An algorithm for computing theory prime implicates in first order logic

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Abstract: An algorithm based on consensus method to compute the set of prime implicates of a quantifier free first order formula X was presented in an earlier work. In this paper the notion of prime implicates is extended to theory prime implicates in the first order case. We provide an algorithm to compute the theory prime implicates of a Knowledge base X with respect to another knowledge base Y where both X and Y are assumed to be unquantified first order formulas. The partial correctness of the algorithm is proved.

Keywords: first order logic; resolution; knowledge compilation; theory prime implicates.

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1 Introduction

Propositional reasoning is a fundamental issue in artificial intelligence due to its high complexity. Checking whether a query is logically entailed by the knowledge base is intractable (Cook, 1971) since every known algorithm takes exponential time in the worst case in the size of the knowledge base. To overcome such computational intractability, the propositional entailment problem is split into two phases such as *off-line* and *online*. In the off-line phase, the original knowledge base X is compiled into another knowledge base X' and in the online phase the query is actually answered from the new knowledge base in polynomial time in the size of X'. In such type of compilation most of the computational overhead is shifted into the off-line phase which is amortised over online query answering. The off-line computation is known as knowledge compilation.

Several approaches for knowledge compilation have been suggested so far. The first kind of approach consists of an equivalence preserving compilation. In one such approach, the propositional knowledge base X is compiled into a logically equivalent knowledge base $\Pi(X)$, the set of *prime implicates* of X (Coudert and Madre, 1992; de Kleer, 1986, 1992; del Val, 1994; Jackson and Pais, 1990; Kean and Tsiknis, 1990; Ngair, 1993; Reiter and de Kleer, 1987; Shiny and Pujari, 1998; Strzemecki, 1992; Slagle et al., 1970; Tison, 1967) with respect to which queries are answered in polynomial time in the size $\Pi(X)$ by a subsumption test. In another approach to equivalence preserving compilation, Marquis suggested the computation of *theory prime implicates* (Marquis, 1995) from a knowledge base X with respect to another knowledge base Y, so that queries can be answered from the set of theory prime implicates in polynomial time. Another kind of knowledge compilation in first order case is given by del Val (1996).

Most of the research work in knowledge compilation have been restricted to propositional knowledge bases. Due to lack of expressive power in propositional logic, first order logic is required to represent knowledge in many problems. We exploit the quantifier free theory of first order logic to store knowledge in a knowledge base. The formulas are assumed to be in Conjunctive Normal Form (CNF). Taking clue from Raut and Singh (2004) we compute the theory prime implicates (Marquis, 1995) of a first order theory X with respect to another theory Y.

This paper is organised as follows. In Section 2, we introduce the definitions and notions for establishing the required results. In Section 3, we review briefly the consensus method in first order logic as presented by Raut and Singh (2004). Section 4 describes the properties of theory prime implicates and presents an algorithm to compute them. Section 5 concludes this paper.

2 Preliminary concepts

The alphabet of first order language contains the symbols x, y, z,… as variables, f, g, h, … as function symbols, P, Q, R,… as predicates, $\neg$, $\wedge$, $\vee$ as connectives, (,) and ‘,’ as punctuation marks and $\forall$ as universal quantifier. Let FM contain the set of formulas built upon this alphabet. We assume the syntax and semantics of first order logic. Formulas are denoted by upper case letters. For an interpretation or a first order structure i and a formula X, we write $i \leq X$ if i is a model of X. For a set of formulas Σ (or a formula) and any formula X we write $\Sigma \leq X$ to denote that for every interpretation i if i is a model of every formula in Σ then i is a model of X. In such a case, we call X. a logical consequence of Σ. If $X \leq Y$ and $Y \leq X$ then $X \equiv Y$. The quotient set of FM induced by the equivalence relation '$\equiv$' is represented as $[\mathrm{FM}]_{\equiv}$.

A literal is an atomic formula or negation of an atomic formula. A disjunctive clause is a finite disjunction of literals which is also represented as a set of literals. A quantifier-free formula is in CNF if it is a finite conjunction of disjunctive clauses. For convenience, a formula is also represented as a set of clauses. In this paper, we consider formulas only in clausal form.

