Sādhanā Vol. 41, No. 7, July 2016, pp. 707–712
DOI 10.1007/s12046-016-0512-9

A simplified approach to solve quasi-statically moving load problems of elastica using end loaded elastica solution

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MS received 5 November 2015; revised 12 January 2016; accepted 4 February 2016

Abstract. Elastica problems with non-conservative moving forces are more complicated as compared to end loaded elastica problems. Established methods exist to solve an end loaded elastica problem. For solving a moving boundary problem, such methods need considerable modification or re-formulation. In this article, results of an end loaded elastica problem which is readily obtainable are used to solve two relatively involved moving boundary cases. The solution methodology involves a unique normalization procedure for the available elastic solution followed by few simple steps. One of the problems considered is three point bending of elastica with finite roller dimension. The other one being cantilever elastica under the action of wedge contact. Structural stiffening is observed in both the cases as a result of moving boundary condition as compared to when roller dimension is negligible or wedge makes only point contact. A structured approach may potentially originate from this kind of procedure to tackle more complicated moving boundary problems of elastica.

Keyword. Elastica; moving load problem; non-linear differential equation; three point bending; contact analysis.

1. Introduction

Large deflection of slender elastic cantilever beam under vertical end load at free end, popularly known as 'elastica' has a long history, see Levien [1]. Keeping the core issue of large deflection of a beam due to finite rotation as same, with time various variations of the traditional elastica problem have evolved. These evolved problems primarily deal with material non-linearity, non-uniform cross section, distributed force, non-conservative force, etc. From method of analysis point of view, various analytical and numerical techniques are developed. The analytical techniques are primarily of three kinds, viz. Legendre elliptic integrals, Jacobi integrals and different series expansions. Numerical techniques employed to solve the elastica involve iterative or direct numerical integration schemes. For a short review of various methods of elastica solution with detailed discussion on Jacobi integral technique, Batista [2] serves as a good reference.

In one of the variants of the elastica, a cantilever experiences a concentrated load of variable magnitude and direction that moves quasi-statically towards the fixed end as the beam deforms, referred here as moving load problem (MLP). Three point bending of slender elastic beam undergoing large deflection is an example of MLP if support rollers have considerable dimension, as shown in figure 1. Incorporation of dimension of support rollers in computation is important for accurate evaluation of elastic properties.

A MLP may also occur when a large wedge descends down along a vertical line, applying contact force on a horizontal cantilever, see figure 2. This problem can be thought of an elastic idealization of industrial sheet metal vee bending process. This problem which is solved by Pandit and Srinivasan [3] has two parts, viz. the large deflection bending differential equation and the constraint equations originating from beam to wedge contact. Here the bending equation is solved by using non-linear finite difference scheme and the coupled constraint equations are solved by secant iterations.

Theocaris *et al* [4] accounted for the role of support radius by suggesting a correction to his formulation of three point bend problem . In solving the three point bending problem, Batista [5] directly incorporated the roller dimension into his solution methodology. In Pandit *et al* [6], elasto-plastic large deflection of three point bending problem of negligible support roller dimension is obtained as a special case solution of the general methodology. The methodology involves solving three linearized differential equations using Runge–Kutta initial value solver in each incremental loading step. Evidently, an end loaded elastica problem referred here as ELE as shown in figure 3 is easier to solve than a MLP. Very much akin to analysis of complex machinery where an "axial force

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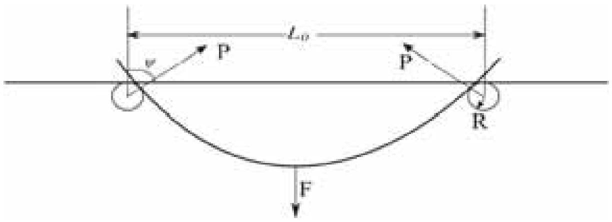

Figure 1. Schematic of three point bending depicting geometry and load.

