Algorithm for Determining Most Reliable Travel Time Path on Network with Normally Distributed and Correlated Link Travel Times

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Transportation networks are subject to significant travel time uncertainty as a result of traveler behavior, recurring congestion, capacity variability (construction zones, traffic incidents), variation in demand, and so on. Therefore, interest is growing in modeling and optimizing travel time reliability in such networks. This paper proposes an efficient algorithm to compute the path of maximum travel time reliability on a network with normally distributed and correlated link travel times. For this optimal reliability path (ORP) problem, it is shown that the subpath optimality condition for the deterministic shortest path problem does not hold, and consequently, a new bounds-based optimality criterion is proposed using the K shortest expected time paths and the minimum path variance on the network. An algorithm is developed to solve the ORP problem on the basis of the proposed optimality criterion and an efficient path generation procedure. Computational experiments on various test networks show the proposed algorithm to be efficient, requiring limited path enumeration. With as few as five shortest paths and 50 Monte Carlo draws, the proposed algorithm is able to find the most reliable path for realistic network sizes. Empirical investigations highlight the unreliability of the least expected time path and suboptimality of the independence assumption. The study also underscores the role of risk attitudes (reflected by reliability threshold) on the benefits of the ORP. The algorithm and empirical results have important applications for developing reliability-based routing applications for congestion mitigation and intelligent transportation systems.

The selection of optimal routes on congested transportation networks has vital applications in advanced traveler information systems, routing and scheduling of freight, and dispatch of emergency services. Transportation networks are subject to significant travel time uncertainty as a result of traveler behavior, recurring congestion, capacity variability (construction zones, traffic incidents), variation in demands, and so forth. Consequently, computing optimal paths on the basis of deterministic criteria such as cost or time is inadequate for real-world networks under stochastic and correlated

travel times. Therefore, there is growing interest in modeling and optimizing travel time reliability in such networks.

In this paper a new algorithm is proposed to compute the path of maximum travel time reliability between a given origin and destination on a network with stochastic and correlated link travel times specified by a multivariate normal distribution. In that context, path travel time reliability is defined as the probability that path travel time is within a suitably defined threshold value T_0.

The motivation for this study is twofold: First, greater levels of unreliability in travel times are observed under congested conditions, which can affect travelers' scheduling and route choices. Empirical studies suggest that commuters attach a high value to the reliability in journey times (*1*). The second motivating factor is the sparsity of literature on the optimal reliability path problem, due partly to its complexity. The path reliability objective is a nonlinear function of means and variances. So, the optimality conditions for shortest path problems based on link separability and linear objective are not applicable. Unfortunately, solution approaches that make restrictive assumptions of link independence for tractability can yield unreliable paths when correlations are significant [e.g., (*2, 3*)].

The interest in this problem of maximizing path reliability is also motivated by the following applications: (*a*) better trip planning and scheduling for commuters, (*b*) improved urban commuting decisions through route guidance and information (advanced traveler information system), and (*c*) development of decision-support tools to evaluate and improve the reliability of travel. These applications can lead to reduced commuter delays, increased travel time savings, and so on.

As a result of these motivating considerations, three objectives are pursued in this study: (*a*) formulate the optimal reliability path (ORP) problem on a network with stochastic, normal, and correlated link travel times and identify an optimality criterion for this problem; (*b*) propose and implement an efficient algorithm to find the ORP for a given origin–destination (O-D) pair; and (*c*) empirically investigate the computational performance of the proposed algorithm on test networks.

This work contributes to the literature in the following ways. First, it is shown that the subpath optimality condition from conventional shortest path problems does not hold for the ORP problem. A new optimality criterion based on reliability bounds is proposed for the ORP problem. An algorithm based on the proposed optimality criterion and an efficient path generation procedure is developed to solve the ORP problem. Through computational experiments on various test networks, the proposed algorithm is found to be practically

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efficient because it does not involve extensive path enumeration. With as few as five shortest paths and 50 Monte Carlo draws, the proposed algorithm is able to find the most reliable path for realistic network sizes. Empirical investigations highlight the unreliability of the least expected time path and suboptimality of the independence assumption. The study also underscores the role of risk attitudes (reflected by reliability threshold) on the benefits of the ORP.

A brief review of existing approaches is presented next, after which the problem is formally defined and formulated. The following section describes the proposed algorithm to compute the path of maximum travel time reliability for the general case with correlations and unequal variances. The proposed approach is illustrated on a simple network, and its computational complexity is analyzed. A discussion of results from a set of computational experiments follow, and the main findings from the study, conclusions, and scope for further research are outlined to conclude the paper.

REVIEW OF LITERATURE

The classical shortest path problem involves computing the path of minimum cost–time on a network with deterministic arc costs. The need to capture inherent network uncertainty led to the study of the stochastic shortest path problem (SSPP) within the context of decision making under uncertainty. Stochastic routing models are intended to provide commuters with either a priori path guidance or adaptive en route guidance. Both versions of the problem have been extensively studied, and existing approaches are reviewed in two parts, the first dealing with the objective of least expected travel time (LET) and the second dealing with reliability-based formulations.

One class of adaptive LET path problems assumes that the traversal time on a link will become known (deterministic) on arrival at its tail (starting) node. Polychronopoulos and Tsitsiklis (4) proposed a dynamic programming (DP) approach (exponential) to compute the adaptive LET path, and Cheung proposed a label correcting algorithm for the same problem (3).

The second class of LET problems assumes that the link travel time distribution is conditional on arrival time at the link entrance [stochastic time varying (STV) networks]. Miller-Hooks and Mahmassani proposed a nondeterministic polynomial label correcting algorithm (discrete travel time distribution) for the a priori path problem (5). The adaptive path variant for a continuous link travel time distribution was examined by Hall, who proposed a non-polynomial DP-based algorithm (6). In addition, Miller-Hooks proposed a label setting algorithm for the discrete version of the problem (7). As a result of the absence of the Markovian property, finding LET paths on STV networks is computationally difficult even with the assumption of independence (7).