Two literals r and s are said to be complementary to each other if the set $\{r, \neg s\}$ is unifiable with respect to a most general unifier ξ. We call ξ a complementary substitution of the set $\{r, \neg s\}$. For example, $Pxf(a)$ and $\neg Pby$ are complementary to each other with respect to the complementary substitution (most general unifier or mgu, for short) $[x/b, y/f(a)]$. So the most general unifier bundles upon infinite number of substitutions to a finite number.

A clause which does not contain a literal and its negation is said to be *fundamental*. Thus a non-fundamental clause is valid. We avoid taking non-fundamental clauses in clausal form because the universal quantifiers appearing in the beginning of the formula can appear before each conjunct of the CNF since $\forall$ distributes over $\wedge$. So each clause in a formula of the knowledge base is assumed to be non-fundamental. Let C_1 and C_2 be two disjunctive clauses. Then C_1 *subsumes* C_2 if there is a substitution σ such that $C_1\sigma \subseteq C_2$, that is, $C_1\sigma \leq C_2$. For example, $\{\neg Rxf(a), \neg Py\}$ subsumes the clause $\{\neg Rg(a)f(a), \neg Py, Qz\}$ with respect to the substitution $\sigma = [x/g(a)]$. A disjunctive clause C is an *implicate* of a finite set of formulas X (assumed to be in CNF) if $X\sigma \leq C$ for a substitution σ. We write $I(X)$ as the set of all implicates of X. A clause C is a *prime implicate* of X if C is an implicate of X and there is no other implicate C' of X such that $C'\tau \leq C$ for a substitution τ (i.e. if no other implicate C' subsumes C). $\Pi(X)$ denotes the set of prime implicates of X. It may be observed that if an implicate C is not prime then there exists a prime implicate D of X that subsumes C, that is, along with D, we have a substitution τ such that $D\tau \leq C$.

Note that the notion of prime implicate is well defined as the knowledge base contains clauses unique up to subsumption. Let Y be a set of fundamental clauses. The *residue of subsumption* of Y, denoted $\mathrm{Res}(Y)$ is a subset of Y such that for every clause $C \in Y$, there is a clause $D \in \mathrm{Res}(Y)$ such that D subsumes C. Moreover, no clause in $\mathrm{Res}(Y)$ subsumes any other clause in $\mathrm{Res}(Y)$.

A clause $C \in \Pi(X)$ is a *minimal element* of $\Pi(X)$ if for all $C \in \Pi(X)$ and for all substitution σ, $C\sigma \leq C'$ implies $C\sigma \equiv C'$. Equivalently, a clause $C' \in \Pi(X)$ is a minimal element of $\Pi(X)$ if there is no $C \in \Pi(X)$ and there is no substitution σ such that $C\sigma \not\equiv C'$ and $C\sigma \leq C'$. Clearly, the prime implicates of a finite set of formulas X are the minimal elements of $I(X)$ with respect to $\leq$. So $\Pi(X) = \min(I(X), \leq)$.

Let C_1, C_2 be two clauses in X and $r \in C_1, s \in C_2$ be a pair of complementary literals with respect to a most general unifier σ. The resolution of C_1 and C_2 is $C = [(C_1 - \{r\}) \cup (C_2 - \{s\})]\sigma$. If C is fundamental, it is called a *consensus of* C_1 and C_2. The set of all consensus of C_1 and C_2 is denoted by $\mathrm{CON}(C_1, C_2)$. C can also be written as $[(C_1\sigma - \{t\}) \cup (C_2\sigma - \{\neg t\})]$ for a literal t, provided $r\sigma = t$ and $s\sigma = \neg t$. In this case, we also say that C is the propositional consensus of $C_1\sigma$ and $C_2\sigma$. For example, if $C_1 = \{Rbx, \neg Qg(a)\}$ and $C_2 = \{Rab, Qz\}$ then $\mathrm{CON}(C_1, C_2) = \{Rbx, Rab\}$ which equals the propositional consensus of $C_1[z/g(a)]$ and $C_2[z/g(a)]$. If C is the consensus of C_1 and

C_2 with respect to a most general unifier σ then C is said to be *associated with* the substitution σ. By default, each clause C of a set of formulas X is associated with the empty substitution ε. Let C_1 and C_2 be two resolvent clauses associated with substitutions σ_1 and σ_2, respectively. Then their consensus with respect to σ is defined provided $\sigma_1\sigma = \sigma_2\sigma$. The consensus is then the propositional consensus of $C_1\sigma$ and $C_2\sigma$ and the consensus is associated with the substitution $\sigma_3 = \sigma_1\sigma = \sigma_2\sigma$.