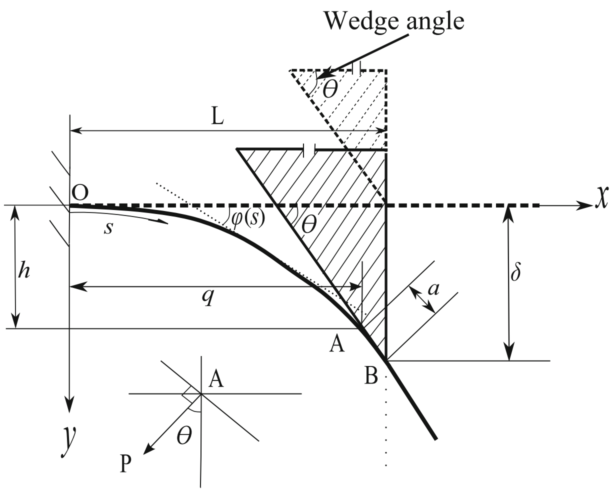

Figure 2. Large cantilever under wedge contact induced deflection.

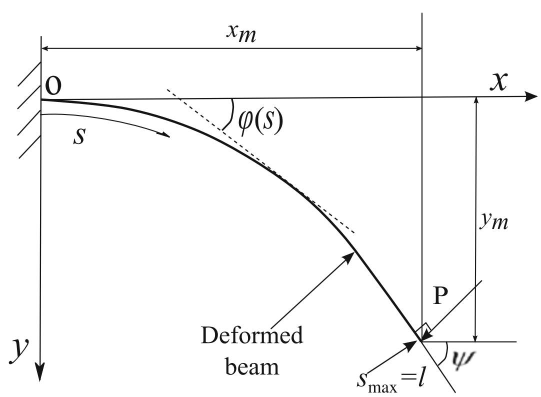

Figure 3. An ELE carrying a follower load acting normally.

member" is first sought for (if it exists), in a given MLP the corresponding ELE may be searched for. We observed that a particular way of normalizing the solution data obtained from an ELE can be directly correlated with that of the MLP in a unique manner. This unique correlation paves the way for solving many MLP by employing simple steps specific to the problem. This inference is supported in Mutyalarao *et al* [7]. In this mentioned article, the authors show that when the end angle is less than $\frac{\pi}{2}$, there exists an unique load to end angle relationship, provided that the load and the end angle are increased monotonically from zero. In many practical cases of large deformation of elastica, the load is increased from zero and the bend profile do not curl back or have points of inflection. The scope of the present method is limited to these problems. The aim of the article is to present easily amenable procedures which uses the result of the ELE as shown in figure 3 to solve MLP as depicted in figure 1 and figure 2, which have practical importance. We hope that such an approach which we will be describing in relevant sections, in effect may help devising general methodology for solving all MLP belonging to a specific ELE. In section 2, the general formulation of an ELE is presented. Here, a follower load acts normally to the elastica at its free end. In section 3 the solution procedure to solve a three point bending of elastica is presented. And in section 4 an elastic slender long cantilever undergoing large deflection under the action of a large wedge is considered. The solution procedure as well as consequent responses are shown.

2. The elastica formulation with normal follower load at its free end

Figure 3 shows a thin elastic cantilever of length l and flexural rigidity EI with a concentrated end load P. The load is of follower type which remains perpendicular to the deformed beam elastic line. A Cartesian coordinate system XOY is chosen wherein OX axis is directed along the undeformed straight beam. The arc length measured along the beam is denoted by s. The angle between the tangent at any point on the beam axis to OX axis is denoted by $\phi(s)$, and the end angle by ψ.

The beam is assumed to satisfy Euler–Bernoulli conditions and hence shear and axial deformations are neglected. Combining the kinetic, kinematic and elastic constitutive law, the governing differential equation is obtained as [8]:

$$EI\frac{d^2\phi}{ds^2} + P\cos(\psi - \phi) = 0; \qquad \phi = \phi(s); \quad 0 \le s \le l. \tag{1}$$

With boundary conditions:

$$\phi(s = 0) = 0, \qquad \frac{d\phi}{ds}(s = l) = 0;$$

Equation 1 may be solved by either analytical or numerical methods of choice. In the presented work, the method proposed in Shvartsman [8] is used. In this methodology, the two point boundary value problem of Eq. (1), is reduced to an initial value problem by change of variables. The initial value problem is subsequently solved by the classical 4th order Runge–Kutta method.

