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A formula for the coefficients of orthogonal polynomials from the three-term recurrence relations

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Abstract

In this work, the coefficients of orthogonal polynomials are obtained in closed form. Our formula works for all classes of orthogonal polynomials whose recurrence relation can be put in the form $R_n(x) = xR_{n-1}(x) - \alpha_{n-2}R_{n-2}(x)$. We show that Chebyshev, Hermite and Laguerre polynomials are all members of the class of orthogonal polynomials with recurrence relations of this form. Our formula unifies the previously known formulas for the coefficients of these familiar polynomial families.

Keywords: Chain sequence; Chebyshev; Hermite and Laguerre Polynomials

1. Introduction

Orthogonal polynomials (OPs) occur often as solutions of mathematical and physical problems. They play an important role in the study of wave mechanics, heat conduction, electromagnetic theory, quantum mechanics and mathematical statistics. They provide a natural way to solve, expand, and interpret solutions to many types of important differential equations. OPs are connected with trigonometric, hypergeometric, Bessel, and elliptic functions and are related to the theory of continued fractions and to important problems of interpolations and mechanical quadrature.

The classical OPs are those named after Jacobi, Laguerre, and Hermite. They can be characterized in a number of ways; their weight functions satisfy first-order differential equations with polynomial coefficients, their derivatives are OPs of the same family [7,11]. The main properties of OPs considered in the literature are related to zeros, generating functions, asymptotic behaviour, expansion problems, connection coefficients, kernel polynomials, integral representations, continued fractions, spectral measures, Rodrigues' formula, etc. As a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in OPs has revived in recent years [2].

An important characterization of monic orthogonal polynomials $\{P_n(x)\}$ is the classical three-term recurrence relation,

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$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),$$
$$P_{-1}(x) = 0, \qquad P_0(x) = 1, \qquad \lambda_{n+1} > 0, \quad n = 1, 2, 3, \dots, \tag{1}$$

where the coefficients are real [5].

It is well known that when $\lambda_n > 0$ for all $n > 1$ the zeros of $P_n(x)$ are real and distinct, and between each pair of consecutive zeros of $P_{n+1}(x)$ there is precisely one zero of $P_n(x)$.

Moreover,

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)\, \mathrm{d}\psi(x) = \delta_{m,n}$$

where ψ is a distribution (bounded, nondecreasing) function with an infinite spectrum (set of points of increase).

In a significant contribution, Chihara [4] has established that a necessary and sufficient condition for the $P_n(x)$ to be orthogonal polynomials whose true interval of orthogonality is a subset of $(0, \infty)$ is that $c_n > 0$ and $\{\lambda_{n+1}/(c_n c_{n+1})\}$ is a chain sequence. A sequence $\{a_n\}_{n=1}^{\infty}$ is a (positive) chain sequence if there exits a second sequence $\{g_n\}_{n=0}^{\infty}$ such that

(i) $0 \le g_0 < 1, \qquad 0 < g_n < 1, \quad n = 1, 2, \dots$

(ii) $a_n = (1 - g_{n-1})g_n, \quad n = 1, 2, \dots.$

The sequence $\{g_n\}$ is called a parameter sequence for $\{a_n\}$. See [8] for an example of a more recent application to a chain sequence. A chain sequence does not, in general, uniquely determine its parameter sequence $\{g_n\}$.

Let $Q_n(x)$ denote the unique nth-degree polynomials with leading coefficients unity which are orthogonal on (a, b) with respect to the distribution, $x\mathrm{d}\psi(x)$. Then there are real constants, d_n and $\nu_{n+1} > 0$ $(n \ge 1)$, such that

$$Q_n(x) = (x - d_n)Q_{n-1}(x) - \nu_n Q_{n-2}(x), \quad n = 1, 2, 3, \dots, \tag{2}$$
$$Q_{-1}(x) = 0, \qquad Q_0(x) = 1, \ \nu_1 \quad \text{arbitrary.}$$

Chihara [4] has defined another set of polynomials, $R_n(x)$, by

$$\begin{aligned} R_{2n}(x) &= P_n(x^2), \\ R_{2n+1}(x) &= xQ_n(x^2), \quad n = 0, 1, 2, \dots. \end{aligned} \tag{3}$$

Then $R_n(x)$ are orthogonal on $(-b^{1/2}, b^{1/2})$ with respect to the distribution

$$\mathrm{d}\phi(x) = (\operatorname{sgn} x)\mathrm{d}\psi(x^2), \qquad -b^{1/2} \le x \le b^{1/2}.$$

Thus there exist constants, $\alpha_n > 0$ $(n \ge 0)$, such that

$$R_n(x) = xR_{n-1}(x) - \alpha_{n-2}R_{n-2}(x), \quad n = 1, 2, 3, \dots, \tag{4}$$

with $R_{-1}(x) = 0$, $R_0(x) = 1$.

