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Acoustic-hydrodynamic-flame coupling—A new perspective for zero and low Mach number flows

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A combustion chamber has a hydrodynamic field that convects the incoming fuel and oxidizer into the chamber, thereby causing the mixture to react and produce heat energy. This heat energy can, in turn, modify the hydrodynamic and acoustic fields by acting as a source and thereby, establish a positive feedback loop. Subsequent growth in the amplitude of the acoustic field variables and their eventual saturation to a limit cycle is generally known as thermo-acoustic instability. Mathematical representation of these phenomena, by a set of equations, is the subject of this paper. In contrast to the *ad hoc* models, an explanation of the flame-acoustic-hydrodynamic coupling, based on fundamental laws of conservation of mass, momentum, and energy, is presented in this paper. In this paper, we use a convection reaction diffusion equation, which, in turn, is derived from the fundamental laws of conservation to explain the flame-acoustic coupling. The advantage of this approach is that the physical variables such as hydrodynamic velocity and heat release rate are coupled based on the conservation of energy and not based on an *ad hoc* model. Our approach shows that the acoustic-hydrodynamic interaction arises from the convection of acoustic velocity fluctuations by the hydrodynamic field and vice versa. This is a linear mechanism, mathematically represented as a convection operator. This mechanism resembles the non-normal mechanism studied in hydrodynamic theory. We propose that this mechanism could relate the instability mechanisms of hydrodynamic and thermo-acoustic systems. Furthermore, the acoustic-hydrodynamic interaction is shown to be responsible for the convection of entropy disturbances from the inlet of the chamber. The theory proposed in this paper also unifies the observations in the fields of low Mach number flows and zero Mach number flows. In contrast to the previous findings, where compressibility is shown to be causing different physics for zero and low Mach number flows, we show that the heat release rate can also introduce the distinction between the zero and low Mach number flows. Therefore, during the thermo-acoustic interaction, we should consider the coupling of convection modes that arise from the interaction of acoustic and hydrodynamic fields with the entropy modes. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4981784]

I. INTRODUCTION

Self sustained acoustic pressure and velocity oscillations are often encountered in gas turbine combustors and rocket motors.[1] These phenomena, known as thermo-acoustic oscillations, are a result of a positive feedback between the acoustic and the reacting flow fields and have been recently studied as arising from the wave-mean flow interaction.[2] Acoustic waves derive energy from the reacting flow field. Unsteady heat release rate drives the acoustic field in a combustor through dilatation.[3] Other sources of acoustic energy, such as those arising from the entropy field-convection interaction, also known as the mixed mode, are also considered to be determining factors of thermo-acoustic instability.[4] Such a mode is found in non-zero Mach number flows. Zero Mach number flows are described by "purely incompressible" flows,[5] where dilatation is zero.[6] The convection of entropy fluctuations, i.e., with non-zero velocity, makes the zero Mach number assumption invalid in the study of low Mach number reacting flows.[7]

In a mathematical treatment of low Mach number flows, the effects of compressibility were analyzed by Klainerman and Majda.[6] They justified the use of linearized acoustic equations for treating the deviations from the zero Mach number assumption. However, various investigations on the acoustic instability in solid rocket motors propose the existence of a "gas dynamic nonlinearity," expressed either as $u \cdot \nabla u$ or $u' \cdot \nabla u'$,[8–10] where u' and u are the acoustic and hydrodynamic fields, respectively. Apart from the nonlinear terms, there could be linear terms of the form $u \cdot \nabla u'$ and $u' \cdot \nabla u$, contributing to the evolution of the acoustic velocity field.[7] These linear terms could be the governing factors of low Mach number flow physics, common to thermo-acoustic systems. However, an investigation on the influence of linear convection operator terms, in governing the evolution of the acoustic field, is currently lacking.

An additional complexity that arises from the inclusion of $u' \cdot \nabla u$ and $u \cdot \nabla u'$ terms is that the hydrodynamic and acoustic time scales are not comparable. One method to circumvent this difficulty is to average the oscillations on the acoustic time scale.[3] Application of the method of multiple scales (MMS) in reacting flows[2] and the subsequent determination of a common time scale[11] is another method. The latter

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method is advantageous, as a common time scale implies a suitable velocity scale, for determining the evolution of the acoustic field.[12] Apart from different time scales, there are different length scales that characterize the thermo-acoustic instability. These length scales correspond to the compact heat source and the long wavelength acoustic wave.[13–16] A relevant length scale where the acoustic-hydrodynamic interaction occurs is the convective length scale.[2] On these scales, a convection-reaction-diffusion (CRD) equation governing the acoustic-hydrodynamic interaction was derived by Pulikkottil and Sujith.[2]

The choice of suitable scales is necessary to resolve the flame-hydrodynamic-acoustic coupling. Various investigators, for representing this coupling, have developed a relation that connects the acoustic velocity, before and after the flame.[13–15] Wu[15] suggested that this jump is proportional to the heat release rate. The length scale of the hydrodynamic region is assumed to be small compared to the length of the duct.[15,17] However, in practical combustors, the hydrodynamic region extends throughout the length of the combustor, and the flow field convects the entropy disturbances to the flame. Therefore, we could use the convective length and time scales to represent the acoustic velocity variation across the flame. We have to determine whether such a formulation captures the influence of heat release rate on the acoustic velocity variation. Further, if the convective length scale is used, we could assess the influence of factors such as convection on the coupling of the acoustic and hydrodynamic fields. These factors could contribute to exciting the acoustic field in the combustors.

