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A DUAL INTEGRAL EQUATION METHOD FOR CAPILLARY-GRAVITY WAVE SCATTERING

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ABSTRACT. A mixed boundary value problem for Laplace's equation involving a higher order boundary condition, associated with scattering (radiation) of capillary-gravity waves in deep water by submerged as well as surface piercing vertical barrier (wave-maker) is considered for its complete solution. Utilizing recently developed mode-coupling relations for eigenfunctions in the expansion formula of the potential function, the boundary value problem has been reduced to solving dual integral equations with kernels comprised of trigonometric functions. A fully analytical solution is derived by the aid of a weakly singular integral equation whose solution has bounded behavior at the end points. The reflection and transmission coefficients, for an incident wave, have been obtained analytically in terms of modified Bessel functions. Numerical results are computed and presented graphically for a surface tension parameter, plotted against a non-dimensional wave parameter. The present method of solution is essentially an extension of the reduction method originally described by Williams [**20**].

1. Introduction.

Mixed boundary value problems occurring in the theory of linear water waves involving vertical barriers have been of interest to many research workers. A number of methods of solution are explained for different barrier topographies by Ursell [**18**], Williams [**20**], Evans [**7**] and others. One method of dealing with such boundary value problems is to reduce them to solving dual integral equations. These equations are often encountered in different branches of mathematical physics, and they generally arise while solving a boundary value problem with mixed boundary conditions (see Sneddon [**16**]). Chakrabarti et al. ([**1, 4**]) studied water wave scattering by

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partial vertical barriers by reducing the corresponding boundary value problem into a dual integral equations problem with a sine kernel.

The effect of surface tension brings out an altogether different kind of mixed boundary value problem such as the higher order boundary conditions for the Laplace or Poisson equation. These problems are not well posed because of the nature of their boundary conditions, and their solution method normally requires certain physical conditions or edge conditions. Evans [**7**] initially investigated the surface tension effect on the two-dimensional transmission problem of time-harmonic water waves for partially immersed vertical barriers. The classical wave-maker problems with surface tension effect in both cases of finite and infinite depths were solved by Rhodes-Robinson ([**13**–**15**]). It was necessary in these solutions to specify certain edge-slope constants or functions a priori for the complete formal wave solution. In [**15**], Rhodes-Robinson studied the influence of surface tension for two-dimensional waves produced by partial vertical wave-makers in an infinite depth of water. These problems involving the incomplete wave-maker were solved analytically by reducing the boundary value problem into a weakly singular integral equation with logarithmic kernel, a method devised originally by Williams [**20**]. Utilizing its solution, he obtained the analytical expressions for reflection and transmission coefficients in wave scattering by a fully submerged vertical barrier. For the surface piercing barrier, Rhodes-Robinson derived these coefficients in terms of certain edge-slope constants which, as pointed out by Hocking [**9**], are presumed to be evaluated at the edges of the barrier on the free surface.

In this present paper, a simple and straightforward method will be demonstrated to solve the mixed boundary value problem associated with the same problem of scattering (radiation) of linear surface waves under surface tension by partial vertical barriers (wave-makers). A general mixed boundary value problem formulation, its analytical solution and the numerical results for a particular physical problem have been derived. In the solution procedure, the mixed boundary value problem has been reduced to the solution of a dual integral equations problem with kernel composed of trigonometric functions. The behavior of one of the integrals in these equations at a point, where the boundary condition changes, plays a crucial role in determining their solution. The reduced dual integral equations are solved by the aid of the bounded solution of a singular integral equation with a kernel of logarithmic type (see [**2, 6**]). In fact, the physical problem requires its solution to be bounded at both the end points and that brings out certain solvability criteria to be satisfied by the forcing function of the integral equation.

The mathematical description of the mixed boundary value problem is given in Section 2. The reduction of the general boundary value problem to dual integral equations and their solution procedure are described in Section 3 and its subsections 3.1 and 3.2 for each specific barrier configuration. The numerical results for the reflection and transmission coefficients for the scattering of an incident wave by both immersed as well as surface piercing barriers are discussed in subsection 3.3.

The method of solution presented here for the mixed boundary value problem arising in the scattering (radiation) of capillary-gravity water waves, propagating in deep water, by partial barriers (wave-makers) is exact and straightforward. In the limiting case where $\beta = MK^2 \to 0$, when the capillary effect is diminishing, numerical values for the quantities of practical importance are discussed and compared with known exact values.

2. Mathematical formulation. In the context of the present study, a general mixed boundary value problem for capillary-gravity wave scattering (radiation), in deep water, by partial vertical barriers (wave-makers) is formulated under the assumptions of linearized water wave theory. A two-dimensional Cartesian coordinate system is used in which the y-axis is taken vertically downward so that $y > 0$, $x \in \mathbf{R}$ is the region occupied by the fluid. Considering the irrotational motion of an incompressible inviscid fluid under the action of gravity and surface tension, the two dimensional time-harmonic motion is described by a velocity potential $\Phi(x, y, t) = \mathrm{Re}\,\{\phi(x, y)e^{-i\omega t}\}$, and the surface elevation $\eta(x, t) = \mathrm{Re}\,\{\eta(x)e^{-i\omega t}\}$ with $\omega(> 0)$ denoting angular frequency and t denoting the time. Also, it is assumed that the wave-maker is oscillating harmonically in time, i.e., $U(y, t) = \mathrm{Re}\,\{U(y)e^{-i\omega t}\}$. The time-dependent factor $e^{-i\omega t}$ is suppressed throughout the analysis. Then $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad x \in \mathbf{R},\ y > 0, \tag{2.1}$$

which is a consequence of the equation of continuity: $\operatorname{Div}\vec{u} = 0$, $\vec{u}$ $(= \operatorname{Grad}\phi)$, denoting the velocity vector (see Stoker [**17**]).