In (4), replace n by $2n$ and $2n + 1$ and use (3); we obtain

$$P_n(x) = xQ_{n-1}(x) - \alpha_{2n-2}P_{n-1}(x), \quad n = 1, 2, 3, \dots,$$

and

$$Q_n(x) = P_n(x) - \alpha_{2n-1}Q_{n-1}(x), \quad n = 1, 2, 3, \dots.$$

From these,

$$P_n(x) = (x - \alpha_{2n-3} - \alpha_{2n-2})P_{n-1}(x) - \alpha_{2n-4}\alpha_{2n-3}P_{n-2}(x), \quad n = 1, 2, 3, \dots, \tag{5}$$

and

$$Q_n(x) = (x - \alpha_{2n-2} - \alpha_{2n-1})Q_{n-1}(x) - \alpha_{2n-3}\alpha_{2n-2}Q_{n-2}(x), \quad n = 1, 2, 3, \dots, \tag{6}$$

with $\alpha_{-2} = 0, \alpha_{-1} = 0$.

From (1), (2), (5) and (6),

$$c_n = \alpha_{2n-3} + \alpha_{2n-2}, \qquad \lambda_{n+1} = \alpha_{2n-2}\alpha_{2n-1}, \qquad \alpha_{-1} = 0, \tag{7}$$

$$d_n = \alpha_{2n-2} + \alpha_{2n-1}, \qquad \nu_{n+1} = \alpha_{2n-1}\alpha_{2n}, \quad n = 1, 2, 3, \ldots. \tag{8}$$

The relations (7) and (8) are identical to those connecting the coefficients in a "corresponding" continued fraction of Stieltjes and the "associated" continued fraction obtained by contraction. The $Q_n(x)$ are the "kernel polynomials" associated with the $P_n(x)$.

Three-term recurrence relations unite classical orthogonal polynomial and birth and death processes (BDPs). This liaison leads to a spectral representation of transient probabilities of BDPs (see, e.g., [9]). Remarkably, for many specific BDPs of practical interest, the weight function $d\psi(x)$ has been explicitly determined and general formulas derived from the OPs. Chebyshev polynomials are associated with a BDP with constant birth and death rates ($M/M/1/N$ queue). Charlier polynomials are associated with a BDP with constant birth rate and linear death rate ($M/M/\infty$ queue). Gilewicz et al. [6] have given the new Nevanlinna matrices for orthogonal polynomials related to cubic birth and death processes.

In this note, we obtain a representation for the coefficients of orthogonal polynomials from the three-term recurrence relations. We give examples to illustrate our result. The results of this work were obtained during an investigation to find the closed form transient solution of birth and death processes with state-dependent birth and death rates [10].

2. Polynomials $R_n(x)$

In this section, we express the coefficients of the OPs $R_n(x)$ in closed form.

Theorem 1. *If*

$$R_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \psi_r(n)\, x^{n-2r}, \tag{9}$$

where $\psi_0(n) = 1$, $\psi_r(n) = 0$ *if* $r > [n/2]$, *for* $n = 1, 2, 3, \ldots$, *and* $r \geq 1$,

$$\psi_r(n) = \sum_{i_1=0}^{n-2r} \alpha_{i_1} \sum_{i_2=i_1+2}^{n-2r+2} \alpha_{i_2} \sum_{i_3=i_2+2}^{n-2r+4} \alpha_{i_3} \cdots \sum_{i_r=i_{r-1}+2}^{n-2} \alpha_{i_r}. \tag{10}$$

Proof. From (4) and (9), for $n = 1, 2, 3, \ldots$, and $r \geq 1$,

$$\psi_r(n) - \psi_r(n-1) = \alpha_{n-2}\psi_{r-1}(n-2). \tag{11}$$

Also, from (3) $R_{2n}(x)$ and $R_{2n+1}(x)$ are monic polynomials in terms of x^2 of degrees $2n$ and $2n+1$ respectively, which gives

$$\psi_0(n) = 1 \qquad \text{and} \qquad \psi_r(n) = 0 \quad \text{if } r > [n/2].$$

We use induction to prove the result (10).