Different sources contributing to the acoustic disturbances act on different scales.[11] However, entropy spots, from the local burning of mixture entrapped in the vortices, are convected by the base flow through the nozzle.[18] The non-uniform base flow that arises from the vortices[19,20] in the flow, from wrinkling of the flame[21] or from the velocity fluctuations, is governed by the convection time scale. Non-uniform base flow causes the coupling of vorticity and entropy modes. The convective velocity, therefore, determines the time scale associated with the vorticity and entropy modes. The mixed convection-entropy mode is a major source for the acoustic field. The frequencies associated with the acoustic-entropy mode, in a combustor nozzle, are experimentally identified by Motheau, Nicoud, and Poinsot.[18] How does the coupling of convection, acoustic, and entropy modes occur in the combustion chamber? This coupling, if possible, could arise from the non-uniform flow field and from the localized heat release rate. A mathematical representation of this phenomenon requires a common time scale, i.e., convective time scale.

When there is acoustic-hydrodynamic interaction, how does the acoustic, hydrodynamic, and entropy modes on various time scales couple with each other? A detailed description of the possible mechanisms coupling the flame with the acoustic, hydrodynamic, and entropy fields is shown in Fig. 1. The presence of the acoustic field implies compressibility in a low Mach number flow. Zank and Matthaeus[5] and Dastgeer and Zank[22] called such a flow as a "nearly incompressible" flow. When we consider this compressibility, the convective terms such as $u' \cdot \nabla u$ and $u \cdot \nabla u'$, even when they are linear, can become a major source for the acoustic field. To what extent

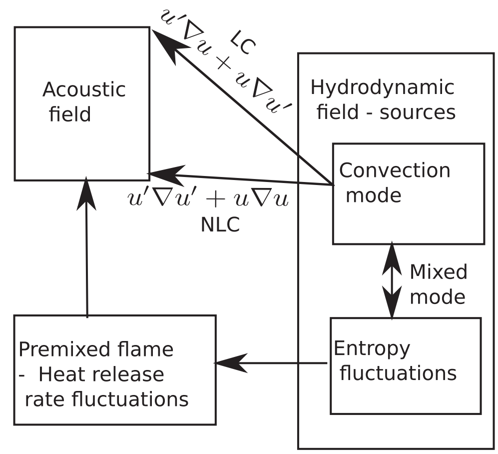

FIG. 1. Possible mechanisms that lead to the coupling of the flame with the acoustic, hydrodynamic, and entropy fields. Both the linear coupling (LC), discussed in this paper, and the non-linear coupling (NLC) are indicated. Heat release rate fluctuation serves as a source for the acoustic field. Hydrodynamic field causes both the convective mechanism and the entropy fluctuations, resulting from the non-uniform base flow, to couple and contribute to the acoustic field as a mixed mode.

does this convective mechanism influence the physics of the low Mach number reacting flow? We will show that this linear mechanism is essential in exciting the acoustic field in a combustor. We show that this mechanism is also responsible for the coupling of density disturbances at the inlet to the acoustic field in the combustion chamber. These density disturbances could lead to localized unsteady heat release rate. Therefore, density disturbances represent the mechanism through which entropy modes exist in a combustor. In this paper, these entropy modes are coupled to acoustic and hydrodynamic fields. Such a coupling is required, in modeling of the acoustic field across the flame, to ensure that the mass conservation law is not violated.[23] An acoustic-hydrodynamic interaction is shown here as arising from a linear convective mechanism, also known as the Reynolds forces in the study of nearly incompressible flows.[22]

The linear convective terms $u' \cdot \nabla u$ and $u \cdot \nabla u'$, appearing in the linearized Navier-Stokes equation, was found to be a non-normal by various researchers.[24,25] The non-normal growth of instability modes in the hydrodynamical systems[26,27] and in the thermo-acoustic systems[28] were recently investigated. In hydrodynamical systems, the existence of the transient non-normal growth of instability modes has been investigated, as arising from the influence of fluid flow on the instability modes. In thermo-acoustics, the influence of mean flow on the non-orthogonality of eigenmodes has been suggested by Nicoud and Wieczorek.[7] However, their work focuses on the nonlinear growth of acoustic amplitudes. The physical mechanism through which these linear convective terms contribute to the acoustic modes in a thermo-acoustic system is not known.