Reliability-based stochastic routing has been studied primarily in the context of finding a priori optimal paths. Frank in his seminal paper, proposed an algorithm to compute the continuous probability distribution of the minimum travel time (8). However, shortest paths are identified through paired comparisons within an already enumerated path set. Nie and Wu introduced a concept of "locally reliable" paths and proposed a label correcting algorithm to compute the set of locally reliable paths (nondominated with respect to reliability at varying thresholds) for static independent link travel time distributions (9). In contrast, the problem of computing an optimal reliability strategy (policy) has received scant attention. A notable exception is Fan and Nie, who proposed a DP-based algorithm (10).

Another widely used approach (incorporating reliability) uses the maximum expected utility (MEU) criterion of Von Neumann and Morgenstern (11). Here, a random utility that is a function of link costs is assigned to each path, with the optimal path being one that maximizes expected path utility. Computing the MEU path for non-linear utility functions is nontrivial, and existing pruning-based approaches assume independence of links (11). The mean variance trade-off was more explicitly addressed in the mean variance routing model proposed by Sen et. al. (12). Their model is noteworthy for its consideration of correlations between link travel times although this advancement comes at the expense of computational efficiency, making the proposed integer programming based solution algorithm ill suited to large networks.

The SSPP has also been studied in the context of robust optimization (13). The robust path is one that minimizes path robust deviation (maximum difference between path cost and the corresponding shortest path cost, over all scenarios) or worst-case performance. Pseudopolynomial algorithms are proposed by Yu and Yang (13). However, such robust routing problems are NP-hard even under restrictive assumptions (5). Other definitions of optimality based on first-order stochastic dominance and definite stochastic dominance are investigated by Miller-Hooks and Mahmassani (5). They propose label correcting algorithms and heuristics to find nondominated paths under the stochastic dominance rules.

In summary, only a few studies consider the objectives of optimizing reliability explicitly or implicitly. Further, most existing approaches make a restrictive assumption of independent link travel times [e.g., (2), (4), (8)]. Many investigations on the optimal reliability problem use a pure Monte Carlo–based approach along with optimization heuristics. There seems to be insufficient understanding of and evidence on the computational performance and accuracy of these heuristics for networks with general correlation patterns. In addition, several empirical issues concerning the ORP problem remain to be addressed systematically. For instance, how many paths and draws are needed to obtain optimal or near-optimal solutions? How well does the least expected time path perform in relation to the reliability objective? What is the consequence of neglecting correlations while determining the ORP? How does the nature and magnitude of benefits of the ORP depend on the threshold for computing reliability? This work seeks to address the limitations and issues above by proposing a new algorithm to determine the ORP and conducting computational experiments on various networks.

PROBLEM DEFINITION AND FORMULATION

The ORP problem considered in this study is concerned with computing the path that maximizes the probability of arrival within a threshold T_0. This formulation allows one to capture features of decision making under uncertainty, particularly the aversion to late arrival and the use of a buffer time to maximize probability of on-time arrival. This problem is defined and formulated as a nonlinear integer programming problem in this section.

Problem Context and Scope

The transportation network of interest is represented as a directed graph or network (cyclic) denoted by $G(N, A)$, where $N = \{1, 2, \ldots, n\}$ represents the set of nodes and A represents the set of m directed arcs. The network is general in the sense that it may include cycles. It is assumed that the link travel times are random and multivariate normally distributed with a general correlation pattern [$\mathbf{t} \sim \text{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$]. This assumption is

empirically supported by travel time studies on freeways and urban arterials (*14*). In addition, this study addresses a static travel time context, and consequently, the most reliable path is considered to be an a priori path without recourse. The a priori path formulation is motivated by the absence of in-vehicle guidance systems–traveler information services to support en route choices in developing countries (study context).

Path with Optimal Travel Time Reliability Between a Given Origin and Destination

The travel time along link (i, j) is represented by a random variable t_{ij} characterized by a mean μ_{ij} and a standard deviation σ_{ij}. Consequently, path travel time is also a random variable, normally distributed with a mean and variance given by

$$
\mu_P = \sum_{(i,j) \in P} \mu_{ij}
$$

$$
\sigma_P^2 = \sum_{(i,j) \in P} \sigma_{ij}^2 + \sum_{(i,j) \neq (k,l) \in P} P_{ij-kl} \sigma_{ij} \sigma_{kl}
$$
(1)

where (i, j) and (k, l) represent links on path P. The travel time reliability (referred to as reliability hereafter for ease of exposition) of path P is defined as the probability of path travel time (t_P) being within a prespecified threshold T_0:

$$
R(P) = P(t_P < T_0) = \Phi\left(\frac{T_0 - \mu_P}{\sigma_P}\right)
$$
(2)

The objective of the problem is to find the most reliable path between a given origin and destination for the network as defined above. In other words, the aim is to determine a path P^* such that its reliability $R(P^*)$ is greater than or equal to the reliability on any other path $R(P)$ connecting origin s to destination t. Thus, the optimal reliability path problem (ORP problem) for a given O-D pair is a nonlinear integer programming problem. Because the normal cumulative distribution function $\Phi(.)$ is monotonic, the objective function may be rewritten as

$$
R'(\mathbf{x}) = \left(\frac{T_0 - \mu}{\sigma}\right)
$$

Failure of Subpath Optimality Property

The objective function $R'(\mathbf{x})$ above is nonlinear in the integer-valued decision variable link flows ($x_{ij} = 0$ or 1). Consequently, the property of subpath optimality (i.e., every subpath of the optimal path is optimal to the respective intermediate node) does not hold. That is illustrated on the network in Figure 1*a* (with given means and unit variance; independent travel times; $T_0 = 4$). Here, path **S**-A-B-**T** is the most reliable between **S** and **T**, whereas subpath **S**-A-B of this path is not optimal to node B (Figure 1*b*). The failure of subpath optimality can also be shown for the case with correlations and unequal variances.