From (11), it is clear that for $r = 1$,

$$\psi_1(n) = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} = \sum_{i_1=0}^{n-2} \alpha_{i_1} \qquad \text{for every } n \geq 2.$$

We assume that this is true up to $r-1$ for every $n \geq 2$,

$$\text{i.e.,}\quad \psi_{r-1}(n) = \sum_{i_1=0}^{n-2r+2} \alpha_{i_1} \sum_{i_2=i_1+2}^{n-2r+4} \alpha_{i_2} \sum_{i_3=i_2+2}^{n-2r+6} \alpha_{i_3} \cdots \sum_{i_{r-1}=i_{r-2}+2}^{n-2} \alpha_{i_{r-1}}. \tag{12}$$

Repeated application of the recurrence relation (11) yields

$$\psi_r(n) = \alpha_{2r-2}\psi_{r-1}(2r-2) + \alpha_{2r-1}\psi_{r-1}(2r-1) + \cdots + \alpha_{n-2}\psi_{r-1}(n-2). \tag{13}$$

Substitute (12) in (13); for every n the result is true for $r \geq 1$. □

This recurrence relation given in (11) has been discussed by Berndt [3, p. 125] in the context of an S-fraction.

Example 1. If $\alpha_0 \neq \alpha$ and $\alpha_i = \alpha$, $i = 1, 2, 3, \ldots$, then for every $n \geq 2$ and $r = 1, 2, 3, \ldots [n/2]$,

$$\psi_r(n) = \alpha_0 \binom{n-r-1}{r-1} \alpha^{r-1} + \binom{n-r-1}{r} \alpha^r. \tag{14}$$

Proof. We prove the result (14) by induction on r.
For $r = 1$,

$$\psi_1(n) = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} = \alpha_0 + \binom{n-2}{1}\alpha \qquad \text{for every } n \geq 2.$$

We assume that this result is valid up to $r-1$ for every $n \geq 2$,

$$\text{i.e., } \psi_{r-1}(n) = \binom{n-r}{r-2}\alpha_0\alpha^{r-2} + \binom{n-r}{r-1}\alpha^{r-1}.$$

From (13),

$$\begin{aligned}
\psi_r(n) &= \alpha_{2r-2}\psi_{r-1}(2r-2) + \alpha_{2r-1}\psi_{r-1}(2r-1) + \cdots + \alpha_{n-2}\psi_{r-1}(n-2) \\
&= \alpha \sum_{k=0}^{n-2r} \psi_{r-1}(2r+k-2) \\
&= \alpha \left[\sum_{k=0}^{n-2r} \binom{r-2+k}{r-2} \alpha_0\alpha^{r-2} + \sum_{k=0}^{n-2r-1} \binom{r-1+k}{r-1} \alpha^{r-1} \right] \\
&= \binom{n-r-1}{r-1}\alpha_0\alpha^{r-1} + \binom{n-r-1}{r}\alpha^r.
\end{aligned}$$

The expression (14) is true for every r. □

If $\alpha_0 = 2\alpha$ then for each $n \geq 2$ and $r \geq 1$,

$$\psi_r(n) = n\alpha^r \frac{(n-r-1)!}{r!(n-2r)!}.$$

Thus,

$$\begin{aligned}
R_n(x) &= n\alpha^{n/2} \sum_{r=0}^{[n/2]} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!} \left(\frac{x}{\sqrt{\alpha}}\right)^{n-2r} \\
&= 2\alpha^{n/2} T_n\left(\frac{x}{2\sqrt{\alpha}}\right) \text{ [1, Equ. 22.3.6],}
\end{aligned}$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.
If $\alpha_0 = \alpha$ then for every $n \geq 2$ and $r \geq 1$,

$$\psi_r(n) = \binom{n-r}{r}\alpha^r,$$

and

$$R_n(x) = \alpha^{n/2} U_n\left(\frac{x}{2\sqrt{\alpha}}\right) \text{ [1, Equ. 22.3.7],}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind.