In this paper, we propose the variation of acoustic velocity amplitude across the flame, as arising from linear convective terms. These terms, defined on the convective time scale, represent the Reynolds forces in nearly incompressible flows. Therefore, the convective length and time scales are the relevant scales for representing the acoustic-hydrodynamic interaction. Dastgeer and Zank[22] had studied the effect of the

Reynolds forces in the excitation of nearly incompressible magnetohydrodynamic (MHD) turbulence flows. Our focus here is the excitation of the acoustic velocity field in the thermo-acoustic instability. In thermo-acoustic systems, u' is of the order ϵ, a small number proportional to the Mach number. The smallness of ϵ is a consequence of the low Mach number of the nearly incompressible flows. We will explain the present understanding of the acoustic-hydrodynamic interaction, in terms of the acoustic velocity "jump" and as arising from the convective influence of acoustic and hydrodynamic fields on each other. We show that the entropy disturbances arising from the second order density fluctuations and the convection of the fluid flow are significant in exciting the acoustic field in a thermo-acoustic system. Using a convection-reaction-diffusion equation (CRDE), we show that the variation in the acoustic velocity is proportional to the heat release rate at the flame. In the conventional CRDE, there is only one convection term, i.e., convection of the perturbation field by the base flow. In our CRDE, there are two linear convective terms. Furthermore, we show that the relevant velocity scale that determines the non-normal acoustic-hydrodynamic interaction is the convective velocity. Furthermore, we illustrate how the heat release rate and compressibility are linked to each other through an example.

II. MATHEMATICAL FORMULATION

Method of multiple scales (MMS) was used recently to decompose a flow field into acoustic and hydrodynamic fields.[2] We will include only the significant parts of that analysis with a required modification suitable for the present analysis. This modification is the change in the heat release rate expansion. In this analysis, we include a first order expansion to the heat release rate which will be shown as a source for the acoustic field. Ordering of flow field variables are introduced in an asymptotic expansion as follows:[2]

$$\rho = \rho_0 + \epsilon^2 \rho_2, \tag{1}$$

$$\vec{u} = \vec{u}_0 + \epsilon \vec{u}_1, \tag{2}$$

$$\dot{Q} = \dot{Q}_0 + \epsilon \dot{Q}'. \tag{3}$$

The hydrodynamic and acoustic field variables are expanded in terms of a small parameter ϵ as shown in Eqs. (1) and (2). These expansions are solutions to the compressible fluid flow equations. Decomposition of the field variables into leading order terms, i.e., the terms with subscript "0," and higher order terms is an integral part of any perturbation methods.[29] The order of perturbation variables are chosen to ensure the convergence of solutions.[30] In the previous derivation by Pulikkottil and Sujith,[2] only the leading order heat release rate and the heat release rate fluctuation were considered. The leading order heat release rate was chosen previously to study the temperature-pressure coupling.[31] In the present study, the heat release rate is expanded, further, to first order in ϵ. To investigate the velocity coupling, the heat release rate and acoustic velocity should be of the same order of magnitude,[31,32] i.e., at $\epsilon = \sqrt{\gamma}M$, and hence, further expansion to the first order in ϵ. The Mach number is denoted by M. Even in the presence of a leading order heat release rate fluctuation, this fluctuation will not appear as a source to the acoustic velocity Eq. (5). Therefore, to isolate and study the velocity coupling, we use Eq. (3) for the heat release rate calculations. In low Mach number flows, ϵ is a small number. The leading order terms represent the mean flow field variables. Perturbation variables $\vec{u}_1$ and ρ_2 represent the acoustic velocity and the density fluctuations, respectively. ϵ determines the ratio between the amplitudes of the perturbation variable and the mean flow field, i.e., $\hat{u}_1/\hat{u_0}$. Since $\epsilon \propto M$, the magnitude of $\hat{u}_1/\hat{u}_0$ is determined by the Mach number of the flow.

Space and time scales are expressed as $\tau = \tau'/\epsilon$ and $\eta = \xi/\epsilon$, respectively. τ' and τ are the convective and the acoustic time scales, respectively. η and ξ are the length scales that represent fluid flow and acoustic waves. Solution expansions in Eqs. (1) and (2) are substituted in the continuity and momentum equations for fluid flow[2] and solutions for acoustic field variables can be obtained as $A_i(\eta, \xi, \tau')e^{i\omega\tau}$, where $A_i = (\hat{\rho}_2, \hat{u}_1)$ represent the amplitudes of acoustic velocity and density perturbations, respectively. However, for ensuring the convergence of the solution expansions given in Eqs. (1) and (2), equations for acoustic field variables are expressed in terms of their amplitudes as follows:[2]

$$\frac{\partial \hat{\rho}_2}{\partial \tau'} + \nabla_\eta \cdot (\hat{\rho}_2 \vec{u}_0) = 0, \tag{4}$$

$$\frac{\partial \hat{\vec{u}}_1}{\partial \tau'} + \vec{u}_0 \cdot \nabla_\eta \hat{\vec{u}}_1 + \hat{\vec{u}}_1 \cdot \nabla_\eta \vec{u}_0 = -\frac{1}{\rho_0 Re} \nabla_\eta^2 \hat{\vec{u}}_1 + \dot{Q}'. \tag{5}$$