After obtaining the solution, i.e. $\phi(s)$, the Cartesian coordinates of the free end of the deformed beam axis is obtained from

$$x(s=l) = x_m = \int_0^l \cos\phi \, \mathrm{d}s, \quad y(s=l) = y_m = \int_0^l \sin\phi \, \mathrm{d}s. \tag{2}$$

We observed that a unique way of normalizing this solution may be used to easily solve relatively complicated cases in which the point of application of force, direction and magnitude changes with deformation of beam. It may be noted here that a given class of ELE can only be applied to solve a particular class of MLP. In this article we consider ELE where the force is always normal to beam axis, likewise the class of MLP that may be solved from these solution need to have force following the beam normally. To achieve the suitable normalization of solution variables, load is normalized as

$$\bar{P} = \frac{P x_m^2}{EI}. \tag{3}$$

The displacement is normalized as

$$\bar{d} = \frac{y_m}{x_m}. \tag{4}$$

Subsequently the normalized solution is stored to be used as a data base. The key relationships are that of $\bar{P}$ vs. $\bar{d}$ and end angle (ψ) vs. $\bar{d}$ and they are depicted graphically in figures 4 and 5 respectively.

It may be noted here that $\frac{x_m}{l}$ or $\frac{y_m}{l}$ are not used individually, but are combined to obtain $\bar{d}$ as given in Eq. (4). Another key point to note here is that the normalization procedure is done using the solution of the large deflection problem (ELE). Here x_m is used to normalize force (P) and not l, the length of the beam. However if the force is such that it produces small deflection then the response obtained by usual way of normalization i.e. end displacement as $\frac{y_m}{l}$ and load as $P^* = \frac{Pl^2}{EI}$ will be very close to that of current approach response. This is because if the force is small, the displacement will be also small and so $x_m \approx l$.

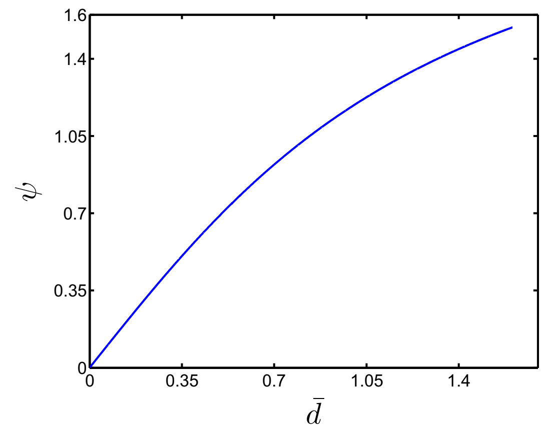

Figure 5. End angle vs. normalized displacement for the ELE.

3. Three point bending of slender elastic beam

In figure 1 a slender elastic beam resting on two circular roller supports is shown. The beam is acted upon by a concentrated vertical force F acting at the center location between the supports. The supports are assumed to be friction-less. Owing to symmetry, this problem is equivalent to the problem of a cantilever under the action of a roller punch moving vertically downward keeping the center of roller always at a fixed distance from the support, see figure 6. The central load application point in the three point bending problem i.e. where F is applied, see figure 1, acts as the fixed point of the equivalent cantilever which is flipped upside down. The support rollers of the three point bending problem acts as the load application roller of the equivalent cantilever which applies a concentrated force P that is perpendicular to the deformed beam. Clearly, the roller center locus line in the equivalent problem and

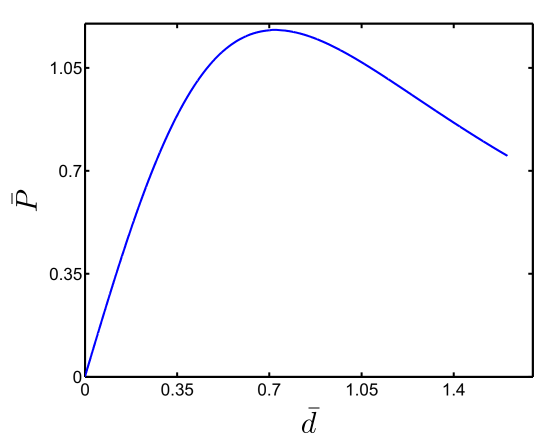

Figure 4. Normalized load vs. normalized displacement for the ELE.

