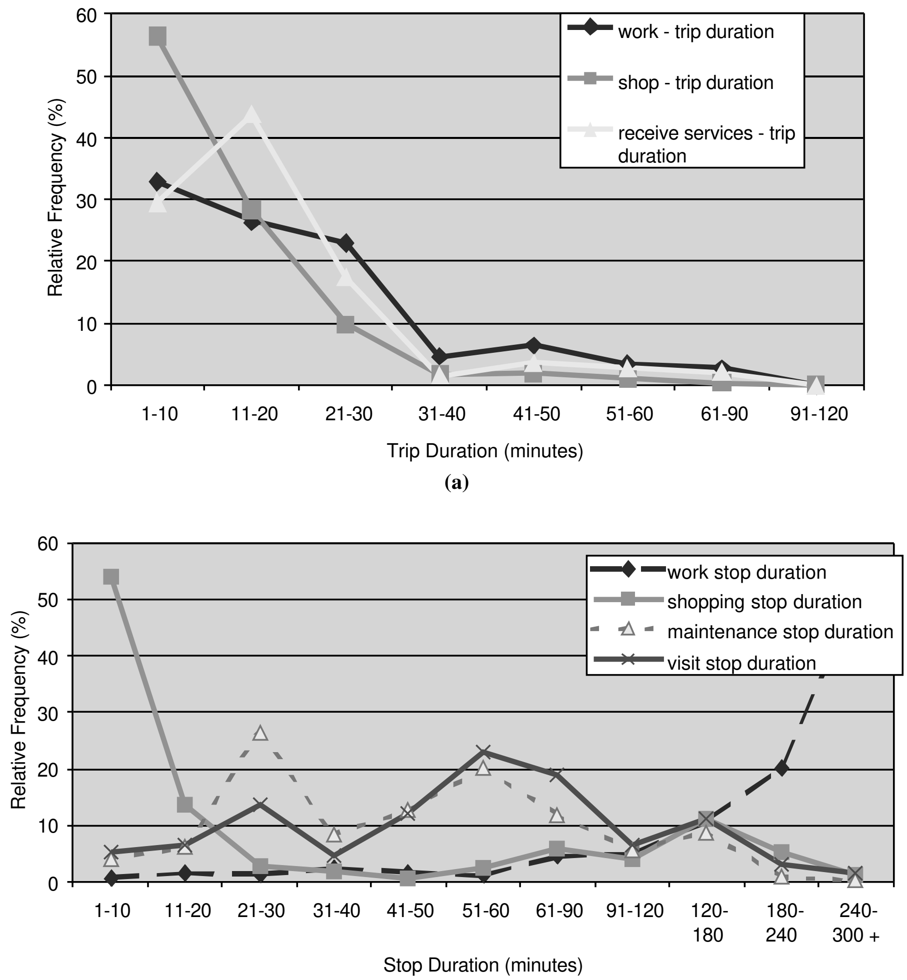

FIGURE 2 Histograms of (*a*) trip duration and (*b*) stop duration by activity type.

Case 1: Proportional Hazard Model

The proportional hazard model is generally used to capture the observed heterogeneity by scaling the baseline hazard up or down on the basis of attributes of individual *i,* as illustrated as follows for duration process S: $h_{si}(u) = h_{s0}(u)\exp(-\beta_s X_{is})$, where β represents the vector of parameters and X is the vector of attributes affecting the hazard function.

Case 2: Correlated Hazard Model System

However, the basic model with observed heterogeneity presented above does not permit a correlation between duration processes S and T or unobserved heterogeneity across individuals. These assumptions can be relaxed by assuming the presence of unobserved random factors (y_1 and y_2) in the two hazard functions. By permitting these unobserved factors to be correlated, the correlation between the processes can be captured. For analytical convenience, the random error terms are assumed to be jointly distributed from a bivariate lognormal distribution. By using a conventional proportional hazard model, the hazard function can now be written in a conditional form as

$$h_{si}(u|y) = h_{s0}(u)\exp(-\beta_s X_{is})y_{1i} = h_{s0}(u)\exp(-\beta_s X_{is} + z_{1i}) \tag{2a}$$

$$h_{ti}(v|y) = h_{t0}(v)\exp(-\beta_t X_{it})y_{2i} = h_{t0}(v)\exp(-\beta_t X_{it} + z_{2i}) \tag{2b}$$