$\rho_2(\hat{\eta}, \tau')$ and $\hat{\vec{u}}_1(\eta, \tau')$ represent the amplitudes of the perturbation variables on the hydrodynamic length scale. In Eqs. (4) and (5), the derivatives in time and space are with respect to the convective time scale and the flow length scale, respectively. Equation (5) implies that the evolution of the acoustic field on the hydrodynamic length scale can be separated from the evolution on the acoustic length scale[2] ξ. Then, we can express $\hat{u}_1(\eta, \xi, \tau') = \hat{u}_1(\eta, \tau') + \hat{u}_1(\xi, \tau')$. This is achieved by applying MMS to the equations governing compressible fluid flow.[2] In agreement with Nicoud and Wieczorek,[7] Eq. (4) shows that the entropy fluctuation in the form of second order density is propagated at the convective velocity of the mean flow. Equation (5) shows the contribution from both acoustic and fluid flow fields, implying that convective time and length scales are the relevant scales. Therefore, in agreement with the finding of Wu *et al.*,[14] we have the acoustic amplitude evolution on a time scale slower compared to the time scale of the acoustic wave propagation. In Eq. (5), $\dot{Q}'$ appears as a source for the evolution of the acoustic velocity amplitude $\vec{u}_1$. We will show further that this direct coupling causes a linear variation of the acoustic velocity across the flame.

In addition to the unsteady heat release rate-acoustic velocity coupling, we have an acoustic-hydrodynamic coupling. This coupling, represented by $\vec{u}_0 \cdot \nabla_\eta \hat{\vec{u}}_1 + \hat{\vec{u}}_1 \cdot \nabla_\eta \vec{u}_0$, i.e., the convective term in Eq. (5), resembles the non-normal operators discussed by Chomaz[26] and Sipp *et al.*[25] The non-normal interaction between the base flow field and the perturbation field can give rise to instability in hydrodynamical systems.[24] This mechanism could contribute to the linear growth of instability modes. The linear convective term also resembles the Reynolds forces from the field of MHD turbulence.[22] While investigating the hydrodynamic stability of a recirculation bubble, Marquet *et al.*[24] studied the effect of the transportation of base flow by the perturbation field and vice versa. They call this mechanism as convective and lift-up non-normality. Here in the study of thermo-acoustic instability, we show such a linear convection term, representing the coupling of base flow and acoustic fields, resembles the Reynolds forces. Equation (5) is also a modified convection-reaction-diffusion equation (CRDE).

III. ENTROPY-CONVECTION-ACOUSTIC COUPLING

There are various mechanisms which generate the acoustic field from the hydrodynamic field.[33,34] Fluctuations in the momentum of the flow field, when the Mach number is "not negligible,"[33] is one of the causes of the generation of the acoustic field. These flow fluctuations can arise from turbulence.[34] In a low Mach number flow, where the Mach number is not negligible, the magnitude of acoustic field variables is small, i.e., the term $\vec{u_0}\cdot\nabla_\eta\hat{\vec{u}}_1+\hat{\vec{u}}_1\cdot\nabla_\eta\vec{u_0}$ is a linear term. The linear mechanisms play a major role in the evolution of disturbances in hydrodynamics. For example, Chagelishvili *et al.*[35] and George and Sujith[27] had explained the role of non-normality in determining the linear dynamics of shear flow disturbances. Marquet *et al.*[24] investigated the non-normality as responsible for the hydrodynamic instability of a recirculation bubble. Equation (5) contains the mathematical representation of this non-normal phenomenon, appearing as the Reynolds forces term. When the magnitude of $\vec{u}_1$ is of the order $O(\epsilon)$, the term $\vec{u_0}\cdot\nabla_\eta\hat{\vec{u}}_1+\hat{\vec{u}}_1\cdot\nabla_\eta\vec{u_0}$ can cause the coupling of acoustic and hydrodynamic modes, which in turn, causes the fluctuations to grow in time. Here, we put forward such a mechanism that generates the acoustic velocity field of non-negligible amplitude causing the acoustic-hydrodynamic interaction. In addition to acoustic and hydrodynamic fields, entropy disturbances exist from the second order density fluctuations $\hat{\rho}_2$. We propose a mechanism of the acoustic-hydrodynamic interaction, mathematically represented by a non-normal convective operator, that exists in the presence of density disturbances. Through this novel mechanism, we highlight the acoustic-hydrodynamic-entropy coupling. Toward this purpose, an example involving the premixed flame-acoustic-hydrodynamic interaction will be discussed.