The linearized surface boundary condition may be derived (see [**12, 19**]) by balancing the hydrodynamic pressure with the net pressure on the free surface of the fluid under surface tension, as

$$M\frac{\partial^3\phi}{\partial y^3} + \frac{\partial\phi}{\partial y} + K\phi = 0, \quad \text{on } y = 0,\ x \in \mathbf{R}, \tag{2.2}$$

where $M = \gamma/(\rho g)$ with γ, ρ, g representing surface tension constant, density of water and acceleration due to gravity, respectively, and $K = \omega^2/g$.

On the rigid vertical structure occupied at $x = 0, y \in (b, \infty)$ or $(0, b)$ with $b > 0$, ϕ satisfies the Neumann boundary condition

$$\frac{\partial\phi}{\partial x} = U(y), \tag{2.3}$$

where $U(y) \to 0$ as $y \to \infty$. Note that, when $U(y) = 0$, the structure represents a vertical barrier and the condition of vanishing normal velocity.

Also, since the fluid flow is continuous across the gap $x = 0$, $y \in (0, b)$, or (b, ∞), the velocity potential $\phi(x, y)$ satisfies

$$\phi(0^-, y) = \phi(0^+, y), \tag{2.4}$$

in the usual notation, and that

$$\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \to 0 \text{as } y \to \infty, \tag{2.5}$$

representing no motion at large depth.

The behavior of $\phi(x, y)$ at the extremities in the horizontal direction is given by

$$\phi(x, y) \to \begin{cases} A_0 e^{i\lambda x - \lambda y} + R\ e^{-i\lambda x - \lambda y} & \text{as } x \to -\infty \\ T\ e^{i\lambda x - \lambda y} & \text{as } x \to \infty, \end{cases} \tag{2.6}$$

representing progressive waves. R and T are two unknown complex constants, known as the reflection and transmission coefficients in wave scattering, to be determined (see Stoker [**17**]), and λ is the positive real root of the polynomial equation $Mx^3 + x - K = 0$.

The constant A_0 is assumed to be known and takes the value of either 0 or 1 depending upon the particular physical problem under consideration. For the wave-maker problem, $A_0 = 0$ and, in this case, R and T represent the radiation constants for waves on both sides of an incomplete vertical wave-maker. Further, when $A_0 = 1$ and $U(y) = 0$, the resulting case is for the scattering of an incident wave by a vertical barrier.

The edge conditions, as required for the energy to be finite in the neighborhood of all edges associated with the flow (see [**11**, subsection 2.4]), are given by

$$\frac{\partial \phi}{\partial x}(0, y) \sim O(|y - t|^{-1/2}) \quad \text{as } y \to t, \tag{2.7}$$

where $t = b^-$ or b^+, the edge point of the thin vertical structure under consideration.

3. The method of solution. It can be shown (Manam et al. [**10**]) that the unknown velocity potentials $\phi(x, y)$ in two regions $x > 0$ and $x < 0$, are expanded as

(3.1)

$$\phi(x,y) = \begin{cases} T\ e^{i\lambda x - \lambda y} + \int_0^\infty B(\xi) \\ [\xi(1 - M\xi^2)\cos \xi y - K \sin \xi y]\ e^{-\xi x}\, d\xi & x > 0, \\ A_0 e^{i\lambda x - \lambda y} + R\ e^{-i\lambda x - \lambda y} \\ + \int_0^\infty A(\xi)\ [\xi(1 - M\xi^2)\cos \xi y - K \sin \xi y]\ e^{\xi x}\, d\xi & x < 0, \end{cases}$$

where $A(\xi)$ and $B(\xi)$ are unknown functions to be determined, along with the unknown constants R and T which are the reflected and transmitted parts of the incident wave $A_0 e^{i\lambda x - \lambda y}$.

The above potential function ϕ automatically satisfies the partial differential equation (2.1) and conditions (2.2), (2.5) and (2.6) for an appropriate choice of the functions $A(\xi)$ and $B(\xi)$, which will be determined.