The corresponding conditional integrated hazard functions and conditional survival functions may be written as (*10*)

$$I_{si}(u|y) = \int_{s=0}^{u} h_{s0}(s)\exp(-\beta_s X_{is})y_{1i}\,ds = \int_{s=0}^{u} h_{s0}(s)\exp(-\beta_s X_{is} + z_{1i})\,ds \tag{3a}$$

$$I_{ti}(v|y) = \int_{t=0}^{v} h_{t0}(t)\exp(-\beta_t X_{it})y_{2i}\,dt = \int_{t=0}^{v} h_{t0}(t)\exp(-\beta_t X_{it} + z_{2i})\,dt \tag{3b}$$

$$S_{si}(u|y) = \exp[-I_{si}(u|y)] \tag{4a}$$

$$S_{ti}(v|y) = \exp[-I_{ti}(v|y)] \tag{4b}$$

where I is the conditional hazard function.

Because the conditional hazard functions are mutually independent, the resulting conditional density functions are also mutually independent and log–logistically distributed. After standard derivations, the relationship between the conditional density function and conditional hazard function can be shown as (*6*)

$$f_{si}(u|y) = h_{si}(u|y)S_{si}(u|y) \qquad \text{and}$$
$$f_{ti}(v|y) = h_{ti}(v|y)S_{ti}(v|y) \tag{5}$$

where $S(\cdot|y)$ represents the conditional survival functions. For notational convenience, the individual specific subscript i is dropped hereafter from the unobserved error terms y_{1i} and y_{2i}.

Assuming that the baseline hazards $h_{s0}(u)$ and $h_{t0}(v)$ are from the log–logistic distribution, the following functional form for the conditional hazard functions can be derived:

$$h_{is}(u|X_{is}, \beta_s, y_1, y_2) = \left[\frac{\alpha_s\lambda_{s0}(\lambda_{s0}u)^{\alpha_s-1}}{1+(\lambda_{s0}u)^{\alpha_s}}\right]\exp(-\beta_s X_{is})y_1 \tag{6a}$$

$$h_{it}(v|X_{it}, \beta_t, y_1, y_2) = \left[\frac{\alpha_t\lambda_{t0}(\lambda_{t0}v)^{\alpha_t-1}}{1+(\lambda_{t0}v)^{\alpha_t}}\right]\exp(-\beta_t X_{it})y_2 \tag{6b}$$

The conditional integrated hazard function is given as

$$I_s(u|y_1, y_2) = \ln(1+\lambda_{s0}u)^{\alpha_s}\exp(-\beta_s X_{is})y_1 \tag{7a}$$

$$I_t(v|y_1, y_2) = \ln(1+\lambda_{t0}v)^{\alpha_t}\exp(-\beta_t X_{it})y_2 \tag{7b}$$

The conditional survival functions may be written as

$$\begin{aligned} S_s(u|y_1, y_2) &= \exp[-I_s(u|y_1, y_2)] \\ &= \exp\left[-\ln(1+\lambda_{s0}u)^{\alpha_s}\exp(-\beta_s X_{is})y_1\right] \end{aligned} \tag{8a}$$

$$\begin{aligned} S_t(v|y_1, y_2) &= \exp[-I_t(v|y_1, y_2)] \\ &= \exp\left[-\ln(1+\lambda_{t0}v)^{\alpha_t}\exp(-\beta_t X_{it})y_2\right] \end{aligned} \tag{8b}$$

Because the conditional density is the product of conditional hazard and survival functions and the conditional densities of S and T are independent given the random errors y, the joint conditional density is given as

$$\begin{aligned} f_{s,t}(u, v|y_1, y_2) &= f_s(u|y_1, y_2)f_t(v|y_1, y_2) \\ &= [h_s(u|y_1, y_2)S_s(u|y_1, y_2)]h_t(v|y_1, y_2)S_t(v|y_1, y_2) \end{aligned}$$

where parametric expressions for h and S are given in Equations 7 and 8, respectively.