In this example, a premixed flame in a duct introduces heat release rate fluctuations. Heat release rate fluctuations may arise from the mass flux fluctuations. McIntosh[36] shows that the mass flux fluctuation can arise from density disturbances. When a density disturbance is introduced in the inlet, the fluid flow transports this disturbance to the flame. In this paper, we have shown the influence of such a density disturbance on the acoustic velocity variation across the flame through Eqs. (4) and (5). The base flow velocity around the flame is obtained by solving Eq. (6), derived from the energy equation.[2] A one dimensional representation of the premixed flame in a duct is shown in Fig. 2. Governing equations for the acoustic-hydrodynamic-flame coupling, i.e., Eqs. (4)–(6), are solved on this 1D grid,

$$\nabla_\eta\cdot\vec{u_0}=\frac{\gamma-1}{\gamma p_0}HDa(\dot{Q}_0)+\frac{\gamma-1}{\gamma p_0 RePr}\nabla^2_\eta T_0-\frac{1}{\gamma p_0}\frac{\partial p_0}{\partial\tau'}s. \quad (6)$$

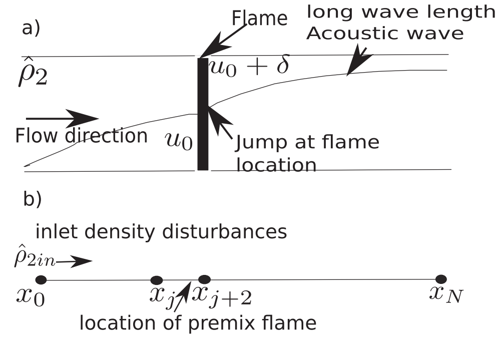

FIG. 2. (a) Illustration of the variation of acoustic and hydrodynamic velocities across the flame. δ denotes the amount of dilatation produced by the flame. This dilatation arises from the gas expansion around the flame. The acoustic field is represented by the superimposition of the amplitude $\hat{u}_1$ on the long wavelength acoustic wave $u_{\xi,t}$. (b) One dimensional representation of the premixed flame in a duct. An inlet density disturbance is provided at the inlet. Here, $j = 25$ and $N = 56$, where $j = 25$ and $N = 56$ represent the location of flame and the total number of grid points in the discretized space, respectively. The premixed flame is confined to 2 grid points.

In this paper, we have not considered the influence of temporal variation in the mean pressure $1/\gamma p_0\partial p_{0_{\tau'}}$, and the thermal diffusion term $(\gamma-1)/(\gamma p_0 RePr)\nabla^2_\eta T_0$ on the acoustic field. We have performed such an analysis to show the influence of the heat source alone on the acoustic field. Now, dilatation is only a function of the mean heat release rate. Figure 3(a) shows the variation in the amplitude of the acoustic velocity that arises from two factors: (1) the dilatation governed by Eq. (6) and (2) the heat release rate fluctuation $\dot{Q}'$ shown in Eq. (5). The heat release rate fluctuation is expressed as $\dot{Q}'=B\rho_2^2XYe^{-E_a/RT}$, where B is the pre-exponential factor. The reaction rate $BXYe^{-E_a/RT}$ is varied to show the effect of intensity of the heat release rate on the magnitude of variation in $\hat{u}_1$.

The first objective of this simulation is to reveal the mechanism through which the "jump" condition, already discussed by various researchers, exists in the acoustic-convection-entropy coupling. We intend to achieve this objective by showing the influence of the non-normal linear convective term on the variation of the acoustic velocity field across the flame. Finally, the distinction between the zero Mach number flow

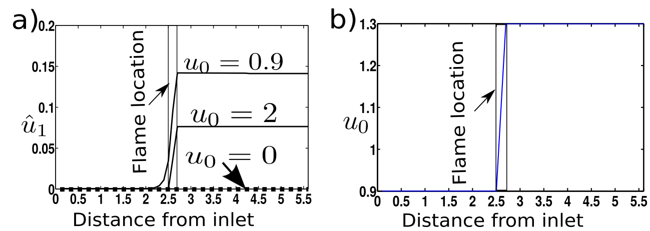

FIG. 3. Solution computed over a 1D grid represented in Fig. 2. The horizontal axis is x_j, which is the spatial location corresponding to the grid number j. The vertical axes show the acoustic velocity amplitude $\hat{u}_1$, and the hydrodynamic velocity u_0, respectively. Boundary condition for the second order density is $\hat{\rho}_{2in}=0.1$. There are three boundary conditions for the hydrodynamic velocity for which the acoustic field is computed. At inlet, we have provided these conditions to be $u_0 = 0$, $u_0 = 0.9$, and $u_0 = 2$. Initial condition is $\hat{u}_1 = 0.1$. (a) Variation in the acoustic velocity amplitude across the flame. The flame region is $2.5 < x_j < 2.7$ and (b) shows the variation in the hydrodynamic velocity across the flame. The spatial growth of acoustic and hydrodynamic velocities is linear, which is an effect of the linear non-normal convection mechanism. The jump condition is naturally recovered from the governing equations for acoustic and hydrodynamic velocities.

and the low Mach number flow in the presence of a heat release source is investigated.