Let Y be a finite set of formulas. We define $\leq_Y$ over FM × FM (as the extension of $\leq$) by $X_1 \leq_Y X_2$ iff $\{X_1\} \cup Y \leq X_2$ where X_1 and X_2 are two formulas in FM. Similarly for a substitution σ we define $X_1\sigma \leq_Y X_2$ if $\{X_1\sigma\} \cup Y\sigma \leq X_2$, (i.e. $(X_1 \cup Y)\sigma \leq X_2$ if X_1 is a set of formulas) where both X_1 and Y are associated with the same substitution σ. If $X_1\sigma \leq_Y X_2$ holds then we say that X_2 is a *Y-logical consequence* of X_1. We define the equivalence relation $\equiv_Y$ over FM by $X_1 \equiv_Y X_2$ if $X_1\sigma_1 \leq_Y X_2$ for all σ_1 and $X_2\sigma_2 \leq_Y X_1$ for all σ_2. In this case, we say that X_1 and X_2 are Y-equivalent. $[\text{FM}]\equiv_Y$ is the quotient set of FM induced by the equivalence relation $\equiv_Y$.

We now extend the definition of prime implicate to theory prime implicate with respect to $\leq_Y$ as follows. Let X and Y be finite sets of formulas. A clause C is a *theory implicate* of X with respect to Y if $X\sigma \leq_Y C$. A clause C is called a *theory prime implicate* of X with respect to Y if C is a theory implicate of X with respect to Y and there is no other theory implicate C' such that $C'\tau \equiv_Y C$ for some substitution τ. We denote by $\Theta(X, Y)$ the set of theory prime implicates of X with respect to Y. Thus, $\Theta(X, Y)$ contains the minimal elements of the set of theory implicates of X with respect to Y. The minimal clauses are considered up to Y-logical equivalence, that is, $\Theta(X, Y)$ contains a clause from each $\equiv_Y$ -equivalence class.

3 Computation of prime implicates

We briefly present the main results for computation of prime implicates (Raut and Singh, 2004) of first order predicate formulas in clausal form. Let $X = \{C_1,\ldots,C_n\}$ be a formula where each clause C_i is fundamental. Then each C_i is an implicate of X with respect to the empty substitution, but each one of them may not be a prime implicate. The key is the subsumption of implicates of X. Since clauses here are disjunctive, we observe that if C_1 subsumes C_2 then there is a substitution σ such that $C_1\sigma \leq C_2$. Our aim is to see how deletion of subsumed clauses leads to the computation of prime implicates. It leads us to explore the relation between consensus closure and the prime implicates.

Lemma 3.1: *A clause C is an implicate of X if and only if there is a prime implicate C′ of X such that C′ subsumes C.*

Lemma 3.2: $X = \Pi(X)$.

Theorem 3.3: *Consensus of two implicates of a formula is an implicate of the formula.*

The computational aspects of prime implicates is described below. For a set of clauses X, let $L(X)$ be the set of all consensus among clauses in X along with the clauses of X, that is, $L(X) = X \cup \{S\colon S$ is a consensus of each possible pair of clauses in $X\}$. We construct the sequence X, $L(X)$, $L(L(X)),\ldots$, that is, $L_{n+1}(X) = L(L_n(X))$ for $n \geq 0$ and $L_0(X) = X$. We put together all the clauses in the sequence of sets and form the consensus closure.

The *consensus closure* of X is written as $L^*(X) = \cup\{L_i(X): i \geq 0\}$. From Theorem 3.3, it follows that $L^*(X) \subseteq I(X)$.

Theorem 3.4: *The set of all prime implicates is a subset of the consensus closure of X, that is, $\Pi(X) \subseteq L^*(X)$. Moreover, $\Pi(X) = Res(L^*(X))$.*

Using the above results on the extended notions of consensus and subsumption, we have suggested (Raut and Singh, 2004) an algorithm to compute prime implicates of first order formulas in clausal form. Correctness of the algorithm has also been proved there. We intend to employ a similar algorithm for computing theory prime implicates.

4 Computation of theory prime implicates

We describe some of the properties of and an algorithm to compute theory prime implicates $\Theta(X, Y)$ of a first order theory X with respect to another theory Y based on the results in Raut and Singh (2004) as described in the last section.