The unconditional joint density function may be obtained by integrating out the random heterogeneity terms (y_1, y_2) as follows:

$$\begin{aligned} f_{s,t}(u, v) &= \int_{y_1}\int_{y_2} f_{s,t}(u, v|y_1y_2)g(y_1, y_2)dy_1dy_2 \\ &= \int_{y_1}\int_{y_2} f_s(u|y_1, y_2)f_t(v|y_1, y_2)g(y_1, y_2)dy_1dy_2 \end{aligned} \tag{9}$$

where $g(y_1, y_2)$ is the joint probability density function of the random error terms (assumed to be lognormal in this study) and the expressions for conditional density functions are given below:

$$\begin{aligned} f_s(u|y_1, y_2) = {} & \left[\frac{\alpha_s\lambda_{s0}(\lambda_{s0}u)^{\alpha_s-1}}{1+(\lambda_{s0}u)^{\alpha_s}}\right]\exp(-\beta_s X_{is})y_1 \\ & \exp\left[-\ln(1+\lambda_{s0}u)^{\alpha_s}\right]\exp(-\beta_s X_{is})y_1 \end{aligned}$$

$$\begin{aligned} f_t(v|y_1, y_2) = {} & \left[\frac{\alpha_t\lambda_{t0}(\lambda_{t0}v)^{\alpha_t-1}}{1+(\lambda_{t0}v)^{\alpha_s}}\right]\exp(-\beta_t X_{it})y_2 \\ & \exp\left[-\ln(1+\lambda_{t0}v)^{\alpha_t}\right]\exp(-\beta_t X_{it})y_2 \end{aligned}$$

Note that the conditional density functions are also conditioned on the parameters θ equal to {β's, λ's, α's} and systematic attributes X_s and X_t as well, although these are not indicated for notational convenience.

The likelihood $L(u^*, v^*)$ of observing an actual termination time vector equal to (u^*, v^*) for individual i is proportional to the joint density function $f_{s,t}(u_i^*, v_i^*)$

$$L_i\left(u_i^*, v_i^*|X_{is}, X_{it}, \theta\right) = kf_{s,t}\left(u_i^*, v_i^*|X_{is}, X_{it}, \theta\right) \tag{10}$$

where k is a proportionality constant.

If it is assumed that individuals in a sample make event termination decisions independently, the likelihood of a sample can be written as the product of individual likelihoods:

$$L(X, \theta) = \prod_{i=1}^{N_{\text{obs}}} kf_{s,t}\left(u_i^*, v_i^*|X_{is}, X_{it}, \theta\right) \tag{11}$$

where N_{obs} is the number of observations.

Since k is a proportionality constant that is not a function of model parameters, maximizing the likelihood is equivalent to maximizing the product of the joint densities. Therefore, the joint density function over the sample is maximized to obtain model parameters in the procedure discussed below. Lognormal unobserved errors are used to capture correlations for analytical and computational convenience. However, the procedure described above may be extended to other correlated error distributions in a straightforward manner.

Case 3: Correlated Hazard Models with Variable Shape Parameters Across Individuals

Note that in the models presented above, the hazard parameters (shape parameter α and intensity parameter λ) are assumed to be fixed and identical across individuals. To represent more flexible hazard profiles, the hazard shape parameters α (α_s and α_t) are permitted to vary across individuals according to the following linear-in-parameters specification:

$$\alpha = \alpha_0 + \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_n Z_n + v_n \tag{12}$$

where the variables $Z_1, \ldots, Z_n$ are individual attributes affecting the shape parameter and the error ν_n represents unobserved variations across individuals. This generic equation can be separately applied to stop or activity and trip shape parameters by including suitable covariates Z. The hypothesis that shape parameters are identical and fixed can be tested in a straightforward manner by setting the coefficients in Equation 12 (e.g., α_1 and α_0) equal to 0. The procedure to estimate the model given in Equation 11 is described below.

Model Estimation Procedure: Maximum Simulated Likelihood Estimation

The parameters in the density function given by Equation 9 are estimated by using a simulated maximum-likelihood estimation procedure, described below:

1. Draw R bivariate lognormal error vectors $y_r = (y_{sr}, y_{tr})$ (where r is a counter running from 1 to R) to account for the correlation between stop and trip duration.

2. The independent components of the baseline hazard distribution for stop and trip durations (in Equations 2a and 2b) are assumed to be log–logistically distributed with parameters (α_s, λ_{s0}) and (α_t, λ_{t0}), respectively. The corresponding conditional hazard functions are given by

$$h_{is0}(u|X_{is}, \beta_s) = \frac{\lambda_{s0}\alpha_s(\lambda_{s0}u)^{\alpha_s - 1}}{1 + (\lambda_{s0}u)^{\alpha_s}}$$

$$h_{it0}(v|X_{it}, \beta_t) = \frac{\lambda_{t0}\alpha_t(\lambda_{t0}v)^{\alpha_t - 1}}{1 + (\lambda_{t0}v)^{\alpha_t}}$$

In the equations that follow, the functions C_{sr}, C_{tr}, I_{tr}, and I_{sr} are all conditional on the correlated lognormal error terms. For notational simplicity, the conditioning term is dropped from the function subscript and C_{sr} is written to imply $C_{sr|y}$ and so on.