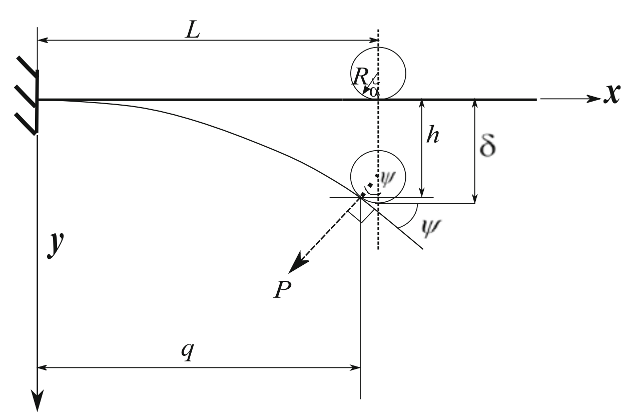

Figure 6. Large cantilever under roller punch contact induced deflection.

support to support center distance in the original problem are related by

$$L_0 = 2L. \quad (5)$$

The forces in the equivalent and original problem are related by

$$F = 2P\cos\psi, \quad (6)$$

where ψ is the inclination of tangent at roller contact point with horizontal, as shown in figure 6.

The kinematics involved in the cantilever problem are given by

$$\delta = h + R(1 - \cos\psi) \quad (7)$$

And

$$L = q + R\sin\psi. \quad (8)$$

In this problem we are given L, EI, R and the final specified displacement of roller bottom point δ_f. We are required to find the force on the beam P for any $0 < \delta \leq \delta_f$. In non-dimensionalized form, the load is given by $\tilde{P} = \frac{PL^2}{EI}$ and displacement by $\tilde{\delta} = \frac{\delta}{L}$. In order to find the $\tilde{P}$ vs. $\tilde{\delta}$ relation we follow the following steps sequentially:

1. Any $\psi^* > 0$ is chosen (as small as required)[1]
2. Corresponding to ψ^*, from figure 5 , $\bar{d}^*$ is determined
3. Corresponding to $\bar{d}^*$, from figure 4, $\bar{P}^*$ is determined
4. q^* determined from Eq. (8)
5. h^* determined from $h^* = q^*\bar{d}^*$, from uniqueness of figure 5
6. δ^* determined from Eq. (7)
7. $\tilde{P}$ determined from $\tilde{P} = \bar{P}^* \frac{L^2}{(q^*)^2}$, from uniqueness of figure 4
8. The above steps are continued for all $\psi > \psi^*$ until the specified δ_f is obtained.

Following the above procedure, excellent agreement with Batista [2] is obtained for the case when $\frac{R}{L} = 0.14$ and rollers are friction-less. In figure 7, the force displacement relationship for varying roller dimension for a specified $\frac{\delta_f}{L}$ (=1) is shown. For higher roller dimension the response curves are seen to have points of inflection. The end loaded cantilever force–displacement response of figure 4 is also represented for comparison purpose. It is observed that, as the roller dimension diminishes, the response curves tend to that of the end loaded cantilever result (of $\bar{P}$ vs. $\bar{d}$). Also it may be noted that under small displacement condition, variation of roller dimension does not affect the response. Small deflection theory predicts linearity in force - end displacement response, this is observed here as well

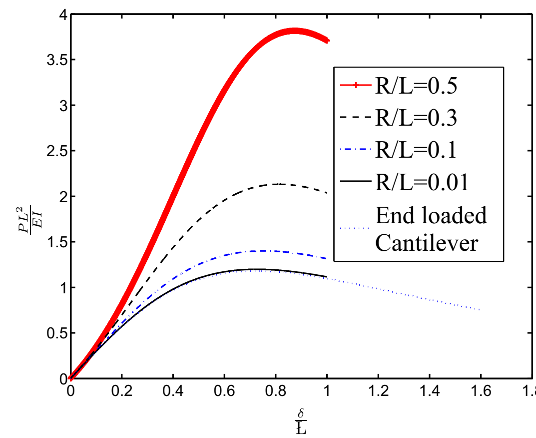

Figure 7. Force response of equivalent cantilever under roller punch displacement.

irrespective of roller dimension. It may be appreciated here that the shape or size of roller can influence the response only when the beam undergoes substantial deflection making it slide along the roller. The results of the equivalent cantilever problem can be readily converted to the three point bend case by employing Eq. (5) and Eq. (6).