Lemma 4.1: *Let X and Y be finite sets of formulas. Then $\Theta(X, Y) \subseteq \Pi(X \cup Y)$.*

Proof: Let $C \in \Theta(X, Y)$. So $X\sigma \leq_Y C$ holds and there is no theory implicate C' of X with respect to Y such that $C'\tau \leq_Y C$ holds for some τ. This implies $X\sigma \cup Y\sigma \leq C$. That is, $(X \cup \mathrm{Y})\ \sigma \leq C$. C is an implicate of $X \cup Y$. If C is not a prime implicate of $X \cup Y$ then there exists a prime implicate C' of $X \cup Y$ such that $C'\tau \leq C$, that is, $C'\tau \leq_Y C$. As C' is a prime implicate of $X \cup Y$, $(X \cup Y)\tau \leq C'$, that is, $X\tau \leq_Y C'$. This implies C' is a theory implicate of X with respect to Y. So we get a theory implicate C' such that $C'\tau \leq_Y C$, that is, C is not a theory prime implicate of X with respect to Y, which is a contradiction.

Lemma 4.2: *If $C, C' \in \Pi(X \cup Y)$ and $C\tau \leq C'$ but $C\tau \not\equiv_{\mathrm{Y}} C'$ for some τ then $C' \notin \Theta(X, Y)$.*

Proof: Let $C' \in \Theta(X, Y)$. $X\sigma \leq_Y C'$ and there is no theory implicate D such that $D\tau \leq_Y C'$ for some τ. In other words, $(X \cup Y)\ \sigma \leq C'$ and there is no theory implicate D (i.e. $(X \cup Y)\ \sigma \leq D$) such that $D\tau \leq_Y C'$. Thus, $(X \cup Y)\ \sigma \leq C'$ and for every D if $(X \cup Y)\tau \leq D$ then $D\tau$ does not entail C' with respect to Y. As C is a prime implicate of $X \cup Y$, C is also an implicate of $X \cup Y$. With $D = C$, $(X \cup Y)\ \sigma \leq C'$ and if $(X \cup Y)\tau \leq C$ then $C\tau$ does not entail C' with respect to Y, which contradicts the hypothesis of the lemma. This implies that $C' \notin \Theta(X, Y)$ completing the proof.

Lemma 4.3: *If $C, C' \in \Theta(X, Y)$ and $C \neq C'$, then either (i) $C\sigma_1 \leq_Y C$ for all σ_1 and it is not that $C'\sigma_2 \leq_Y C$ for some σ_2 or (ii) $C'\sigma_2 \leq_Y C$ for all σ_2 and it is not that $C\sigma_1 \leq_Y C'$ for some σ_1.*

Proof: If $C\sigma_1 \leq_Y C'$ for all σ_1, C' is a Y-logical consequence of C. If $C'\sigma_2 \leq_Y C$ for all σ_2 then C is a Y-logical consequence of C'. So $C \equiv_Y C'$. Thus either C or C' belongs to $\Theta(X, Y)$ but not both which is a contradiction to the hypothesis. Similarly the other part is proved. This completes the proof.

The following result is obtained by using the above Lemmas.

Theorem 4.4: *$\Theta(X, Y) = min\ (\Pi(X \cup Y), \leq_Y)$.*

Theorem 4.5: *Let X and Y be finite sets of formulas and C be any clause. $X\sigma \leq_Y C$ holds if there exists a theory prime implicate C' of X with respect to Y such that $C' \tau \leq_Y C$ holds.*

Proof: Suppose $X\sigma \leq_Y C$ holds. C is an implicate of $X \cup Y$. If C is not prime then there is an implicate C × (≠ C) of $X \cup Y$ and a substitution τ such that $C \times \tau \leq C$. Let $A^* = \{C^*$: C^* is an implicate of $X \cup Y$ and $C^*\tau \leq_Y C$ for some $\tau\}$. We can obtain a finite subset $A = \{C_1,\ldots, C_n\}$ of A^* such that for each C^* in A^* there is a C_i in A and a substitution η such that $C^* = C_i\eta$, since there are only a finite number of variables in C^* and C is finite. Now A, being a finite set, has a strict partial order as logical consequence with respect to Y. Each element of A is an implicate of $X \cup Y$. Any minimal element C' of A is a prime implicate of $X \cup Y$, that is, a theory prime implicate of X with respect to Y.