3. For each draw, the covariate effects (C_{sr} and C_{tr}) are then computed:

$$C_{sr} = \exp(-\beta_s Z_s)y_{sr} \text{ and } C_{tr} = \exp(-\beta_t Z_t)y_{tr}$$

4. The conditional integrated hazard functions $I(\cdot)$ can be computed as

$$I_{tr}(v|X_{it}, \beta_t, y_r) = \ln[1 + (\lambda_{t0}v)^{\alpha_t}]C_{tr}$$

$$I_{sr}(u|X_{is}, \beta_s, y_r) = \ln[1 + (\lambda_{s0}u)^{\alpha_s}]C_{sr}$$

5. The conditional joint density function $f_{r|y}(u, v|y)$ is given by Equation 9 as

$$f_{r|y}(u, v|y_r) = [h_{is0}(u)h_{it0}(v)C_{sr}C_{tr}]\exp[-(I_{sr} + I_{tr})]$$

6. The unconditional density for an individual i (joint stop and trip duration) is estimated by averaging the conditional density across draws as follows:

$$f_i(u, v) = \frac{\sum_r f_{r|y}(u, v|y_r)}{R_{\text{draws}}}$$

where R_{draws} is the total number of Monte Carlo draws for each observation. The function f_r is conditional on lognormal errors (y_{sr}, y_{tr}), and both the functions $f_i(\cdot)$ and $f_r(\cdot)$ are conditioned on the hazard parameters [β and data values (X)].

7. The likelihood (L_i) of observing activity and trip durations of u_i, v_i is proportional to the joint density function evaluated at u_i, v_i. This may be expressed as

$$L_i(\theta|u_i, v_i, X_{si}, X_{ti}) = kf_i(u_i, v_i|X_{si}, X_{ti}, \theta)$$

where k is a constant of proportionality.

8. The sample likelihood $L(\theta|u_i, v_i, X_{si}, X_{ti})$ given in Equation 11 is obtained as the product of the estimate in Step 7 under the assumptions that (*a*) the decisions across individuals are independent and (*b*) k is set equal to 1 (without a loss of generality, since the constant does not affect the maximization procedure).

9. This likelihood function $L(\cdot)$ is maximized to determine the vector of parameters θ by standard nonlinear optimization techniques from simulated maximum-likelihood estimation procedures (*27*).

MODELING RESULTS AND DISCUSSION OF RESULTS

To test whether duration dependence effects are significant in activity and trip duration processes, the predictive performance of the "best" independent hazard-based model specification is compared with a corresponding regression-based model (in which the same covariates were used). The calibration data set consisted of 1,234 shopping episode records from the data for the San Francisco Bay Area described earlier. Because of the different parametric assumptions in the two models, a direct comparison of the likelihood can be misleading. Therefore, the observed trip and activity durations were grouped into discrete duration intervals (bins), and the probability of selecting the chosen bin was estimated for the two models. The discrete duration log likelihood for the regression-based model was −6,192, whereas the hazard-based model provided a much better fit of the data, with a log likelihood of −4,196. With equal numbers of variables, the hazard-based model outperformed the simultaneous regression model (Akaike information criterion), and therefore, the hypothesis of the absence of duration dependence must be rejected. Thus, it is essential to explicitly account for duration dependence in stop and trip times, at least in the context of shopping activities.