PROPOSED ALGORITHM FOR COMPUTING ORP

In view of the failure of the subpath optimality property, an algorithm to compute the ORP is proposed that draws on principles of bounding, simulation, and network optimization. A brief overview of the proposed approach is first outlined followed by a detailed algorithm description, illustration on an example network, and analysis of computational complexity.

Rationale and Overview

For the general case of correlations and unequal variances, an algorithm is proposed for the ORP problem based on convergence of lower and upper bounds on reliability. The proposed approach involves computing the K shortest expected time paths, on the basis of which are estimated a lower bound on optimal path reliability and a series of progressively decreasing upper bounds on path reliability (of the K paths). Subsequently it is shown that when the lower bound (LB) on optimal path reliability exceeds the smallest of the upper bounds [minimum UB (min UB)], then the most reliable path is contained in the K-path set. If the sufficient condition does not hold, then an efficient path generation procedure is proposed to identify paths of maximum conditional reliability given a set of random draws on the correlated component of the link travel time vector. These paths are augmented to the K set, and the path generation procedure is iteratively repeated until either the bounds converge or a prespecified number of Monte Carlo draws have been completed. The draws can be chosen to limit the probability of the optimal solution falling outside the K set.

Algorithm Description

The key steps of the proposed algorithm include

1. Determine K shortest expected time paths, and evaluate path reliability. Hence, compute LB (see section on computation of shortest expected time paths).
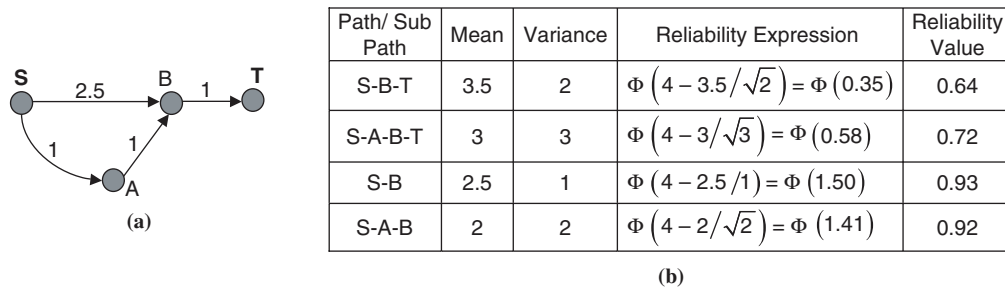

Path/ Sub Path	Mean	Variance	Reliability Expression	Reliability Value
S-B-T	3.5	2	$\Phi\left(4 - 3.5/\sqrt{2}\right) = \Phi\left(0.35\right)$	0.64
S-A-B-T	3	3	$\Phi\left(4 - 3/\sqrt{3}\right) = \Phi\left(0.58\right)$	0.72
S-B	2.5	1	$\Phi\left(4 - 2.5/1\right) = \Phi\left(1.50\right)$	0.93
S-A-B	2	2	$\Phi\left(4 - 2/\sqrt{2}\right) = \Phi\left(1.41\right)$	0.92

(b)

FIGURE 1   Illustration of failure of subpath optimality: (*a*) example network and (*b*) failure of subpath optimality.

2. Determine lower bound on minimum path variance (see section on computation of a lower bound on minimum path variance), and compute path reliability upper bounds. Hence, compute min UB (see section on computation of minimum path reliability UB for K-path set).

3. Perform optimality–termination check (see section on sufficient condition for optimality of ORP).

If LB $\geq$ min UB or number of iterations $l > L$, then optimal reliability estimate = LB; terminate.

Else (path generation—see section of Monte Carlo–based path generation procedure).

– Perform a random draw of the multivariate normal component of the link travel time vector.

– Compute path of maximum conditional reliability P_{opt} (reliability R_{opt}) by using the proposed algorithm for the independent and identically distributed (i.i.d.) problem.

– Update lower bound: if $R_{opt} >$ LB, set LB = R_{opt}.

– Set $l = l + 1$, and repeat Step 3.

Computation of Shortest Expected Time Paths

Because the mean path travel time is one determinant of reliability, paths with small expected travel times are potential candidates for the ORP. However, because the shortest expected time paths may not be reliable, a set of K shortest expected time paths is considered that must contain the optimal path for some K, although in the worst case this requires enumerating all paths. The K shortest expected time paths problem is solved by using the standard label correcting algorithm (15). A lower bound on optimal reliability is the largest path reliability from the K-path set.