3.1 *Generalization of the method for non-circular roller*

In figure 6 a circular roller is shown of constant radius of curvature as R. Now suppose we imagine to have rollers where the center of curvature is constant but the radius changes with increase of α (see figure 6); this is clearly a case of non-uniform curvature roller where radius R is a function of α, i.e $R = R(\alpha)$. In this case, the kinematic conditions of Eqs. (7) and (8) reads:

$$\delta = h + R(0) - R(\psi)\cos\psi \quad (9)$$

And

$$L = q + R(\psi)\sin\psi. \quad (10)$$

To obtain the response of the non-circular roller problem, the subsequent steps to be followed are the same as that for the circular roller case. It may be pointed out here that, $R(\alpha)$ is a known function i.e. the geometry of the roller should be known and be independent of deformation. Clearly, for other types of roller, the kinematic conditions relating L and δ with h, q and ψ will be different, but the subsequent steps will be same as that of circular roller support, provided the rollers are friction-less.

4. Wedge contact with elastic cantilever beam

In figure 2 a long elastic cantilever is deformed by the action of a large wedge descending vertically downward. The wedge is assumed to be rigid. The plane slant edge of the

[1] The first choice of end angle.

wedge is assumed to be inclined by θ radian with respect to x axis. The vertical displacement of its tip is denoted by δ. In the deflected configuration, the wedge is assumed to apply a concentrated force P at a point denoted by A on the beam. The surface of the wedge is assumed friction-less and so the force acts in a direction normal to the wedge face. If the surface of the wedge applies a distributed force on the beam, then the beam in this portion would have non-zero bending moment which would bend it and thereby make it loose contact with the wedge. Hence the force is assumed to be concentrated and the point where its concentrated is denoted by A. Beyond point A along AB since no bending moment is present, the beam remains straight with point B on the beam coinciding with the wedge tip. The straight portion of the deflected beam, denoted by $AB = a$ is defined as the contact region of the beam with the wedge.

In this problem, similar to that in section 3, we are given L, EI, θ and the final displacement of wedge tip : δ_f. As in the previous case, we are required to find P for any $0 < \delta \leq \delta_f$. Here, additionally we are interested in the relationship of contact length (a) with the wedge tip displacement (δ).

In order to solve the given problem, we first note that line contact will commence for a given δ only when the inclination of the deformed beam at B of figure 2 just equals θ. The corresponding displacement is denoted as δ^*. Response of an elastic cantilever is unique and hence the force response of the beam for $\delta \leq \delta^*$ is that given in figure 4. This implies when contact do not develop, $\tilde{P} = \bar{P}$ and $\tilde{\delta} = \bar{d}$. However, if $\delta_f > \delta^*$, the responses are not the same. This is because from the point of view of the ELE as discussed in section 2, the free end is not where the wedge tip is. Rather, the domain is defined by the fixed end (O) and A of figure 2. Since displacement is specified for the wedge tip (i.e. B), a relationship is required to connect the ELE and this wedge contact problem. In developing such a relationship, the key thing to observe is that the slope at A is given by $\tan(\theta)$ for any $\delta \geq \delta^*$. Let the displacement corresponding to $\psi = \theta$ from figure 5 be denoted as $\bar{d}^*$ and subsequently from figure 4 the force be denoted as $\bar{P}^*$.

Clearly,

$$\frac{\delta^*}{L} = \bar{d}^*. \tag{11}$$

Now from simple kinematics of figure 2, we have the following for $0 \leq \delta \leq \delta_f$:

$$q = L - a\cos\theta \tag{12}$$

And

$$h = \delta - a\sin\theta. \tag{13}$$

Again, for all $\delta > \delta^*$:

$$\frac{q}{h} = \bar{d}^*. \tag{14}$$

Using Eq. (12) and Eq. (13) into Eq. (14) and simplifying, we get:

$$a = \frac{\delta - \delta^*}{\sin\theta - \bar{d}^*\cos\theta}. \tag{15}$$

Now, since during contact deformation, the slope at A of figure 2 remains $\tan\theta$, the force is given by

$$\bar{P}^* = \frac{Pq^2}{EI}. \tag{16}$$

Using $\tilde{P} = \frac{PL^2}{EI}$ and Eq. (12) into Eq. (16), we get

$$\tilde{P} = \frac{\bar{P}^*}{(1 - \frac{a}{L}\cos\theta)^2}. \tag{17}$$

The commencement of contact deformation is governed by Eq. (11). During contact deformation the complete response of force and contact length is given by Eq. (17) and Eq. (15) respectively. Following the procedure described in this section, the force and contact length responses are depicted