Next, three alternative hazard-based models of shopping episodes are compared (Table 1) to test for a correlation between duration processes and heterogeneity in shape parameters. The models include (*a*) the independent hazard model (Model 1), (*b*) the correlated hazard model with fixed shape parameters (Model 2), and (*c*) the correlation duration process with heterogeneous and flexible shape parameters (Model 3). The log likelihoods of the three models are −10,099 (37 parameters), −10,066 (40 parameters), and −10,032 (67 parameters), respectively. Because both Models 1 and 2 can be obtained as parametric restrictions of Model 3, the likelihoods (in terms of density) are compared by chi-square (χ^2) tests. χ^2 tests with Models 1 and 2 indicate that activity and trip durations are correlated at the 1% level [$\chi^2_{\text{actual}} = -2(10{,}066 - 10{,}099) = 66 > \chi^2_{\text{critical}}$ (0.01, 3 degrees of freedom) = 11.34]. Furthermore, a comparison of Models 2 and 3 implies that the hypothesis of fixed and identical shape parameters across individuals must be rejected [$\chi^2_{\text{actual}} = -2(10{,}032 - 10{,}066) = 68 > \chi^2_{\text{critical}}$ (0.01, 27 degrees of freedom) = 46.96]. The effect of varying the α parameter is depicted graphically in Figure 3 for shopping activity durations for a single sociodemographic segment (males ages 30 to 40 years who are full-time workers, use the automobile mode, and are from multiperson, multicar households). Figure 3 illustrates that as the shape parameters vary with time of day, the peak of the hazard function and slopes change accordingly. In view of the findings, only the joint model with heterogeneity and variable shape parameters (Model 3) is discussed below (Table 2). The role of covariates is first analyzed, followed by a discussion of the role of the

TABLE 1 Comparison of Simultaneous Regression Model, Independent Hazard Model, and Simultaneous Hazard Model

Model	Simultaneous Regression	Fixed Alpha Independent Hazard Model	Variable Alpha and Simultaneous Hazard Model
Calibration Log-Likelihood (1234 observations)		-10099 (37)	-10032 (67)
Aggregated Calibration Log-Likelihood of Selecting Discrete Duration Bin	-6192.95	-4340.92	-4311.11
Predicted Log-Likelihood (1236 observations from holdout data set)		-10433 (37)	-10399 (67)
Aggregated Calibration Log-Likelihood of Selecting Discrete Duration Bin	-6451.09	-4629.20	-4602.75
Prediction Error Trip Model			
RMS Error (%)	NA	53.62	48.19
Mean Absolute Percentage Error (%)	NA	62.28	59.82
Max. Absolute Percentage Error (%)	NA	86.34	79.56
Prediction Error Stop Model			
RMS Error (%)	NA	34.12	26.06
Mean Absolute Percentage Error (%)	NA	23.49	20.53
Max. Absolute Percentage Error (%)	NA	83.10	47.98

NOTE: RMS = root mean square; Max. = maximum; NA = not applicable.

α (shape) parameters. The results are interpreted relative to a baseline respondent with the following characteristics: a young male (age 20 to 30 years) who is a nonworker from a multiperson and multicar household.

Effect of Covariates on Hazard Rate

The covariate effect directly scales the baseline hazard rate (up or down) by a multiplier $\exp(-\beta X)$. Thus, if the coefficient is positive, the hazard rate decreases (because of the negative sign) compared with the baseline rate and the duration of "survival" increases.

Individual Attributes

The hazard rate for part-time workers was higher than that for nonworkers, suggesting a shorter shopping trip duration for part-time workers, on average, possibly because of work-related time constraints. Individuals with disabilities have lower hazard rates or