3.1. Submerged wave-maker or barrier. The definition sketch of the problem in this case is shown in Figure 1 (a). Since the horizontal

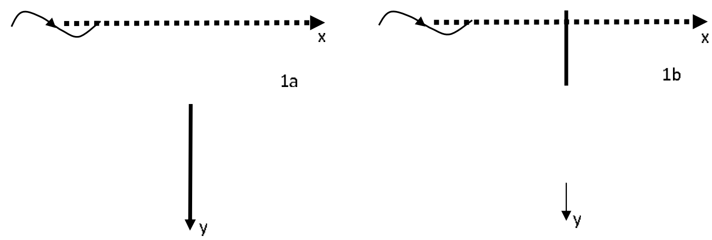

FIGURE 1. Schematic diagram for membrane covered surface with (a) submerged, (b) surface piercing vertical barrier.

velocity component is continuous across the positive y-axis, it may be shown that

$$\int_0^\infty \xi[A(\xi)+B(\xi)]\,\Big[\xi(1-M\xi^2)\cos\xi y - K\sin\xi y\Big]\,d\xi = i\lambda(T+R-A_0)\ e^{-\lambda y}, \quad \text{for } y\geq 0.$$

The functions $[\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]$, $\xi, y\geq 0$ and $e^{-\lambda y}$ are orthogonal with respect to the mode-coupling relation(see Manam et al. [**10**])

$$\langle f,g\rangle = \int_0^\infty f(y)g(y)\,dy + \frac{M}{K}f'(0)g'(0),$$

where $'$ denotes the derivative, and hence we find that

$$T = A_0 - R; \qquad A(\xi) = -B(\xi).$$

Conditions (2.3) and (2.4) along with relations (3.1) give rise to a pair of integral equations given by

$$\int_0^\infty A(\xi)\ [\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]\,d\xi = -R\ e^{-\lambda y}, \quad \text{on } y\in(0,b),$$

$$\int_0^\infty \xi A(\xi)\ [\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]\,d\xi = -i\lambda(A_0-R)\ e^{-\lambda y} + U(y), \quad \text{on } y\in(b,\infty),$$

and these can be rewritten as

(3.2)
$$\mathcal{L}\int_0^\infty A(\xi)\ \sin \xi y\, d\xi = -R\ e^{-\lambda y}, \quad \text{on } y \in (0,b)$$
(3.3)
$$\mathcal{L}\int_0^\infty \xi A(\xi)\ \sin \xi y\, d\xi = -i\lambda(A_0 - R)\ e^{-\lambda y} + U(y), \quad \text{on } y \in (b,\infty),$$

where
$$\mathcal{L} = \left(M\frac{d^3}{dy^3} + \frac{d}{dy} - K\right).$$

The above ordinary differential equations (3.2) and (3.3) can be easily solved to give the following dual integral equations:

$$\int_0^\infty A(\xi)\ \sin \xi y\, d\xi = C_1 e^{\lambda y} + C_2 e^{\lambda_1 y} + C_3 e^{\overline{\lambda}_1 y} + \frac{Re^{-\lambda y}}{Q(\lambda)} \equiv f(y), \quad \text{for } y \in (0,b) \tag{3.4}$$

and

(3.5)
$$\int_0^\infty \xi A(\xi)\ \sin \xi y\, d\xi = D_1 e^{\lambda_1 y} + D_2 e^{\overline{\lambda}_1 y} + D_3 e^{\lambda y} + \frac{i\lambda(A_0 - R)}{Q(\lambda)} e^{-\lambda y} + \mathcal{L}^{-1}[U(y)] \equiv h(y), \quad \text{for } y \in (b,\infty),$$

where $C_1, C_2, C_3, D_1, D_2, D_3$ are arbitrary constants, $\mathcal{L}^{-1}[U(y)]$ is the particular integral of $U(y)$ with respect to the differential operator $\mathcal{L}$, $Q(\lambda) = \lambda(1+M\lambda^2)+K$ and $\lambda_1, \overline{\lambda}_1$ are complex roots of $Mx^3+x-K = 0$.

Accommodating zero and infinity along the positive y-axis, the arbitrary constant D_3 in (3.5) must be taken as zero. Also, since $f(y) = \int_0^\infty A(\xi)\ \sin \xi y\, d\xi$, for $y \in (0,b)$ is the solution of the third order differential equation (3.2), one must have $f(0) = f''(0) = 0$.

That is,

$$C_1 + C_2 + C_3 + \frac{1}{Q(\lambda)} R = 0 \tag{3.6}$$

and

$$\lambda^2 C_1 + \lambda_1^2 C_2 + \bar{\lambda}_1^2 C_3 + \frac{\lambda^2}{Q(\lambda)} R = 0. \tag{3.7}$$

Now we define

$$\int_0^\infty \xi A(\xi) \sin \xi y \, d\xi = g(y), \quad \text{for } y \in (0, b), \tag{3.8}$$

where $g(y)$ is an unknown function to be determined.

Relations (3.5) and (3.8), as an application of the Fourier sine transform, yield

$$A(\xi) = \frac{2}{\pi \xi} \int_0^\infty P(y) \ \sin \xi y \, dy, \tag{3.9}$$

where

$$P(y) = \begin{cases} g(y) & \text{for } y \in (0, b), \\ h(y) & \text{for } y \in (b, \infty). \end{cases}$$

Putting $A(\xi)$ into equation (3.4) and, after utilizing the standard integral (see Gradshteyn and Ryzhik [**8**, equation 3.741 (1)])

$$\int_0^\infty \frac{\sin \xi y \ \sin \xi t}{\xi} d\xi = -\frac{1}{2} \log \left| \frac{y - t}{y + t} \right|, \quad \text{for } y, t \in (0, \infty),$$

the unknown function $g(y)$ satisfies the weakly singular integral equation

$$\frac{1}{\pi} \int_0^b g(u) \log \left| \frac{u + x}{u - x} \right| du = f_1(x), \quad \text{for } x \in (0, b), \tag{3.10}$$

where

$$f_1(x) = f(x) - \frac{1}{\pi} \int_b^\infty h(t) \log \left| \frac{x + t}{x - t} \right| dt.$$