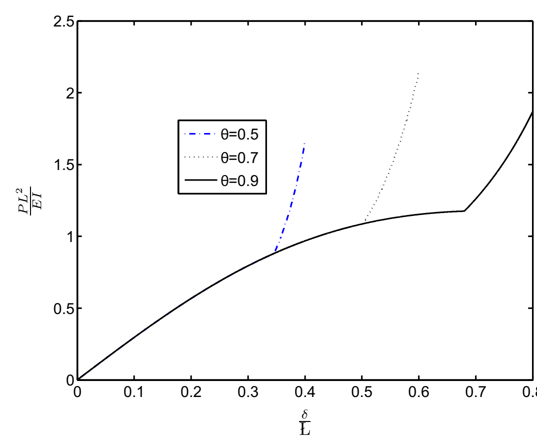

Figure 8. Force–displacement response for various wedge angles.

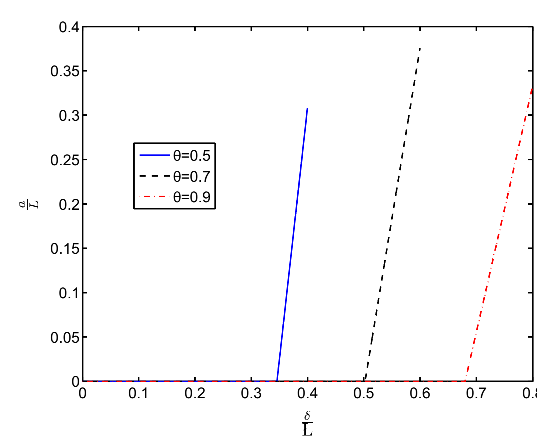

Figure 9. Dependence of contact length on wedge tip displacement, for various wedge angles.

in figure 8 and figure 9 respectively. It is found that these results are in excellent agreement with Pandit and Srinivasan [3] . It may be noted here that these results are obtained without the need to solve non-linear equations arising due to contact constraints. In the force response curves, it is observed that when contact develops, the structural response is stiffer. It is seen that for the same force, the structure is stiffer for smaller wedge angle. Smaller wedge angle is also seen to induce contact development at smaller wedge tip displacement.

In figure 9, the contact length is seen to be varying linearly with displacement, which is expected from Eq. (15). The rate of increase of contact length with respect to wedge tip displacement is seen to be higher for smaller wedge angle.

5. Conclusion

Traditionally MLP are solved directly by employing a single special formulation suitable for the specific problem only. In this article we focused on solving a couple of MLP by first normalizing the relevant ELE results in a novel way and then follow a few simple steps to tackle the constraint equations. It may be said that this way of solving reduced the formulation complexity and computation time (since stored ELE results are used from data base) without compromising on the accuracy of results.

In this study, two moving boundary problems are considered: three point bending of elastica with finite roller dimension and elastica deformed by wedge action leading to line contact between beam and wedge. In the case of three point bending, the force–displacement response showed that with increase in roller dimension, structure stiffens up. Hence in accurate estimation of elastic properties of a material via a three point bend test, accounting for support roller dimension is necessary. Similarly in the case of wedge contact induced deflection of elastica, the force response is seen to stiffen up abruptly from that of point-contact response curve, giving a look akin to bifurcation phenomenon. Since the wedge contact problem discussed here can be assumed to be an elastic idealization of sheet metal bending problem, the force response obtained via this simplified technique employing ELE results may be considered to form the upper bound of maximum force the die may ever experience.

In future a more structured approach using the results of ELE may open the door to efficiently solve more complicated MLP.

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