The acoustic velocity field depends upon the local heat release rate fluctuations, which arise from the premixed flame. The local heat release rate fluctuations amplify the acoustic velocity field to a non-negligible amplitude. The increase in the acoustic velocity amplitude may seem to be a consequence of nonlinear processes. Such nonlinear processes were discussed by Wu.[15] However, we show that the linear terms arising from the acoustic-hydrodynamic interaction, however small the terms $\hat{u}_1 \cdot \nabla_\eta u_0$ and $u_0 \cdot \nabla_\eta \hat{u}_1$ are, can cause a significant change in the acoustic velocity field. The hydrodynamic velocity field also exhibits the change in the amplitude, which arises from the local heat release rate. Figures 3(a) and 3(b) show the spatial distributions of acoustic velocity and hydrodynamic velocity fields, respectively. The spatial distribution for the hydrodynamic velocity field is obtained by solving Eq. (6) for dilatation[2] and the spatial distribution of the acoustic velocity field is obtained by solving Eqs. (4) and (5). The variation of acoustic and hydrodynamic velocities is linear in the flame region. In Fig. 3(a), for $u_0 = 0.9$ the acoustic velocity tends to vary before the flame location, thereafter increasing linearly in the flame region. For $u_0 = 2$, the variation starts only at the beginning of the flame location. This is a consequence of the competition between the convective and diffusion phenomena. This competition can be mathematically represented by the non-normal operators $\hat{u}_1 \cdot \nabla_\eta u_0$ and $u_0 \cdot \nabla_\eta \hat{u}_1$. As the magnitude of u_0 increases, the diffusion effects, i.e., the slow variation before the flame location, disappear. The diffusive mechanism could influence the spatial variation of the acoustic velocity amplitude, since there exists a preheat zone which could also act as a source to the acoustic field. Figure 3 shows that the non-normal convective term, which is expressed as a spatial gradient, tends to cause a spatial linear variation of the acoustic field. Therefore, as opposed to the nonlinear mechanisms discussed in aforementioned researches, the linear non-normal convective mechanism also deserves much emphasis in the future research.

The jump conditions state that, across the flame, the acoustic velocity undergoes a change in the amplitude. This change in the amplitude, for the acoustic velocity field, is caused by the unsteady heat release rate. Unsteady heat release rate acts as a proportionality constant in the *ad hoc* models for jump conditions.[14,15] Our convection-reaction-diffusion equation makes no assumption about jump conditions. However, the proportionality is evident from the solutions to the governing equations. These solutions, plotted with respect to the heat release rate in Fig. 4, show that the jump is acoustic velocity is proportional to the heat release rate. A similar proportionality has been reported by Matalon and Matkowsky[13] and Wu[15] when they performed the method of matched asymptotic expansion with single time scale. In their analysis, the reaction, hydrodynamic, and acoustic zones are spatially separated. These zones are represented by different length scales. A "jump" condition couples acoustic and hydrodynamic fields. This jump condition arises either from the change in the surface area of the flame, caused by the wrinkling, or from the enthalpy fluctuations arriving at the flame. They cause unsteady heat release rate (Eq. (3.14) in Wu[15] and Eq. (3.17) in Wu *et al.*[14]).

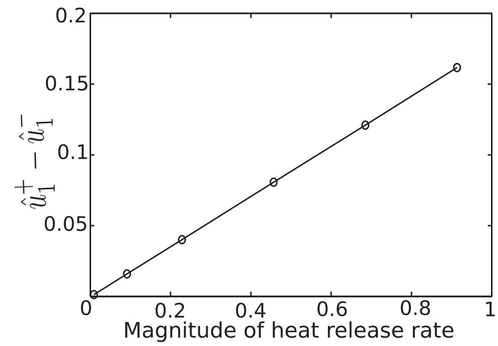

FIG. 4. Acoustic velocity jump is expressed as the difference between the value of $\hat{u}_1$ after the flame ($\hat{u}_1^+$) and the value of $\hat{u}_1$ before the flame location ($\hat{u}_1^-$), where $\hat{u}_1$ represents the amplitude of the non-dimensional acoustic velocity. In agreement with the previous researchers, acoustic velocity jump is proportional to the heat release rate. The solution is determined after solving Eqs. (4)–(6) for different heat release rates. Therefore, in addition to the acoustic and hydrodynamic fields, entropy fluctuations—essential source of the acoustic field in a thermo-acoustic system—are also incorporated in the analysis. This implies that the acoustic velocity jump is a consequence of the acoustic-convection-entropy coupling.

This unsteady heat release rate is linearly proportional to the acoustic velocity jump.