Example 2. Let $\alpha_i = (i+1)\alpha, i = 0, 1, 2, \ldots$.
Then,

$$\psi_1(n) = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} = \alpha(1+2+3+\cdots+n-1) = \binom{n}{2}\alpha.$$

Assume that, for every $n \geq 2$, this result is valid up to $r-1$,

$$\text{i.e., } \psi_{r-1}(n) = \frac{(2r-3)!}{2^{r-2}(r-2)!}\binom{n}{2r-2}\alpha^{r-1}.$$

Then,

$$\begin{aligned}
\psi_r(n) &= \alpha_{2r-2}\psi_{r-1}(2r-2) + \alpha_{2r-1}\psi_{r-1}(2r-1) + \cdots + \alpha_{n-2}\psi_{r-1}(n-2) \\
&= \frac{(2r-3)!}{2^{r-2}(r-2)!}\left[(2r-1)\binom{2r-2}{2r-2} + 2r\binom{2r-1}{2r-2} + \cdots + (n-1)\binom{n-2}{2r-2}\right]\alpha^r \\
&= \frac{(2r-1)!}{2^{r-1}(r-1)!}\binom{n}{2r}\alpha^r. \\
\psi_r(n) &= \frac{(2r)!}{2^r\, r!}\binom{n}{2r}\alpha^r = \frac{n!}{r!(n-2r)!}\left(\frac{\alpha}{2}\right)^r.
\end{aligned}$$

Thus,

$$R_n(x) = \left(\frac{\alpha}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2\alpha}}\right) \text{ [1, Equ. 22.3.10]},$$

where $H_n(x)$ are the Hermite polynomials.

3. Polynomials $P_n(x)$, $Q_n(x)$

In this section, we present the closed form coefficients of the OPs $P_n(x)$ and the corresponding kernel polynomials $Q_n(x)$ using the theorem given in the previous section. This closed form expression leads to the transient solution of state-dependent BDPs [10].

Theorem 2. *We have*

$$\begin{aligned}
P_n(x) &= \sum_{r=0}^{n}(-1)^r\psi_r(2n)x^{n-r}, \\
Q_n(x) &= \sum_{r=0}^{n}(-1)^r\psi_r(2n+1)x^{n-r}
\end{aligned} \tag{15}$$

where $\psi_r(n)$ is given in (10).

The result (15) follows by substitution of Theorem 1 in (3).

Example 3. If $\alpha_i = -\alpha, i = 0, 1, 2, \ldots$, then

$$\psi_r(n) = (-1)^r\binom{n-r}{r}\alpha^r.$$

Thus,

$$\begin{aligned}
P_n(x) &= \alpha^n\sum_{r=0}^{n}\binom{2n-r}{r}\left(\frac{x}{\alpha}\right)^{n-r} \\
&= \alpha^n\left[U_n\left(\frac{x+2\alpha}{2\alpha}\right) - U_{n-1}\left(\frac{x+2\alpha}{2\alpha}\right)\right] = \alpha^n W_n\left(\frac{x+2\alpha}{2\alpha}\right),
\end{aligned}$$

where $W_n(x)$ are the Chebyshev polynomials of the fourth kind [2, p. 102] and

$$Q_n(x) = \alpha^n U_n\left(\frac{x+2\alpha}{2\alpha}\right).$$

If $\alpha_0 = 2\alpha$, and $\alpha_i = \alpha, i = 1, 2, 3, \ldots$, then

$$P_n(x) = 2\alpha^n\ T_{2n}\left(\frac{1}{2}\sqrt{\frac{x}{\alpha}}\right)$$

$$Q_n(x) = 2\alpha^n\sqrt{\frac{\alpha}{x}}\ T_{2n+1}\left(\frac{1}{2}\sqrt{\frac{x}{\alpha}}\right).$$

Example 4. If $\alpha_i = -(i+1)\alpha, i = 0, 1, 2, \ldots$, then

$$\psi_r(2n) = (-1)^r\frac{(2r)!}{2^r\ r!}\binom{2n}{2r}\alpha^r.$$

Thus,

$$P_n(x) = \left(\frac{\alpha}{2}\right)^n\sum_{r=0}^{n}\frac{(2n)!}{r!(2n-2r)!}\left(\frac{2x}{\alpha}\right)^{n-r}.$$

Replacing x by $-x^2$,

$$P_n(-x^2) = \left(\frac{\alpha}{2}\right)^n(-1)^n\sum_{r=0}^{n}(-1)^r\frac{(2n)!}{r!(2n-2r)!}\left(\sqrt{\frac{2}{\alpha}}x\right)^{2n-2r}.$$

Thus,

$$P_n(x) = (2\alpha)^n n! L_n^{(-1/2)}\left(\frac{-x}{2\alpha}\right) \text{ [1, Equ. 22.5.38]},$$

and

$$Q_n(x) = (2\alpha)^n n! L_n^{(1/2)}\left(\frac{-x}{2\alpha}\right) \text{ [1, Equ. 22.5.39]},$$

where $L_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials.

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