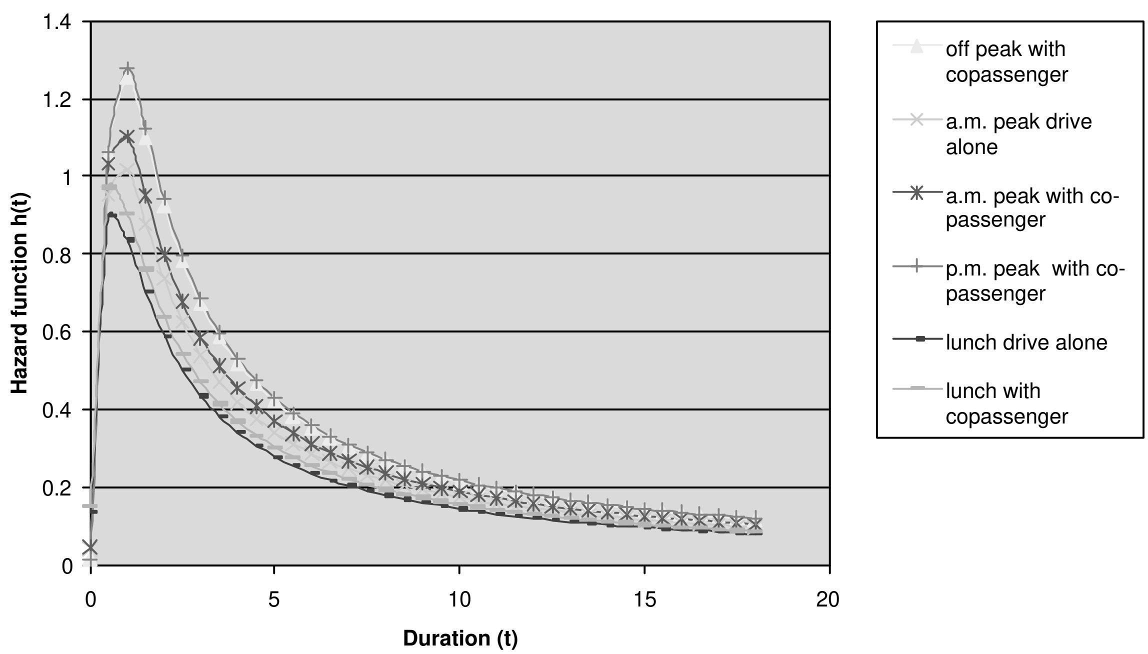

FIGURE 3 Log–logistic hazard functions for shopping stop duration.

TABLE 2 Effects of Covariates on Hazard Rate and Shape Parameters

Variable Description	Trip Covariate		Stop Covariate		Trip Shape - α_t		Stop Shape - α_s	
	Coefficient	t-stat	Coefficient	t-stat	Coefficient	t-stat	Coefficient	t-stat
Baseline parameters:								
λ_0	0.305	5.85	0.155	5.48				
α_0	2.659	3.52	2.167	13.72				
Person Attributes:								
age (20-30)	0.049	3.15						
age (30-40)	0.063	6.14	-0.118	-2.86				
age (40-50)	0.077	2.78	-0.127	-3.34	-0.116	-3.37	0.040	3.95
age (50-60)	0.259	2.72	-0.140	-8.71	-0.116	Same	0.040	Same
age (60+)	0.055	4.99			-0.099	-5.97	0.012	5.6
disability indicator	0.270	4.34						
female			0.156	5.047	-0.135	-4.83	0.230	1.49
Household Attributes:								
medium income (30-60k)	-0.125	-4.4			0.038	3.61		
high income (60k or more)								
single parent	-0.331	-2.84	-0.144	-5.64	0.116	7.86	-0.296	-34.5
one person household	-0.071		-0.221	-3.1	-0.206	-3.84	-0.198	-2.88
single car household	-0.083	-2.34	0.184	3.81			-0.114	
presence of small children (age 1-5)					-0.391	-3.21	0.033	3.61
presence of school children (5-16)			-0.025	-3.37			0.065	2.77
elderly head of household indicator			0.194	1.58				
part-time worker	-0.331	-2.85						
full-time worker					-0.171	-6.71	-0.113	-5.35
household with four or more members					0.025	2.99	0.159	3.61
Situational and Trip Attributes								
mass transit indicator	1.457	3.39						
walk mode indicator					0.068	6.8	0.177	3.28
free parking indicator	-0.519	-4.2						
parking cost in $			0.345	7.25				
presence of co-passenger	0.080	2.93	-0.011	-11.43				
morning peak indicator (6:00 - 9:00 a.m.)			0.212	1.87	0.177	2.81	-0.228	-3.37
travel during noon indicator (11:00 a.m. - 2:00 p.m.)			0.083	4.15	-0.260	-3.12	0.034	1.78
evening peak indicator (4:00 - 7:00 p.m.)			-0.144	-11.84	0.059	2.4	-0.528	-2.75
correlation between trip & stop hazard	0.140	4.14						

NOTE: k = thousands of dollars.

longer trip durations to shopping activities, possibly because of access difficulties and use of public transit modes. While the hazard rate for trip duration appears to decrease slowly with age, the rate was the lowest for individuals in the age group of 40 to 50 years.

In contrast to the effect on trip duration, older respondents, particularly those in the age groups of 20 to 30, 30 to 40, and 40 to 50 years, have higher activity duration hazard rates (or shorter shopping times). However, females in all age groups have lower hazard rates for activity duration, implying longer shopping durations. This finding is consistent with other empirical studies and reflects the strong role of women in meeting household shopping needs.

Household Attributes

Respondents in the middle income category have hazard rates (shorter trip durations) higher than those for the other income groups, suggesting differences in shopping destination choice and the effect of value of time across various income segments. Individuals who own homes are found to travel longer (lower hazard rates) to participate in shopping activities. This suggests a significant linkage between land development pattern and lower prices in the suburbs and the demand and need for longer travel times to access activity locations.