In our analysis using MMS, the acoustic and hydrodynamic fields are represented by the same space scale η, i.e., the hydrodynamic length scale. The unsteady heat release rate, that arises from the density fluctuations, acts as a source for the acoustic field when the same length scale is used for the acoustic-hydrodynamic interaction. We have demonstrated that a linear relationship exists, between the unsteady heat release rate and the acoustic velocity jump across the flame, in Fig. 4. The unsteady heat release rate also modifies the magnitude of velocity jump across the flame, which, in turn, causes the variation in the Mach number of the flow, since $\hat{u}_1/u_0$ represents the Mach number of the flow. Nicoud and Wieczorek[7] have shown that the increase in the Mach number also causes the variation in the acoustic velocity amplitude across the flame. Therefore, as shown in Table I, the relation between the magnitude of velocity jumps, which we obtained by increasing

TABLE I. Comparison with the 1D numerical simulation of Nicoud and Wieczorek.[7] In this paper, the acoustic velocity jumps (*VJ*) are obtained by solving non-dimensional CRD equations. We have compared these values with the values obtained from Fig. 6 of Nicoud and Wieczorek.[7] They have plotted the spatial variations of the acoustic velocity for $M = 0$, $M = 0.05$, and $M = 0.11$. From Nicoud and Wieczorek,[7] the variation in the acoustic velocity field from the zero Mach number flow is linearly increasing with the Mach number. For example, the relation between the velocity jumps at M = 0.05 ($VJ_{M=0.05}$) and M = 0.11 ($VJ_{M=0.11}$) is same for our simulation using CRD and the simulation of Nicoud and Wieczorek.[7] The relation is such that at M = 0.1 *VJ* is twice that of *VJ* at M = 0.05. Therefore, *VJ* at M = 0.11 is approximately 2.2 times that of *VJ* at M = 0.05. In CRD equations, Mach number M is equivalent to the ratio of $\hat{u}_1/u_0$.

	Velocity jump (VJ)		
M	Nicoud *et al.*	CRD	Relation
0	0	0	
0.05	1.5×10^{-4}	0.0484	
0.1		0.0925	$\approx 2 \times VJ_{M=0.05}$
0.11	3.4×10^{-4}	0.0998	$\approx 2.2 \times VJ_{M=0.05}$
0.15		0.1419	$\approx 3 \times VJ_{M=0.05}$

the unsteady heat release rate, is similar to that of Nicoud and Wieczorek,[7] where the acoustic velocity change is a function of the Mach number. The relations shown in Table I reveal a linear relation between the magnitude of velocity jumps at various Mach numbers. Therefore, our analysis, without using *ad hoc* models, has proved the existence of a linear relation between the unsteady heat release rate and the acoustic velocity jump. Previously, various researchers[15,32] had assumed that the onset of instability is an outcome of the discontinuity in the acoustic velocity amplitude across the flame. However, we show that the acoustic velocity amplitude is not discontinuous when a common length scale is assumed. Using a common length scale, acoustic velocity can be seen as varying linearly across the flame. Also, since the theory is based on the conservation of mass and momentum, mechanism of acoustic and hydrodynamic velocity variation in response to the heat release rate fluctuation intensity is shown in this paper as a result of convective effects of the fluid flow.

The convective effects of the fluid flow cause the heat release rate fluctuations. These fluctuations in the heat release is a consequence of the incoming density fluctuations $\hat{\rho}_2$ that arrive at the flame location at the convective speed, i.e., when $u_0 \neq 0$. Therefore, the acoustic velocity amplitude vanishes as shown in Fig. 3(a) when $\hat{\rho}_2 = 0$. The decay of the acoustic velocity to zero amplitude is a consequence of the elimination of the source of heat release rate fluctuation $\dot{Q}'$, which, in turn, is a function of $\hat{\rho}_2$. The Reynolds force terms play the role of convecting the inlet density fluctuation to the flame location. Here, we have (1) a mean flow velocity ($u_0 \cdot \nabla_\eta \hat{u}_1 \neq 0$) and (2) a gradient of u_0 at the flame location as shown in Fig. 3(b), i.e., $\hat{u}_1 \cdot \nabla_\eta u_0 \neq 0$. These factors cause the non-normal terms to be significant in the study of the acoustic-hydrodynamic interaction. Now, what is the influence of the hydrodynamic field on the acoustic field? The acoustic-hydrodynamic interaction occurs on the same length scale, i.e., η, of the heat release source. Entropy fluctuations are propagated by the hydrodynamic field. Therefore, the entropy fluctuations leading to $\dot{Q}'$ will establish a spatial evolution of $\hat{u}_1$, which, in turn, establishes the acoustic-convection-entropy coupling. Therefore, there is a possibility that the mixed mode acoustic instability, discussed by Motheau, Nicoud, and Poinsot,[18] is a consequence of the linear coupling between acoustic and hydrodynamic fields. Also, the entropy field can cause deviation from the zero Mach number flow physics.