The dual integral equations (3.4) and (3.5) can be differentiated twice as they satisfy a third order differential equation, and hence

$$\int_0^\infty \xi^2 A(\xi) \ \sin \xi y \, d\xi = -\frac{d^2 f}{dy^2}, \quad \text{for } y \in (0, b) \tag{3.11}$$

and

$$\int_0^\infty \xi^3 A(\xi)\ \sin \xi y \, d\xi = -\frac{d^2 h}{dy^2}, \quad \text{for } y \in (b, \infty).$$

Also, since

$$\int_0^\infty \xi^3 A(\xi)\ \sin \xi y \, d\xi = -\frac{d^2 g}{dy^2}, \quad \text{for } y \in (0, b),$$

it is clear that

$$\xi^2 A(\xi) = -\frac{2}{\pi \xi} \int_0^\infty \frac{d^2 P}{dy^2}\ \sin \xi y \, dy. \tag{3.12}$$

Inserting $\xi^2 A(\xi)$ from relation (3.12) into relation (3.11), it is found that $g_1 = d^2 g/dy^2$ also satisfies the weakly singular integral equation

$$\frac{1}{\pi} \int_0^b g_1(u) \log \left| \frac{u+x}{u-x} \right| du = f_2(x), \quad \text{for } x \in (0, b), \tag{3.13}$$

where

$$f_2(x) = \frac{d^2 f}{dx^2} - \frac{1}{\pi} \int_b^\infty \frac{d^2 h}{dt^2} \log \left| \frac{x+t}{x-t} \right| dt.$$

It may be noted that the logarithmic singular integral equation (3.10) or (3.13) has many forms of solution depending upon its solution behavior at the endpoints. Here, the behavior of $g(u)$ at the endpoint $u = b$ is investigated by letting

$$\frac{\partial \phi}{\partial x}(0, y) = F(y), \quad \text{for } y \in (0, b).$$

Then, we have from relation (3.1) that

$$\left(M \frac{d^3}{dy^3} + \frac{d}{dy} - K \right) \int_0^\infty \xi\ A(\xi)\ \sin \xi y \, d\xi = -i\lambda(A_0 - R)\ e^{-\lambda y} + F(y), \quad \text{for } y \in (0, b). \tag{3.14}$$

Clearly, relation (2.7) gives the behavior of $F(y)$ at $y = b$ and relation (3.14) makes the functions $g(y)$ and $d^2 g/dy^2$ bounded at the endpoint $y = b$ with a behavior as described by

$$\frac{d^2 g}{dy^2} \sim O(|y - t|^{1/2}) \quad \text{as } y \to b^-.$$

Therefore, the bounded solution of integral equation (3.10) is given by (see Chakrabarti et al. [**2**])

$$g(u) = \frac{2}{\pi} u\sqrt{b^2-u^2}\int_0^b \frac{f_1'(t)}{\sqrt{b^2-t^2}(u^2-t^2)}\,dt, \quad u \in (0,b), \tag{3.15}$$

with the condition on the forcing function that

$$\int_0^b \frac{f_1'(t)}{\sqrt{b^2-t^2}}\,dt = 0. \tag{3.16}$$

Also, the bounded solution $g_1(u)$ of integral equation (3.13) is

$$g_1(u) = \frac{d^2 g}{du^2} = \frac{2}{\pi} u\sqrt{b^2-u^2}\int_0^b \frac{f_2'(t)}{\sqrt{b^2-t^2}}(u^2-t^2)\,dt, \quad u \in (0,b), \tag{3.17}$$

provided that

$$\int_0^b \frac{f_2'(t)}{\sqrt{b^2-t^2}}\,dt = 0. \tag{3.18}$$

Equating $A(\xi)$ in the relations (3.9) and (3.12), one may derive by integration by parts that the following conditions must be satisfied by the functions g and h: $h(b)=0$, i.e.,

$$e^{\lambda_1 b}D_1 + e^{\overline{\lambda}_1 b}D_2 - \frac{i\lambda}{Q(\lambda)}e^{-\lambda b}R = -A_0\frac{i\lambda}{Q(\lambda)}e^{-\lambda b} - \mathcal{L}^{-1}[U(y)]\Big|_{y=b} \tag{3.19}$$

and

$$g'(b) = h'(b). \tag{3.20}$$

The following integrals (see [**8**, 3.387 (5, 6)]) are used to express conditions (3.16) and (3.18) in terms of the unknown constants:

$$(3.21)\quad \text{(i)}\quad \int_0^b \frac{dt}{\sqrt{b^2-t^2}(x^2-t^2)} = \frac{\pi}{2x\sqrt{x^2-b^2}}, \quad \text{for } x > b,$$

$$\text{(ii)}\quad J_1(x) \equiv \int_0^b \frac{e^{xt}}{\sqrt{b^2-t^2}}\,dt = \frac{\pi}{2}\big[I_0(bx) + L_0(bx)\big],$$