Computation of a Lower Bound on the Minimum Path Variance

An upper bound on the actual reliability R'_P of any path P can be obtained by substituting path variance with the minimum path variance on the network σ^2_{min} (for the given O-D pair). The main idea here involves decomposition of the variance objective into an equivalent sum-of-squares problem (i.e., a minimum cost norm problem) involving a multiobjective cost vector. For this, the Cholesky decomposition of the variance–covariance matrix is performed yielding Cholesky matrix $A = [a_{ij}]$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$. Path variance can be expressed as a sum of link separable components based on the Cholesky coefficients (Equation 3) and consequently can be restated as the norm or sum of squares of m cost elements (Equation 4). Thus, the problem of computing the minimum variance path is equivalent to finding the minimum cost norm path on a network with multiple (m) objectives.

$$\sigma^2_P = \left(\sum_{i \in P} a_{i1}\right)^2 + \left(\sum_{i \in P} a_{i2}\right)^2 + \cdots + \left(\sum_{i \in P} a_{im}\right)^2 \qquad (3)$$

$$= \left(\sum_{k=1}^{m} q^2_{pk}\right) \text{ where } q_{p1} = \sum_{i \in P} a_{i1}, \ldots, q_{pm} = \sum_{i \in P} a_{im} \qquad (4)$$

Unfortunately, the minimum cost norm problem is difficult to solve exactly because it is a multiobjective problem. Therefore, a lower bound on σ^2_{min} that can be easily computed is used instead. For that

purpose, consider the problem of minimizing $Z = (\sum_{k=1}^{m} q_{pk})^2$ (the square of the sum of Cholesky coefficients), which is referred to as the minimum sum objective problem. This problem can be solved by using shortest path algorithms because it involves a single objective (link cost equals sum of Cholesky coefficients for that link, $\sum_j a_{ij}$). Although optimizing the minimum sum objective (optimal path denoted P_{sum}) does not minimize variance per se, studies show that it gives a fairly tight upper bound (typically within 0.5%) on minimum variance (16).

If λ represents the maximum value of $(\sigma_{Psum}/\sigma_{min})$ across various instances, then $\sigma_{min} \geq (1/\lambda) \sigma_{Psum}$. Thus, $\sigma^* = c\sigma_{Psum}$ is a lower bound on σ_{min} whenever the constant $c < (1/\lambda)$. Empirical evidence that is consistent with the authors' own observations, indicates that a value of $c = 0.75$ provides a valid lower bound for most real networks (16).

To summarize, the proposed procedure to estimate a lower bound on minimum path variance σ^2_{min} involves (a) Cholesky decomposition of the travel time covariance matrix, (b) formulation and solution of the min sum path problem, and (c) computation of a lower bound $\sigma^* = c\sigma_{Psum}$ by applying a suitable scaling factor c to standard deviation of the optimal min sum path.

Computation of a Minimum Path Reliability UB for the K-Path Set

By using this lower bound on σ_{min}, an upper bound for path reliability is computed for each path in the K-path set, and the smallest of these path bounds is taken as minimum UB (min UB). Because the mean travel times in the K-path set progressively increase, the corresponding upper bounds on path reliability progressively decrease. Consequently, the smallest upper bound on path reliability in this set corresponds to the Kth path.

Sufficient Condition for Optimality of ORP

When the lower bound on optimal path reliability exceeds the smallest of the upper bounds (min UB), then the most reliable path is contained in the K-path set. This follows because the path upper bounds progressively decrease. Therefore, min UB must be larger than all upper bounds of paths outside the set, which in turn are larger than the corresponding actual path reliabilities outside the set. Thus, a sufficient condition for finding the ORP is, if LB $\geq$ min UB in the set above, then the most reliable path is contained in the set. If this condition is satisfied, optimality is guaranteed and the algorithm terminates. However, the optimality condition above is sufficient, but not necessary. In other words, if LB < min UB, then either optimality is achieved but is not verifiable because of loose upper bound, or optimality is not achieved in the given K set.

Monte Carlo–Based Path Generation Procedure

If the optimality criterion above is not met, a Monte Carlo–based path generation procedure is proposed to generate new candidate paths that may be more reliable than the K-path set. The proposed path generation procedure involves (a) decomposing the travel time variance–covariance matrix into a correlated component and an independent component and generating several Monte Carlo draws (vectors) of the correlated component to produce conditionally independent instances of link travel time distributions and (b) for each of the conditionally independent instances, determining the condi-

tionally most reliable path by using a network optimization procedure. The conditionally most reliable paths constitute the additional potentially reliable candidate paths and are likely to be generated frequently with increasing draws.

Error Components Scheme for Decomposing the Travel Time Distribution

The link travel time vector ($\mathbf{t}$) is decomposed into two components: a correlated component across links ($\mathbf{z}$) and an i.i.d. component ($\mathbf{\eta}$) across links.

$$t_i = \mu_i + z_i + \eta_i \text{ or } \mathbf{t} = \mathbf{\mu} + \mathbf{z} + \mathbf{\eta} \text{ for the vector of link travel times}$$

(5)

where

$$\mathbf{t} \sim \text{MVN}(\mathbf{\mu}, \mathbf{\Sigma}_T),$$
$$\mathbf{z} \sim \text{MVN}(\mathbf{0}, \mathbf{\Sigma}_Z), \text{ and}$$
$$\mathbf{\eta}_i \sim \text{N}(0, \sigma^2).$$

The major advantage of this decomposition is that given a draw of the correlated component vector $\mathbf{z}$, the resulting conditional link travel times ($\mathbf{t}|\mathbf{z}$) are i.i.d. across links with variance σ^2.

Label Correcting Algorithm to Compute the Conditionally Most Reliable Path for the i.i.d. Case

On the basis of the error components scheme, several instances (L) of the multivariate correlated component $\mathbf{z}$ are randomly drawn from its distribution (Equation 5). Next, the path of maximum conditional reliability is computed for each instance (draw) of the conditional link travel time vector $\mathbf{t}|\mathbf{z}$. The i.i.d. nature of the conditional travel time vector makes it easier to solve the ORP problem for this case. The following modified label correcting algorithm is proposed.

The path variance in the i.i.d. case (with link variance σ^2) is simply σ^2 times path length (number of arcs). Consequently, the ORP problem objective (Equation 2) reduces to

$$R(\mathbf{x}) = \Phi\left(\frac{T_0 - \sum_{(i,j)\in A} x_{ij}\mu_{ij}}{\sigma\sqrt{\sum_{(i,j)\in A} x_{ij}}}\right)$$

(6)

Paths with equal length have equal variance. Hence, among paths of equal length (number of arcs), the path with maximum reliability is the one with the least expected time.