Conversely, there exists a prime implicate C' of $X \cup Y$ such that $C'\tau \leq_Y C$, that is, $C'\tau \cup Y\tau \leq C$ holds. As $(X \cup Y)\ \sigma \leq C'$, $C' \leq C'\tau$ and $C'\tau$ and $Y\tau$ are disjunctive, $(X \cup Y)\ \sigma \leq C'\tau \cup Y\tau$. Thus $(X \cup Y)\ \sigma \leq C$, showing that $X\sigma \leq_Y C$. This completes the proof.

Lemma 4.6*: If $X \equiv X'$, then $\Pi(X) \equiv \Pi(X')$.*

Proof: Let $C \in \Pi(X)$; that is, $X\sigma \leq C$ and there exists no implicate C' of X such that $C'\sigma \leq C$. Since $X \equiv X'$, we have $X'\sigma \leq C$. Also, there exists no implicate C' with $C'\sigma \leq C$. This implies $C \in \Pi(X')$. Thus, $\Pi(X) \subseteq \Pi(X')$. Similarly, $\Pi(X') \subseteq \Pi(X)$.

Theorem 4.7: *If $X \equiv X'$ and $Y \equiv Y'$, then $\Theta(X, Y) \equiv \Theta(X', Y')$, upto Y-equivalence.*

Proof: Let $C \in \Theta(X', Y')$, that is, for all $C' \in \Pi(X' \cup Y')$, if $C'\sigma \leq_{Y'} C$ then $C'\sigma \leq_Y C$. By hypothesis, $X' \cup Y' \equiv X \cup Y$ and by Lemma 4.6, $\Pi(X' \cup Y') = \Pi(X \cup Y)$. So, for all C' in $\Pi(X \cup Y)$ if $C'\sigma \leq_{Y'} C$ then $C'\sigma \leq_Y C$. As $C' \in \Theta(X, Y)$, we see that $\Theta(X', Y') \subseteq \Theta(X, Y)$. Similarly, the other inclusion $\Theta(X, Y) \subseteq \Theta(X', Y')$ follows.

To see the computational aspects of prime implicates, let X and Y be finite sets of formulas and $Z_1 = X \cup Y$. Define $L_1(Z_1) = Z_1 \cup \{C_1$: C_1 is the consensus of a pair of clauses from $Z_1\}$. Construct Z_2 by deleting those clauses from $L_1(Z_1)$ which are Y-logical consequences of C_1. Let $L_2(Z_1) = Z_2 \cup \{C_2$: C_2 is the consensus of a pair of clauses from $Z_2\}$. Construct Z_3 like Z_2, but from $L_2(Z_1)$. In general, write $L_n(Z_1) = Z_n \cup \{C_n$: C_n is the consensus of two clauses from $Z_n\}$. At one stage for some $m > n$, $L_m(Z_1) = L_n(Z_1) = Z_{n+1}$ happens, in the propositional case. Unfortunately this need not be so in a first order knowledge base as explained by an example in Raut and Singh (2004). However, each clause in each set $L_n(X \cup Y)$ is a Y-logical consequence of X, as the following result shows.

Lemma 4.8: *$X \leq_Y L_n(X \cup Y)$.*

Proof: First, we show that $X \cup Y \leq L_1(X \cup Y)$. Let i be a model of $X \cup Y$; that is, $i \leq C$ for all C in $X \cup Y$. If $X \cup Y = L_1(X \cup Y)$, then $X \cup Y \leq L_1(X \cup Y)$. If not, there exists a clause D in $L_1(X \cup Y)$ such that D is the consensus of two clauses, say, of C_j and C_k from $X \cup Y$. As $i \leq C_j \wedge C_k$, it follows that $i \leq D$ and $i \leq L_1(X \cup Y)$. This implies that $X \cup Y \leq L_1\ (X \cup Y)$, that is, $X \leq_Y L_1(X \cup Y)$. Similarly we can show that $L_1(X \cup Y) \leq_Y L_2(X \cup Y) \leq_Y L_2(X \cup Y)\ldots\leq_Y L_n(X \cup Y)$. By induction, it follows that $X \leq_Y L_n(X \cup Y)$.

This implication can be strengthened to an equivalence as in the following.

Theorem 4.9: $X \equiv_Y L_n(X \cup Y)$.