$$\text{(iii)}\quad J_2(x) \equiv \int_0^b \frac{t^2 e^{xt}}{\sqrt{b^2-t^2}}\,dt = \frac{b^2\pi}{2}\big[I_0(bx) + L_0(bx)\big] - \frac{b\pi}{2x}\big[I_1(bx) + L_1(bx)\big],$$

$$\text{(iv)}\quad J_3(x) \equiv \int_b^\infty \frac{e^{xt}}{\sqrt{t^2-b^2}}\,dt = K_0(-bx),\ \operatorname{Re}(x) < 0,$$

$$\text{(v)}\quad J_4(x) \equiv \int_b^\infty \frac{t^2 e^{xt}}{\sqrt{t^2-b^2}}\,dt = -\frac{b}{x}K_1(-bx) + b^2 K_0(-bx),\ \operatorname{Re}(x) < 0,$$

where K_0, K_1, I_0 and I_1 are the modified Bessel functions and L_0, L_1 are the Struve functions.

Then, relations (3.16) and (3.18) become

$$(3.22)\quad \lambda J_1(\lambda)C_1 + \lambda_1 J_1(\lambda_1)C_2 + \overline{\lambda}_1 J_1(\overline{\lambda}_1)C_3 - \left[\frac{\lambda}{Q(\lambda)}J_1(-\lambda) + \frac{i\lambda}{Q(\lambda)}J_3(-\lambda)\right]R - J_3(\lambda_1)D_1 - J_3(\overline{\lambda}_1)D_2 = \frac{i\lambda}{Q(\lambda)}J_3(-\lambda)A_0 + \int_b^\infty \mathcal{L}^{-1}[U(t)]\frac{1}{\sqrt{t^2-b^2}}\,dt$$

and

$$(3.23)\quad \lambda^3 J_1(\lambda)C_1 + \lambda_1^3 J_1(\lambda_1)C_2 + \overline{\lambda}_1^3 J_1(\overline{\lambda}_1)C_3 - \left[\frac{\lambda^3}{Q(\lambda)}J_1(-\lambda) + \frac{i\lambda^3}{Q(\lambda)}J_3(-\lambda)\right]R - \lambda_1^2 J_3(\lambda_1)D_1 - \overline{\lambda}_1^2 J_3(\overline{\lambda}_1)D_2$$

$$= \frac{i\lambda^3}{Q(\lambda)} J_3(-\lambda) A_0 + \int_b^\infty \frac{d^2}{dt^2}\Big\{\mathcal{L}^{-1}[U(t)]\Big\} \frac{1}{\sqrt{t^2-b^2}}\, dt.$$

At this stage, the unknown constants C_1, C_2, C_3, D_1, D_2 and R can be determined, in principle, from relations (3.6), (3.7) and (3.19)–(3.23). However, condition (3.20) is not in a suitable form for algebraic manipulation. Condition (3.20) can be modified into a suitable form by integrating the relation (3.17) from 0 and b. This results in

$$-bg'(b) = \int_0^b \frac{t^2 f'''(t)}{\sqrt{b^2-t^2}}\, dt - \frac{4}{\pi^2}\int_b^\infty \frac{t^2 h''(t)}{\sqrt{t^2-b^2}}\, dt + \frac{4}{\pi^2}\int_b^\infty t\; h''(t)\, dt, \tag{3.24}$$

upon utilizing the integrals

(vi) $$\int_0^b \frac{t^2\sqrt{b^2-t^2}}{(t^2-x^2)}\, dt = -\frac{\pi}{4}[2x^2-b^2], \quad \text{for } 0 < x < b,$$

(vii) $$\int_0^b \frac{t^2}{\sqrt{b^2-t^2}(t^2-x^2)}\, dt = -\frac{\pi}{2}\left[\frac{x}{\sqrt{x^2-b^2}} - 1\right], \quad \text{for } x > b.$$

Thus, we derive from equations (3.20) and (3.24) that

$$\begin{aligned}
&\lambda^3\left[J_2(\lambda) - \frac{1}{2}J_1(\lambda)\right]C_1 + \lambda_1^3\left[J_2(\lambda_1) - \frac{1}{2}J_1(\lambda_1)\right]C_2 \\
&\quad + \overline{\lambda}_1^3\left[J_2(\overline{\lambda}_1) - \frac{1}{2}J_1(\overline{\lambda}_1)\right]C_3 - \left[\frac{\lambda^3}{Q(\lambda)}\left\{J_2(-\lambda) - \frac{1}{2}J_1(-\lambda)\right\}\right. \\
&\quad \left. - \frac{i\lambda^3}{Q(\lambda)}\left\{\frac{4}{\pi^2}J_4(-\lambda) - \frac{1}{2}J_3(-\lambda)\right\} - \frac{ib\lambda^2}{Q(\lambda)}e^{-\lambda b}\left(1 - \frac{4}{\pi^2}\right)\right]R \\
&\quad - \left[\lambda_1^2\left\{\frac{4}{\pi^2}J_4(\lambda_1) + \frac{1}{2}J_3(\lambda_1)\right\} - b\lambda_1 e^{\lambda_1 b}\left(1 - \frac{4}{\pi^2}\right)\right]D_1 \\
&\quad - \left[\overline{\lambda}_1^2\left\{\frac{4}{\pi^2}J_4(\overline{\lambda}_1) + \frac{1}{2}J_3(\overline{\lambda}_1)\right\} - b\overline{\lambda}_1 e^{\overline{\lambda}_1 b}\left(1 - \frac{4}{\pi^2}\right)\right]D_2 \\
&= A_0\frac{i\lambda^3}{Q(\lambda)}\left[\frac{4}{\pi^2}J_4(-\lambda) + \frac{1}{2}J_3(-\lambda)\right]
\end{aligned} \tag{3.25}$$