With this principle, the conditionally most reliable path is solved in two stages for each draw: first, the most reliable path and corresponding reliability values R_a^* are determined for every feasible path length a (number of arcs = 1, 2, . . . , up to a maximum of $n-1$). This is done by using a modified label correcting procedure in which a set of up to $n-1$ labels are maintained at each node i. The kth label ($k = 1, 2, \ldots, n-1$) of node i, $d_k(i)$ represents the current estimate of minimum mean travel time (and hence maximum reliability) to node i among subpaths with exactly k arcs. The optimality condition can be stated as

$$d_k(j) \le d_{k-1}(i) + \mu_{ij} \qquad \forall (i,j)\in A; \quad k = 2, \ldots, n-1$$

(7)

Ties between multiple subpaths of equal length may be broken arbitrarily. At optimality, the maximum reliability for a given path length a, R_a^* is computed from the label of destination node t as

$$R_a^* = \Phi\left(\frac{(T_0 - d_a(t))}{\sigma\sqrt{a}}\right)$$

where σ^2 is the variance of the i.i.d. components (η_i in Equation 5). Next, the optimal path across all path lengths is found as the path with the largest reliability from the first stage across varying number of arcs a, and thus optimal path reliability $R^* = \max_a R_a^*$. Thus, the path of maximum conditional reliability is computed for each draw.

With probability theory, it can be shown that the probability of not encountering the true optima in L random draws (δ) decreases as per the following equation:

$$L = \left(\frac{\log\delta}{1 - \left(pz_{a/2}\sqrt{\frac{p(1-p)}{R}}\right)}\right)$$

(8)

where p is an estimate (obtained from R draws) of the probability that the path of maximum conditional reliability coincides with the true optimum on a single draw, and the confidence level is $100(1-\alpha)$.

Illustrative Example

The proposed algorithm is illustrated on the simple example network shown (mean, variance in Figure 2a) for origin node **1** and destination node **5.** The O-D threshold is assumed to be 12, and for simplicity, links are assumed to be equi-correlated with correlation coefficient $\rho = 0.25$. The results of each step are summarized in Figure 2c.

Computational Complexity

The proposed ORP algorithm involves (a) identification of K shortest expected time paths requiring $O[Kn(m + n\log n)]$ computations (O = order of) (*14*); (b) computation of lower bound on variance requiring $O[m^3] + O[m] + O[n^2]$ computations for the Cholesky decomposition and computation of arc level min sum costs and the corresponding shortest path; and (c) path generation procedure requiring $O[mn(n-1)]$ computations per iteration. Thus, the overall computational complexity of the proposed algorithm is given by $O[Kn(m + n\log n)] + O[m^3 + n^2 + m] + O[Lmn(n-1)]$. The algorithm is polynomial in m and n, but may increase significantly depending on K and L (pseudopolynomial).

COMPUTATIONAL EXPERIENCE

This section reports the results from a set of computational experiments on different synthetic and real networks. The computational experiments are intended to

1. Study the effect of choice of parameters L and K on algorithm performance,
2. Investigate algorithm performance with increasing network sizes and densities, and
3. Analyze the performance of the ORP in relation to other benchmarks for a real-world network with empirical travel time data.

(a)
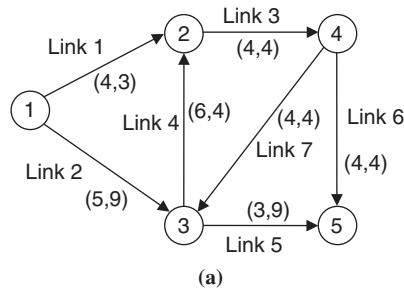
Arc	Draw				
	1	**2**	**3**	**4**	**5**
1-2	2.78	3.79	2.24	5.54	1.85
1-3	4.03	1.02	9.56	16.04	3.81
2-4	3.91	6.10	0.31	2.57	5.35
3-2	7.93	4.58	4.93	5.13	2.68
3-5	0.10	8.42	4.02	11.04	2.66
4-5	4.45	6.33	3.70	1.53	4.59
4-3	1.61	4.44	6.02	3.82	4.64

(b)

Algorithm Step	Result
1. Lower bound computation ($K = 1$)	$K = \{1\text{-}3\text{-}5\}$; L.B $= R_{1\text{-}3\text{-}5} = \mathbf{0.80}$

2. Computation of lower bound on σ_{min}
   a. Cholesky decomposition
   b. Compute P_{sum}, $\sigma_{P_{sum}}$
   c. Compute $\sigma^* = c\sigma_{P_{sum}}$
	a. min sum objective costs
	
	b. P_{sum}: path 1-2-4-5, $\sigma^2_{P_{sum}} = 16.46$
	
	c. $\sigma^* = 0.75 \times \sqrt{16.46} = 3.043$

3. Computation of min UB ($c = 0.75$)	min UB $= \mathbf{0.905}$
4. Path generation ($L = 5$)	Paths of maximum conditional reliability 1. 1-3-5 4. 1-2-4-5 2. 1-3-5 5. 1-3-5 3. 1-2-4-5
5. Termination	Max reliability $= 0.80 =$ LB Optimal path 1-3-5; reliability $= 0.80$

(c)

FIGURE 2   Illustrative example: (a) example network, (b) mean conditional link travel times for $L = 5$ draws, and (c) stepwise results for illustrative example.