Proof: Due to Lemma 4.8, it is enough to show that $L_n(X \cup Y) \leq_Y X$. For this, let i be a model of $L_1(X \cup Y)$, that is, $i \leq C$ for all C in $L_1(X \cup Y) \cup Y$. If $L_1(X \cup Y) \cup Y = X$ then it is clear that $L_1(X \cup Y) \leq_Y X$. If not, then there exists a clause C^* in X such that $C^* \notin L_1(X \cup Y) \cup Y$. This implies that there exists a clause $D^* \in L_1(X \cup Y) \cup Y$ such that C^* is a Y-logical consequence of D^*, that is, $D^*\sigma\sigma \leq_Y C^*$. As $i \leq C$ for all $C \in L_1(X \cup Y) \cup Y$, $i \leq D^*$. Thus, $i \leq D^*\sigma$. As $i \leq Y$, $i \leq C^*$, we have $i \leq X$. Hence, $L_1(X \cup Y) \cup Y \leq X$ that is, $L_1(X \cup Y) \leq_Y X$. It follows by induction that $L_n(X \cup Y) \leq_Y X$.

Theorem 4.10: $I(X \cup Y) \equiv L_n(X \cup Y)$.

Proof: From Lemma 4.8, we see that all the clauses of $L_n(X \cup Y)$ are implicates of $X \cup Y$. That is, $L_n(X \cup Y) \subseteq I(X \cup Y)$. Therefore, $I(X \cup Y) \leq L_n(X \cup Y)$ as the sets of clauses are interpreted as CNF.

Conversely, let i be a model of $L_n(X \cup Y)$. If all the clauses of $I(X \cup Y)$ are in $L_n(X \cup Y)$, then the result is obvious; otherwise, there exists a clause C in $I(X \cup Y)$ such that $C \notin L_n(X \cup Y)$. Then there exists a clause $D \in L_n(X \cup Y)$ such that C is a Y-logical consequence of D, that is, $D\sigma \leq_Y C$. Since $i \leq L_n(X \cup Y)$, $i \leq D$ for all D in $L_n(X \cup Y)$. Thus, $i \leq D\sigma$. As $i \leq Y$, we have $i \leq C$ and $i \leq I(X \cup Y)$. Hence, $L_n(X \cup Y) \leq I(X \cup Y)$.

It may be noted that $L_n(X \cup Y)$ contains implicates up to the relation of Y-equivalence, that is, it contains one representative per equivalence class.

Theorem 4.11: $L_n(X \cup Y) \subseteq \Theta(X, Y)$ *for every n. Moreover, if* $L_{m+1}(X \cup Y) = L_m(X \cup Y)$ *holds for some m, then* $L_m(X \cup Y) = \Theta(X, Y)$.

Proof: Let $C \in L_n(X \cup Y)$. Then, C is an implicate of $X \cup Y$. Since there does not exist any clause C' in $L_n(X \cup Y)$ with $C'\sigma \leq_Y C$ (otherwise C would not have been in $L_n(X \cup Y)$), $C \in \Pi(X \cup Y)$, by Theorem 4.10. Moreover, C is a minimal element of $\Pi(X \cup Y)$ with respect to $\leq_Y$. By Theorem 4.4, $C \in \Theta(X, Y)$. So, $L_n(X \cup Y) \subseteq \Theta(X, Y)$ for every n.

Now, suppose $L_{m+1}(X \cup Y) = L_m(X \cup Y)$ holds for some m. Let $C \in \Theta(X, Y)$. Since $C \in \Pi(X \cup Y)$, we have $C \in I(X \cup Y)$. If $C \notin L_m(X \cup Y)$, then there exists a clause $C' \in L_m(X \cup Y)$ such that C is a Y-logical consequence of C'. Thus C is not prime, that is, $C \notin \Theta(X, Y)$. This shows that $C \in L_m(X \cup Y)$. Consequently, $\Theta(X, Y) \subseteq L_m(X \cup Y)$. Equality follows from the previous paragraph.

It may be observed that in case $L_{m+1}(X \cup Y) = L_m(X \cup Y)$ holds for some m, then $\Theta(X, Y)$ is finite as $L_n(X \cup Y)$ is finite, for each n.