Household structure and interactions also play a key role in determining shopping trip duration. As expected, single parents have considerably shorter trip durations (higher hazard rates) than the baseline duration, possibly because of child care obligations. Individuals from single-person households also exhibit higher hazard rates. In this case, the shorter duration is possibly a reflection of consumption needs lower than those of multiperson households.

As with shopping trips, the presence of children (particularly children who go to school) tends to inhibit shopping activity duration, whereas the shopping durations of elderly heads of households are longer (lower hazard rates) because of a greater degree of time flexibility and increased household responsibility. The shopping durations of single individuals and single parents are shorter than those of other individuals, consistent with the explanations presented above. Individuals from single-car households had longer shopping episodes than the baseline respondents. A plausible explanation is that these households may shop less frequently but for a longer duration to efficiently use the single car to satisfy activity participation needs.

Trip and Situational Attributes

Other situational and travel-related factors also influence the trip duration, consistent with a priori expectations. For instance, transit users have lower hazard rates and longer trip durations than automobile users, a reflection of the large out-of-vehicle trip times (transfer and waiting times) associated with transit. Interestingly, whether or not free parking is available at the destination also affects the hazard rate of shopping trip duration. When free parking is available, a higher hazard rate (trip termination probability) is observed. This implies that if parking is not free, the trip is likely to be extended, perhaps in search of free parking.

Surprisingly, as the cost of parking increases, longer stop durations and lower hazard rates are observed. However, this finding may be an indication of the willingness of users who shop for longer durations to pay more toward parking costs. The results also indicate longer shopping activity durations in the morning peak than in the evening peak and in the afternoon than in other off-peak periods and shorter stop durations in the evening peak. These results are consistent with the lower shopping propensities reported for workers who leave during the evening peak period (*25*).

Role of Covariates on α

Unlike the simple multiplicative effects of covariates in the previous section, the effects of covariates on α must be interpreted more carefully for the following reasons. For nonmonotonic log–logistic hazard functions ($\alpha > 1$), the shape parameter determines both the peak hazard intensity and the time-to-peak characteristics (*10*). With increasing α, the time to peak increases, suggesting longer durations, whereas the peak hazard intensity also increases, suggesting that event (duration) termination probability increases once the peak is reached. Thus, with increasing α, the probability of activity termination increases up to a critical time threshold, while the probability decreases thereafter (Figure 3).

Individual Attributes

The shape factor for trip duration is smaller for females than for males and for older respondents than younger respondents. Females are also seen to have a higher shape parameter for activity duration than males, suggesting lower hazard peak intensity and longer shopping survival propensity, consistent with the findings of other empirical studies. Workers, on the other hand, have a larger α for the activity duration than nonworkers, indicating a greater propensity to terminate shopping activity once the peak hazard rate is reached.

Household Attributes

Individuals in single-person households have smaller shape parameters for trip duration than baseline respondents (suggesting shorter trips, on average, consistent with lower consumption needs). Larger households (with four or more members) also display larger α-values, indicating later (positive coefficient) and higher peak hazard intensities (longer trip durations, on average). On the other hand, the greater time constraint on single parents is reflected in a smaller shape parameter and may lead to shorter trip times. Interestingly, the shape parameter for stops increases for these individuals, suggesting that the probability of duration termination increases, once the peak is reached, compared with that for the baseline case. In contrast, the shape parameter for multiperson households with children shows an opposite trend. A lower trip shape parameter and a larger activity shape parameter are found, suggesting that these respondents shop closer to home or work locations but are likely to shop for longer times (until the peak hazard rate is reached), consistent with the increased consumption needs of larger households. Note that the net impact of children on both trip and stop durations must be obtained by considering the impact of covariates on hazard intensity and shape parameters.

Situational Attributes

The hazard shape parameter is also influenced by trip and time-of-day characteristics. The shape parameter for walking trips is larger than that for automobile trips, indicating (*a*) longer trip times because of slower speeds (longer time to peak) and (*b*) an increased likelihood of duration termination with increasing trip time (the chance of longer-duration walking trips decreases with an increase in the distance or duration). Trips in the afternoon have a smaller shape parameter (lower durations) than shopping trips during morning and evening peak periods, reflecting lower congestion in the afternoon. In contrast, the shape parameters for activity duration increase for afternoon shopping episodes and decrease in the morning and evening peak periods. These results are consistent with expectations of the shorter times spent on shopping activities during the peak period because of congestion avoidance and trip-chaining constraints.