The experimental factors for the first two experiments are summarized below:

- Five randomly generated synthetic networks: size (500 or 1,000 nodes); density (1, 3, 5 links per node)
- Parameter levels: K paths (0/5/10/15); L draws (25/50/100)
- Network link attributes: mean travel times (random 5–15 units), variance (random 0–100 units^2), and correlations (random 0–1)
- Performance measures: average values of (a) computational time(s) and (b) accuracy (percent deviation from true optimum) from a series of 10 runs for each configuration
- Benchmark models
  – Bounding heuristic: computation of K shortest expected time paths where K is iteratively doubled until convergence of bounds
  – Conditional reliability optimization heuristic: algorithm with $K = 0$ (purely draw-based simulation procedure—see section on Monte Carlo–based path generation procedure)
- Benchmark solutions
  – True optimum (computed by using bounding heuristic)
  – Optimal reliability path assuming independence (ORP-I)

Effect of Algorithm Parameters *K* and *L*

Results indicate that the proposed algorithm is accurate even for very small values of K and L (see Tables 1 and 2). With as few as five K-shortest paths and 25 draws, the optimal path is found in four of the five networks, whereas, with five paths and 50 draws the optimal path is found in the remaining network (Network 4). The computational time is quite acceptable (59 s) for realistic networks (1,000 nodes, 3,000 links) and a general correlation pattern. Further, the computational time increases as K and L increase, with a larger increase noted for L for a given network size. Across network sizes for a given number of draws, the effect of K becomes more significant, particularly as the number of links increases (Networks 3 and 5). Thus, K-shortest path computations tend to be more expensive for larger networks.

Results support the conjecture that the K-best expected time paths serve as a good initial candidate set, but are not necessarily optimal. (For example, in Network 4, the optimal path is outside the $K = 15$ least-expected time paths.) Furthermore, the effectiveness of augmenting the K-path set with conditionally reliable paths is also observed. In Networks 3 and 5, with $K = 5$ and 25 augmented draws, the optimal path is identified in each of 10 sets of replications. Also, in Network 4, increasing K from 5 to 15 does not find the optimal solution, while even with $K = 5$, increasing L from 25 to 50 draws ensures optimality. In this network, the optimal path lies outside $K = 15$ and yet is consistently identified as the conditionally most reliable path within a set of 50 draws.

Effect of Network Size and Density

The examination of algorithm performance for varying network sizes (nodes) and densities (links) and comparisons with the bound-

TABLE 1   Effect of Parameters *K* and *L* on Algorithm Performance

K	L	Network 1 (500 nodes, 625 links)		Network 2 (500 nodes, 1,530 links)		Network 3 (500 nodes, 2,515 links)		Network 4 (1,000 nodes, 1,250 links)		Network 5 (1,000 nodes, 3,055 links)	
		Time (s)	% Deviation	Time (s)	% Deviation	Time (s)	% Deviation	Time (s)	% Deviation	Time (s)	% Deviation
0	25	2.34	1.73	12.08	3.23	28.12	9.39	8.92	4.50	55.56	4.08
0	50	2.81[a]	0.00[a]	15.17	1.38	34.24	7.91	9.532[a]	0.00[a]	58.27	4.08
0	100	3.75	0.00	21.37[a]	0.92[a]	46.17[a]	0.00[a]	10.79	0.00	70.99[a]	0.00[a]
5	25	2.35[b]	0.00[b]	12.10[b]	0.00[b]	29.32[b]	0.00[b]	8.94	4.50	58.83[b]	0.00[b]
5	50	2.81	0.00	15.22	0.00	35.44	0.00	9.55[b]	0.00[b]	61.94	0.00
5	100	3.75	0.00	21.49	0.00	46.77	0.00	10.80	0.00	72.81	0.00
10	25	2.35	0.00	12.13	0.00	30.88	0.00	8.97	4.50	62.33	0.00
10	50	2.82	0.00	15.52	0.00	36.43	0.00	9.58	0.00	65.97	0.00
10	100	3.76	0.00	21.95	0.00	47.11	0.00	10.66	0.00	75.38	0.00
15	25	2.35	0.00	12.20	0.00	31.75	0.00	8.98	4.50	66.19	0.00
15	50	2.81	0.00	15.89	0.00	36.41	0.00	9.60	0.00	69.14	0.00
15	100	3.76	0.00	22.79	0.00	47.29	0.00	10.66	0.00	79.89	0.00

[a]Value for conditional reliability optimization heuristic used in comparison.
[b]Algorithm value used for comparison with bounding heuristic and conditional reliability optimization heuristic.

TABLE 2   Comparison of Bounding Heuristic with Proposed Algorithm

Performance Measure	Network 1	Network 2	Network 3	Network 4	Network 5
K for convergence	10	20	40	40	40
% diff. in time	34.5	45	66.3	38.4	142.2
% diff. in accuracy	0	0	0	0	0

ing heuristic and conditional reliability optimization heuristic indicated the following:

• The number of network links has a more significant effect than the number of network nodes. Computational time increases 13-fold (as links increase from 625 to 2,510–Networks 1 and 3), whereas as nodes increase from 500 to 1,000, the increase is 4.6 times (Networks 2 versus 5).

• Compared with the bounding heuristic, the proposed algorithm (parameters chosen to ensure 100% accuracy) is computationally more efficient (34% to 142% faster). This illustrates the efficiency of the conditional path reliability optimization scheme, which leads to the enumeration of fewer paths.

• The performance of the ORP algorithm (bounds + path generation) is compared against the pure path generation procedure. The proposed algorithm is computationally more efficient (20% to 75%) in four of the five networks, suggesting the need to combine bounds and path generation procedures in many networks. In contrast, in Network 4, the algorithm is marginally slower (by 0.19%) than the Monte Carlo–based conditional reliability-based optimization algorithm.