$$+ A_0 \frac{ib\lambda^2 e^{-\lambda b}}{Q(\lambda)}\left(1 - \frac{4}{\pi^2}\right)$$
$$+ \frac{4}{\pi^2}\int_b^\infty \frac{d^2}{dt^2}\Big\{\mathcal{L}^{-1}[U(t)]\Big\}\frac{t^2}{\sqrt{t^2-b^2}}\, dt$$
$$- b\left(1 - \frac{4}{\pi^2}\right)\frac{d}{dt}\Big\{\mathcal{L}^{-1}[U(t)]\Big\}\Big|_{t=b},$$

where $J_1(x)$, $J_2(x)$, $J_3(x)$ and $J_4(x)$ are given in relation (3.21).

Hence, we have obtained the description of the analytical solution for the general mixed boundary value problem posed in Section 2, corresponding to the case of the submerged wave-maker or barrier. Clearly, all the unknown constants in relations (3.4) and (3.5) can be obtained from expressions (3.6), (3.7), (3.19), (3.22), (3.23) and (3.25).

3.2. Surface piercing wave-maker or barrier. Unlike in the previous case, the normal component of the velocity is not continuous at the intersection of the free surface and the rigid barrier. A schematic diagram can be seen in Figure 1 (b). Writing the difference of the velocity components on both sides of the barrier interface $x = 0$, it may be obtained that

$$\int_0^\infty \xi[A(\xi) + B(\xi)]\ \Big[\xi(1 - M\xi^2)\cos\xi y - K\sin\xi y\Big]\, d\xi$$
$$= (\phi_x(0^-, y) - \phi_x(0^+, y)) + i\lambda(T + R - A_0)\ e^{-\lambda y}, \quad \text{for } y > 0.$$

Since the functions $[\xi(1 - M\xi^2)\cos\xi y - K\sin\xi y]$, $\xi, y > 0$ and $e^{-\lambda y}$ are orthogonal with respect to the mode-coupling relation(see Manam et al. [**10**]

$$\langle f, g\rangle = \int_0^\infty f(y)g(y)\, dy + \frac{M}{K} f'(0)g'(0),$$

it can be obtained that

(3.26)
$$T = A_0 - R + \frac{2iM(\mu^+ - \mu^-)}{1 + 3M\lambda^2}; \qquad B(\xi) = -A(\xi) - \frac{2M(\mu^+ - \mu^-)}{\pi\Delta(\xi)},$$

where $\Delta(\xi) = \xi^2(1 - M\xi^2)^2 + K^2$ and $\mu^\pm = \phi_{xy}(0^\pm, 0)$ are the prescribed edge slope constants.

In this case, the pair of integral equations resulting from conditions (2.3) and (2.4) is

(3.27)
$$\mathcal{L}\int_0^\infty \xi A(\xi)\ \sin\xi y\, d\xi = -i\lambda(A_0 - R)\ e^{-\lambda y} + U(y), \quad \text{on } y \in (0, b)$$

and

$$(3.28)\quad \mathcal{L}\int_0^\infty A(\xi)\ \sin\xi y\, d\xi = -\frac{M}{2}(\mu^+ - \mu^-)\int_0^\infty \frac{\xi(1 - M\xi^2)\cos\xi y - K\sin\xi y}{\Delta(\xi)}\, d\xi - \left[R - \frac{iM(\mu^+ - \mu^-)}{1 + 3M\lambda^2}\right]\ e^{-\lambda y}, \quad \text{on } y \in (b, \infty).$$

Upon integrating the equations (3.27) and (3.28), the following dual integral equations are obtained:

$$(3.29)\quad \int_0^\infty \xi A(\xi)\ \sin\xi y\, d\xi = C_1 e^{\lambda y} + C_2 e^{\lambda_1 y} + C_3 e^{\overline{\lambda}_1 y} + \frac{i\lambda(A_0 - R)}{Q(\lambda)} e^{-\lambda y} + \mathcal{L}^{-1}[U(y)] \equiv f(y), \quad \text{for } y \in (0, b),$$

$$(3.30)\quad \int_0^\infty A(\xi)\ \sin\xi y\, d\xi = D_1 e^{\lambda_1 y} + D_2 e^{\overline{\lambda}_1 y} + D_3 e^{\lambda y} + \left[R - \frac{iM(\mu^+ - \mu^-)}{1 + 3M\lambda^2}\right]\frac{e^{-\lambda y}}{Q(\lambda)} - \frac{M}{2}(\mu^+ - \mu^-)\int_0^\infty \frac{\sin\xi y}{\Delta(\xi)}\, d\xi \equiv h(y), \quad \text{for } y \in (b, \infty),$$