Definition 4.1: *Let X and Y be finite sets of formulas such that* $X \leq Y$. *The theory prime implicate compilation of X with respect to Y is defined by* $\Omega_Y(X) = \Theta(X, Y) \cup Y$.

Theorem 4.12: $\Omega_Y(X) \equiv X$ *provided that* $\Theta(X, Y)$ *is finite.*

Proof: By Theorems 4.9 and 4.11, $\Theta(X, Y) \cup Y \leq X$. Conversely, let i be a model of X. Since $X \leq Y$ (as implicitly assumed in Definition 4.1), X entails each clause obtained by taking consensus of any pair of clauses from $X \cup Y$, that is, $X \leq L_n(X \cup Y)$. Thus, $X \leq \Theta(X, Y)$, again by Theorem 4.11, this shows that i is a model of $\Theta(X, Y) \cup Y$. Hence $X \leq \Theta(X, Y) \cup Y$.

We compute the theory prime implicates Θ of a set of formulas *X* with respect to *Y* by computing the implicates of $X \cup Y$. The latter computation uses a consensus based prime implicate algorithm (de Kleer, 1992; Kean and Tsiknis, 1990; Raut and Singh, 2004) and only the representatives of *Y*-logical equivalent clauses are kept in the set Θ. The algorithm is described as follows:

Algorithm TPI

Input: The set of clauses *X* and *Y*

Output: (X, Y), the set of theory prime implicates of *X* with respect to the theory *Y*

begin

$:= X \cup Y$

If $X \cup Y = \varnothing$

$:= \varnothing$;

else

$Z_0 = \varnothing$;

$Z_1 :=$;

$i := 1$;

while $Z_i \neq Z_{i-1}$

do

compute $L_1(Z_i)$;

compute Z_{i+1};

$i := i+1$;

od

$:= Z_{i+1}$;

endif

return (X, Y);

end

In the above algorithm we compute the set of theory prime implicates of *X* with respect to *Y* by computing the prime implicates of $X \cup Y$ and keeping only the *Y*-logical equivalent clauses in the set. We know, apart from the set of clauses Z_1, that $L_i(Z_1)$ contains the consensus CON of a pair of clauses from Z_i. The set Z_{i+1} contains the clauses that remain after discarding the Y-logical consequence of CON. The process is repeated till $Z_n = L_n(Z_1) = Z_{n+1} = L_{n+1}(Z_1)$ holds. After the termination of the algorithm, the set $L_n(Z_1)$ contains the set of theory prime implicates.

Theorem 4.13: *The algorithm TPI correctly computes the set of theory prime implicates of a clausal formula provided it terminates.*

Proof: Let $X \cup Y$ be the given set of clauses. If the set $X \cup Y$ is empty, obviously there are no theory prime implicates. If $X \cup Y$ is non-empty, assign $X \cup Y$ to Z_1. Then, $L_1(Z_1)$ is computed. Those clauses D of $L_1(Z_1)$ will be discarded from the set which are Y-logical consequences of the newly added clauses in CON to get Z_2 because anything derivable from D can be derived from CON. By Theorems 4.4 and 4.11, $L_1(Z_1) \subseteq \Pi(X \cup Y)$. Similarly, $L_n(Z_1)$ is computed repeating the steps. Due to the assumption that *TPI* terminates, we see that for some n and for all $m > n$, $L_n(Z_1) = L_m(Z_1)$. Hence all the minimal elements of $\Pi(X \cup Y)$ has been computed in $L_n(Z_1)$. Hence, by Theorem 4.4, $\Theta(X, Y)$ has been computed.

We remark that partial correctness of TPI is the best possible. In fact such algorithms cannot be totally correct due to the undecidability of first order logic.

5 Conclusions

In this paper, the notion of prime implicates is generalised to the theory prime implicates and an algorithm for computing the theory prime implicates has been established. The algorithm computes the set of theory prime implicates $\Theta(X, Y)$ of a first order knowledge base X with respect to another knowledge base Y. The correctness of the algorithm has been proved.

The size of the compilation is exponential in the size of the original knowledge base. If we take $Y = \emptyset$, then the theory prime implicate compilation coincides with the prime implicates compilation. Since the compilation takes a long time to obtain Θ, it is desirable to ask queries at any time while the compilation is in progress. Though all the queries cannot be answered before the off-line computation is completed, the possibility of answering the number of queries increases. The off-line computation could be avoided partially but how it can be done efficiently is not yet known.

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