Empirical Application

In this section the proposed algorithm is applied to a network (33 nodes, 98 links) of major roads in Chennai, India (Figure 3). The coefficients of variation of link travel time (based on empirical data) are consistent with those (0–0.7) observed in other urban transporta-

tion networks (*13*). The correlation coefficient between adjacent links is taken as 0.75, and correlation between nonadjacent links is taken as zero. The travel time threshold is chosen to be 1.5 times the shortest expected time for the corresponding O-D pair (allowing for 50% buffer over the least expected time due to various sources of uncertainty). The optimal travel time reliability measure is compared with the least expected time path for five randomly selected O-D pairs on the basis of mean, variance, and reliability of path travel times.

The results illustrate that minimizing expected travel time (LET path) alone may result in choice of unreliable paths (three out of five O-D pairs). In contrast, maximizing reliability results in a modest increase in mean travel time (between 0.91% and 9.29% compared with LET path). However, the decrease in standard deviation is quite substantial (17.5% to 75%) resulting in a reliability improvement of 5% to 35%.

The effect of assuming independence was investigated by comparing actual path characteristics of the ORP-I against the true optimum. The optimal path ORP-I was found to be different from the true optima (ORP) in three of the five O-D pairs, all of which result in choice of suboptimal or unreliable paths. The difference in reliability of ORP-I from true ORP ranges between 11% and 17%, and the mean travel times were suboptimal by between 3% and 4.5%. The standard deviations in the independent case were higher by 15% to 25% than the optimal reliability solutions. Thus, the assumption of independence may lead to suboptimal results in some cases.

The comparison was also performed by systematically varying the range of threshold values T_0 (1.1, 1.25, 1.5, 1.75, and
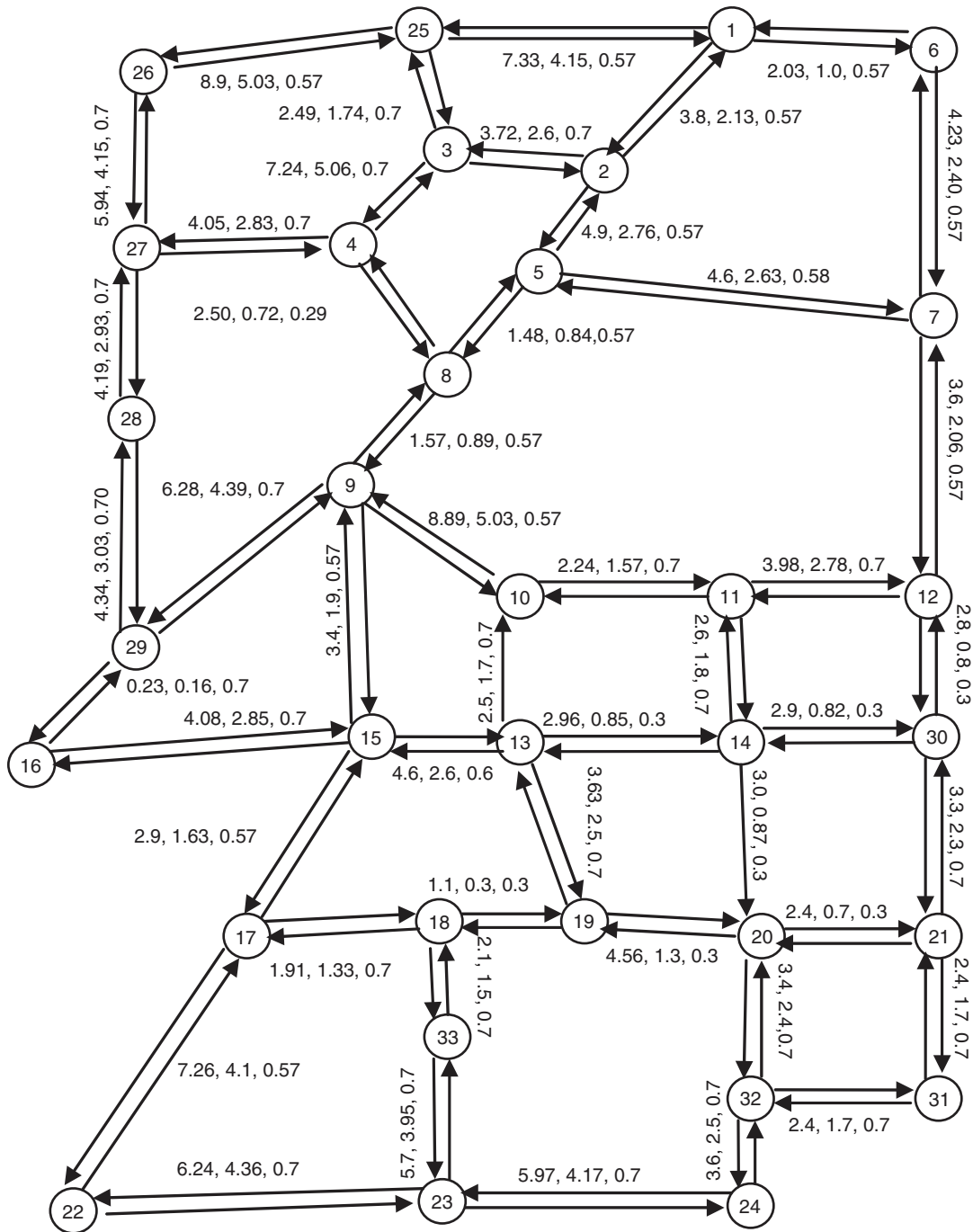
FIGURE 3   Chennai road network topology.

2.0 times the least expected time T_{min}). The threshold T_0 represents the total time budgeted by a commuter to arrive before a preferred arrival time. This includes the expected travel time for the journey plus an extra buffer to account for unexpected delays and congestion. The threshold is a measure of commuters' risk-taking propensity with low thresholds representing risk-seeking (or risk-prone) commuters and high thresholds representing risk-averse commuters.