The spatial distribution of the acoustic velocity amplitude (Fig. 5) is obtained after superimposing the solution of Eq. (5), i.e., $\hat{u}_1(\eta, \tau')$, with $\hat{u}_1(\xi, \tau')$. This superimposing is possible as the evolution of the acoustic velocity is governed by the linear processes, i.e., the non-normal convection operator. Figure 5 shows that the amplitude of $\hat{u}_1$ increases as the intensity of heat release rate increases. Then, the non-normal effect induced by $\vec{u_0} \cdot \nabla_\eta \hat{\vec{u}}_1 + \hat{\vec{u}}_1 \cdot \nabla_\eta \vec{u_0}$ becomes significant with the increase in the intensity of the unsteady heat release rate. This effect arises from the deviation from the zero Mach number assumption. When the Mach number is not zero, the acoustic and hydrodynamic fields cannot be viewed as different from each other. The presence of the acoustic field introduces the effect of compressibility and causes the transition to nearly incompressible flows.[5,6] Therefore, the increase in the magnitude of

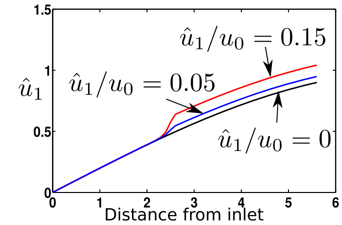

FIG. 5. Variation in the acoustic velocity amplitude across the flame for different heat release rates. For each of this distribution, $\hat{u}_1/u_0$ is also marked. This quantity denotes the magnitude of ϵ which, in turn, is a measure of the Mach number. Heat release rate is obtained after solving Eqs. (6) and (4) simultaneously. Amplitudes of the unsteady heat release rate ($\dot{Q}'$) are 0.04 and 0.16, respectively, for $\hat{u}_1/u_0 = 0.05$ and $\hat{u}_1/u_0 = 0.15$. Heat release intensity increases the ratio of amplitude of the acoustic velocity to the flow field velocity. The increase in the ratio also implies deviation from the incompressible physics of the fluid flow. Therefore, the heat release rate can cause different physics for incompressible and nearly incompressible flows. Variation in the acoustic amplitude is computed by solving Eqs. (5) and (6).

$\hat{u}_1$ changes the physics of the flow. Nicoud and Wieczorek[7] show that, with the increase in the Mach number, the jump across the flame increases. Wu[15] claims that the increase in the acoustic velocity amplitude arises from the unsteady heat release rate. There could be a link between the unsteady heat release rate and the compressibility that arises from increasing the Mach number. Our investigation relates the theories of Wu[15] and Nicoud and Wieczorek,[7] by relating the influence of the heat release rate with the influence of compressibility in a low Mach number reacting flow. We find that increasing the heat release rate intensity increases the acoustic velocity amplitude. The ratio $\hat{u}_1/u_0$, therefore increases, implying that the heat release rate can cause different physics for the zero Mach number and low Mach number flows.

IV. CONCLUSION

A mechanism which couples the entropy and convection modes, thereby generating an acoustic field in the combustor, is proposed. This mechanism, mathematically represented by a convection reaction diffusion (CRD) equation, explains the role of the Reynolds forces as a linear non-normal mechanism causing the evolution of the acoustic field. The linear mechanism governing the spatial variation of the acoustic velocity amplitude is mathematically represented here as a non-normal convection operator. The amplitude of the acoustic velocity is traditionally treated as discontinuous at the flame location. This jump in the acoustic velocity, which was modeled to be abrupt in various investigations, is resolved using the hydrodynamic length scale. Therefore, the variation of the acoustic velocity amplitude is found to be linear in the flame region. Also, in agreement with the previous investigations, the difference in the acoustic velocity amplitude before and after the flame is proportional to the heat release rate. We have shown that this proportionality arises from the presence of a first order heat release rate fluctuation, which, in turn, is the source of acoustic velocity field. Increase in the intensity of heat release rate increases the acoustic velocity amplitude. This implies that the ratio of acoustic velocity amplitude to hydrodynamic velocity increases with the increase in the heat release rate. A coupling between the acoustic and hydrodynamic fields, described in this paper, is therefore necessary to represent the thermo-acoustic system. The coupling term $\hat{u}_1 \cdot \nabla_\eta u_0 + u_0 \cdot \nabla_\eta \hat{u}_1$ is linear. Then, the deviation from the zero Mach number to the high Mach number should occur through a linear mechanism, which causes the acoustic and hydrodynamic fields to grow to sufficiently high amplitudes until nonlinearity can govern the growth process. This linear coupling term is the novelty of our theory.

CRD equation, with linear coupling terms, is based on the conservation of momentum. Therefore, the theory developed in this paper could be applied to general combustion systems. Further, the heat release rate is shown to introduce similar effects of compressibility in low Mach number reacting flows. In a previous investigation,[7] increasing the Mach number increased the amplitude of acoustic velocity across the flame. Increasing the heat release rate exhibits the same phenomenon. Heat release source is an integral part of a thermo-acoustic system. Therefore, we show that the deviation from the zero Mach number should be considered while mathematically representing a thermo-acoustic system. Therefore, the coupling of entropy mode with the convection mode is more evident in the example illustrated in this paper. As a consequence, the convective velocity determines the relevant velocity scale that determines the acoustic velocity variation when the acoustic-hydrodynamic interaction is considered.

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