where C_1, C_2, C_3, D_1, D_2 and D_3 are arbitrary constants and

$$\mathcal{L}^{-1}[U(y)] = e^{\lambda y}\int \frac{U(y)W_1(y)}{W(y)}\, dy + e^{\lambda_1 y}\int \frac{U(y)W_2(y)}{W(y)}\, dy + e^{\overline{\lambda}_1 y}\int \frac{U(y)W_3(y)}{W(y)}\, dy,$$

with

$$\begin{aligned}W_1(y) &= (\overline{\lambda}_1 - \lambda_1)e^{(\overline{\lambda}_1+\lambda_1)y},\\ W_2(y) &= (\lambda - \overline{\lambda}_1)e^{(\overline{\lambda}_1+\lambda)y},\\ W_3(y) &= (\lambda_1 - \lambda)e^{(\lambda_1+\lambda)y},\\ W(y) &= [\overline{\lambda}_1^2(\lambda_1 - \lambda) + \lambda_1^2(\lambda - \overline{\lambda}_1) + \lambda^2(\overline{\lambda}_1 - \lambda_1)]e^{(\lambda+\lambda_1+\overline{\lambda}_1)y}.\end{aligned}$$

Clearly, the arbitrary constant D_3 in (3.30) may taken to be zero and the function $f(y)$ must satisfy $f(0) = f''(0) = 0$, i.e.,

$$C_1 + C_2 + C_3 - \frac{i\lambda}{Q(\lambda)}R = -A_0\frac{i\lambda}{Q(\lambda)} - \mathcal{L}^{-1}[U(y)]\Big|_{y=0} \tag{3.31}$$

and

$$\lambda^2 C_1 + \lambda_1^2 C_2 + \overline{\lambda}_1^2 C_3 - \frac{i\lambda^3}{Q(\lambda)}R = -A_0\frac{i\lambda^3}{Q(\lambda)} - \frac{d^2}{dy^2}\Big\{\mathcal{L}^{-1}[U(y)]\Big\}\Big|_{y=0}. \tag{3.32}$$

Letting

$$\int_0^\infty \xi A(\xi)\ \sin\xi y\, d\xi = g(y), \quad \text{for } y \in (b,\infty),$$

where $g(y)$ is an unknown function to be determined, it may be obtained from relation (3.29) that

$$A(\xi) = \frac{2}{\pi\xi}\int_0^\infty P(y)\ \sin\xi y\, dy, \tag{3.33}$$

where

$$P(y) = \begin{cases} f(y) & \text{for } y \in (0,b)\\ g(y) & \text{for } y \in (b,\infty).\end{cases}$$

By putting $A(\xi)$ into the relations (3.30), the unknown function $g(y)$ satisfies the weakly singular integral equation

$$\frac{1}{\pi}\int_b^\infty g(u)\log\left|\frac{u+x}{u-x}\right| du = h_1(x), \quad \text{for } x \in (b,\infty), \tag{3.34}$$

where

$$h_1(x) = h(x) - \frac{1}{\pi}\int_0^b f(t)\log\left|\frac{x+t}{x-t}\right| dt.$$

Differentiating relations (3.29) and (3.30) twice, one can obtain

$$\xi^2 A(\xi) = -\frac{2}{\pi\xi}\int_0^\infty \frac{d^2P}{dy^2}\ \sin\xi y\, dy, \tag{3.35}$$

where the function $g_1 = (d^2g)/(dy^2)$ satisfies the weakly singular integral equation

$$\frac{1}{\pi}\int_b^\infty g_1(u)\log\left|\frac{u+x}{u-x}\right| du = h_2(x), \quad \text{for } x \in (b,\infty), \tag{3.36}$$

with

$$h_2(x) = \frac{d^2h}{dx^2} - \frac{1}{\pi}\int_b^\infty \frac{d^2f}{dt^2}\log\left|\frac{x+t}{x-t}\right| dt.$$

It can be shown, by a similar argument given for the previous case, that the unknown functions $g(y)$ and $(d^2g)/(dy^2)$ are bounded at the endpoint $y = b$ with the behavior

$$\frac{d^2g}{dy^2} \sim O(|y-t|^{1/2}) \quad \text{as } y \to b^+.$$

Hence, the bounded solutions for the above integral equations (3.34) and (3.36) are given by

$$g(u) = \frac{2}{\pi}\frac{\sqrt{u^2-b^2}}{u^2}\int_b^\infty \frac{t\ h_1'(t)}{\sqrt{t^2-b^2}(t^2-u^2)}\,dt, \quad u \in (b,\infty),$$

provided that

$$\int_b^\infty \frac{t\ h_1'(t)}{\sqrt{t^2-b^2}}\,dt = 0 \tag{3.37}$$

and

(3.38)

$$g_1(u) = \frac{d^2g}{du^2} = \frac{2}{\pi}\frac{\sqrt{u^2-b^2}}{u^2}\int_b^\infty \frac{t\ h_2'(t)}{\sqrt{t^2-b^2}(t^2-u^2)}\,dt, \quad u \in (b,\infty),$$

provided that

$$\int_b^\infty \frac{t\ h_2'(t)}{\sqrt{t^2-b^2}}\,dt = 0. \tag{3.39}$$

Utilizing the following integrals (see [**8**, 3.389 (3, 4)])