The data also revealed the presence of unobserved heterogeneity in the hazard rates of trip and activity durations. The standard deviation of the lognormal error terms for stop duration was found to be 0.98, and the corresponding standard deviation for trip duration was 0.77. Both were significant at the 1% level (t-statistics = 5.32 and 3.78, respectively), indicating considerable unobserved variation across individuals. The correlation between the two duration processes was found to be small (0.14), but significant (t-statistic = 4.14), which confirms the endogeneity of durations for the two processes.

By using a prediction data set, the performance of the proposed model is compared with those of the simultaneous regression model and the independent hazard model for purposes of validation. With both the calibration and the prediction data sets, the proposed model outperforms the independent hazard-based model and the simultaneous regression model in terms of the prediction of the log likelihood and the goodness of fit of predicted market shares with the observed data (Table 1).

These results were valid for both the disaggregated and the aggregated (into duration intervals) likelihood values. The closeness of fit with the empirical data is assessed by using three measures: root mean squared deviation, mean absolute percentage error, and maximum percentage error between the predicted and the observed market shares. The predicted market shares obtained from the proposed model were closer to the observed market shares from the proposed model than those from the independent hazard model for all three measures of comparison (by 4% to 8%, roughly consistent with the bias corresponding to a correlation of 0.14 in Figure 1). These statistics collectively confirm that the proposed model provides a more accurate representation of observed shopping activity and trip duration data than alternative specifications.

The proposed empirical models can be improved further by the addition of more policy-sensitive and behavioral factors not considered in this study. For instance, the performance of the trip duration model can be enhanced by the consideration of destination choice decision, access measures, and inclusion of network level of service and congestion-related variables. Similarly, the activity duration model can be refined by consideration of trip chain, time constraint, and interhousehold interdependencies. Furthermore, because of the focus on episode-level analysis, the interdependency between tour level and past state dependence (activity type choice decisions) have not been investigated but are needed for a richer understanding of the tour- and pattern-level duration decision processes.

The joint hazard model can be included as a part of planning and travel demand models to address two key shortcomings of trip-based models. First, the explicit analysis of durations (of activities and trips) enables the representation of temporal aspects that are needed in modeling congestion. For instance, this model may be used to obtain the distribution of durations of trips for various origin–destination pairs instead of the static origin–destination table typically used in planning. Second, this framework can be extended to model trip-chaining effects by modeling the interdependence between all activity episodes and trips in a tour. Finally, the proposed models may also be used in air quality assessment studies. Note that while the use of these general duration models in the planning process can lead to increased sensitivity and accuracy in policy making, the lack of closed forms will necessitate the use of microsimulation-based approaches to analyze policy impacts.

CONCLUSIONS

In this paper, a joint hazard-based model has been proposed to analyze mutually interdependent (simultaneous) duration processes. The proposed model generalizes independent hazard-based models by accounting for correlations between simultaneous duration processes. Furthermore, the model also permits the use of flexible and variable hazard profile parameters to capture realistic features observed empirically in activity duration data (e.g., bimodal peaks). The proposed model relies on an implicit component of error structure that combines a baseline hazard function (log–logistic distribution) with a mixing (lognormal distribution) to account for correlations. This model is estimated by using simulated maximum-likelihood techniques based on Monte Carlo simulation of random components to analyze activity and trip durations for shopping activities.

The results highlight the need to account for duration dependence effects in activity–travel duration decisions. Furthermore, hazard-based models that disregard the correlation across joint duration processes (by the use of hazard-based models) can provide biased estimates and inaccurate forecasts. The results imply that stop and trip durations for shopping activities are positively correlated. The hazard rate shape parameters and covariates also vary significantly across individuals, suggesting the need for targeted demand management measures. At a substantive level, the results indicate the roles of personal factors, household attributes, and situational attributes on activity and trip duration decisions.

This study illustrates the value of the proposed framework at the episode level for a single activity type and can be generalized to capture duration process dynamics at the tour and pattern levels across multiple activity types. The proposed framework has important applications and extensions in the travel behavior arena for the development of models of timing and duration and other multiple-spell duration data. A comparison of the theoretical and empirical properties of the more flexible parametric framework proposed in this study with nonparametric methods forms an interesting line of further research.

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