In general as the threshold value increases (quantified by ratio of T_0 to T_{min}), the improvement in reliability initially increases and then decreases (Figure 4). The point of maximum improvement varies

between 1.25 and 1.5 times T_{min} for different O-D pairs. Significant variation is seen in the extent of reliability improvement (relative to the least expected time path) for different O-D pairs. The improvement is quite significant (20% to 40%) for most O-D pairs, with the exception of O-D Pair 2 (in which maximum improvement is below 5%). That is attributable to the differences in the coefficient of variation and expected travel times between the ORP and least expected time path for these O-D pairs.

In contrast to the general trend of small reliability improvement at low and high thresholds, significant improvement is possible in some cases. Notable exceptions include O-D Pairs 1 and 4, which
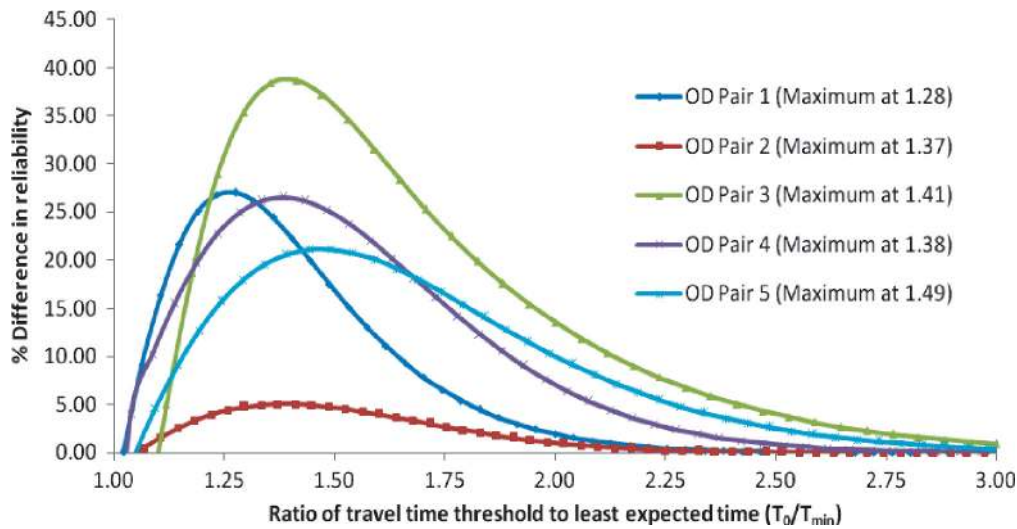
FIGURE 4   Effect of threshold on percentage of reliability improvement.

show improvement of 16% and 10% at $T_0 = 1.1T_{min}$. This is possible when the shortest expected time path has a moderate coefficient of variation (0.48 and 0.51 for O-D Pairs 1 and 4) that is significantly higher than that of the ORP (60% and 45% higher for O-D Pairs 1 and 4). However, O-D Pairs 3 and 5 show improvement of 13.5% and 10.2% at $T_0 = 2T_{min}$, illustrating that a large improvement in reliability is possible even at high thresholds. This occurs when the shortest expected time path has a high coefficient of variation (0.66 and 0.63 for O-D Pairs 3 and 5) that is again, significantly higher than the ORP (77% and 56% higher for O-D Pairs 3 and 5).

These findings have the following implications: the largest improvements in reliability are attained at moderate thresholds ($1.25–1.5T_{min}$), which correspond to uncertain system performance and moderately risk-averse travelers. To the extent that the threshold is a measure of risk aversion, benefits of the ORP tend to be lower for both risk-seeking and extremely risk-averse travelers. For some O-D pairs, and threshold values, the least expected time path may serve as a good proxy for reliability, whereas for many others the ORP may provide significant reduction in variability and improvement in reliability. The magnitude of reliability improvement depends on the differences in coefficient of variation and mean travel time between the shortest expected time path and ORP, as well as the threshold for reliability.

CONCLUSIONS

This paper proposes a new bounds-based optimality criterion for the ORP problem. On the basis of the bounds, an algorithm is proposed to find the path with maximum travel time reliability on a network with stochastic, normal, and correlated link travel times. Computational experiments demonstrate the accuracy and efficiency of the proposed approach and its applicability to realistic network sizes. The computational time is affected by the number of K shortest paths, number of draws, and network size. For realistic network sizes, it is found that as few as five paths and 50 draws are sufficient to find the ORP. Further, the proposed algorithm is computationally faster than both a bounding heuristic (involving doubling

K-shortest expected time paths until the bounds converge) and a pure Monte Carlo draw based simulation procedure (conditional reliability optimization heuristic).

In addition, experiments on a real-world network (Chennai) using empirical travel time data yielded the following insights:

• Existing practice of using shortest expected time paths can lead to the choice of highly unreliable paths, whereas using travel time reliability can yield significant improvements in reliability (more than 35%) without excessively compromising on mean travel time (less than 10%).

• Assuming independence of link travel times can result in the choice of suboptimal paths with a significant compromise on path reliability (between 11% and 17% from the ORP).

• Results also underscore the role of risk attitudes on benefits due to reliability-based optimization. The maximum benefits in regard to an improvement in reliability are obtained by moderately risk-averse travelers.

• The magnitude of maximum reliability improvement depends on the threshold value and differences in coefficient of variation and mean travel time between the ORP and the shortest expected time path.

The proposed algorithm will have important applications in the context of congestion mitigation and ITS applications related to route guidance on networks with significant travel time uncertainty. Potential directions for future research include extending the proposed algorithm to incorporate nonnormal distributions and routing in stochastic and dynamic networks. Another interesting line for further work is the development of reliability-based algorithms for the context of routing with recourse under stochastic travel times.

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