$$\text{(viii)} \quad \int_b^\infty \frac{t}{\sqrt{t^2-b^2}(t^2-x^2)}\,dt = \frac{\pi}{2\sqrt{b^2-x^2}}, \quad \text{for } x<b,$$

$$\text{(ix)} \quad J_5(x) \equiv \int_0^b \frac{te^{xt}}{\sqrt{b^2-t^2}}\,dt = b\left[1+\frac{\pi}{2}\Big(I_1(xb)+L_1(xb)\Big)\right],$$

$$\text{(x)} \quad J_6(x) \equiv \int_b^\infty \frac{te^{xt}}{\sqrt{t^2-b^2}}\,dt = bK_1(-xb), \quad \operatorname{Re}(x)<0,$$

$$\text{(xi)} \quad \int_b^\infty \frac{\sin xt}{\sqrt{t^2-b^2}}\,dt = \frac{\pi}{2}J_0(xb),$$

with J_0 being a Bessel function of the first kind, conditions (3.37) and (3.39) can be expressed in terms of the unknown constants as

$$\begin{aligned}(3.40)\quad & J_5(\lambda)C_1 + J_5(\lambda_1)C_2 + J_5(\overline{\lambda}_1)C_3 \\ & - \left[\frac{i\lambda}{Q(\lambda)}J_5(-\lambda) - \frac{\lambda}{Q(\lambda)}J_6(-\lambda)\right]R - \lambda_1 J_6(\lambda_1)D_1 \\ & - \overline{\lambda}_1 J_6(\overline{\lambda}_1)D_2 = -A_0\frac{i\lambda}{Q(\lambda)}J_5(-\lambda) \\ & + \frac{M}{2}(\mu^+ - \mu^-)\int_0^\infty \frac{\Delta(\xi)-\xi\Delta'(\xi)}{[\Delta(\xi)]^2}J_0(\xi b)\,d\xi \\ & + \frac{iM(\mu^+-\mu^-)\lambda}{(1+3M\lambda^2)Q(\lambda)}J_5(-\lambda) + \int_0^b \Big\{\mathcal{L}^{-1}[U(t)]\Big\}\frac{t}{\sqrt{b^2-t^2}}\,dt\end{aligned}$$

and

$$\begin{aligned}(3.41)\quad & \lambda^2 J_5(\lambda)C_1 + \lambda_1^2 J_5(\lambda_1)C_2 + \overline{\lambda}_1^2 J_5(\overline{\lambda}_1)C_3 \\ & - \left[\frac{i\lambda^3}{Q(\lambda)}J_5(-\lambda) - \frac{\lambda^3}{Q(\lambda)}J_6(-\lambda)\right]R \\ & - \lambda_1^3 J_6(\lambda_1)D_1 - \overline{\lambda}_1^3 J_6(\overline{\lambda}_1)D_2 \\ = & -A_0\frac{i\lambda^3}{Q(\lambda)}J_5(-\lambda) - \frac{M}{2}(\mu^+-\mu^-)\int_0^\infty \frac{3\xi^2\Delta(\xi)-\xi^3\Delta'(\xi)}{[\Delta(\xi)]^2}J_0(\xi b)\,d\xi \\ & + \frac{iM(\mu^+-\mu^-)\lambda^3}{(1+3M\lambda^2)Q(\lambda)}J_5(-\lambda)\int_0^b \frac{d^2}{dt^2}\Big\{\mathcal{L}^{-1}[U(t)]\Big\}\frac{t}{\sqrt{b^2-t^2}}\,dt.\end{aligned}$$

Also, matching relations (3.33), (3.35) and using the integration by parts result the following conditions are satisfied by functions g and h:

$$f(b) = 0, \quad \text{i.e.,} C_1 e^{\lambda b} + C_2 e^{\lambda_1 b} + C_3 e^{\overline{\lambda}_1 b} - R\frac{i\lambda}{Q(\lambda)} e^{-\lambda b}$$

$$(3.42) \qquad = -A_0 \frac{i\lambda}{Q(\lambda)} e^{-\lambda b} - \mathcal{L}^{-1}[U(y)]\Big|_{y=b}$$

and

$$(3.43) \qquad g'(b) = f'(b).$$

It may be pointed out that, contrary to the case of the immersed structure, condition (3.43) cannot be expressed in a suitable form by integrating the relation (3.38).

Instead, relation (3.30) is twice differentiated and integrated with respect to y, from b to ∞, so that

$$\int_0^\infty \xi^2 A(\xi) J_0(\xi b)\, d\xi = -\int_b^\infty \frac{h''(y)}{\sqrt{y^2 - b^2}}\, dy.$$

Utilizing the substitution from equation (3.35) and the integral (see [**8**, 6.693 (7)])

$$\int_0^\infty J_0(xb) \frac{\sin xy}{x}\, dx = \frac{\pi}{2}, \quad \text{for } y > b,$$

we derive that

$$\int_0^b f''(y)\, dy + \int_b^\infty g''(y)\, dy = \int_b^\infty \frac{h''(y)}{\sqrt{y^2 - b^2}}\, dy.$$

After elementary integration and using equation (3.44), the above equation reduces to

$$(3.44) \qquad \int_b^\infty \frac{h''(y)}{\sqrt{y^2 - b^2}}\, dy + f'(0) = 0.$$

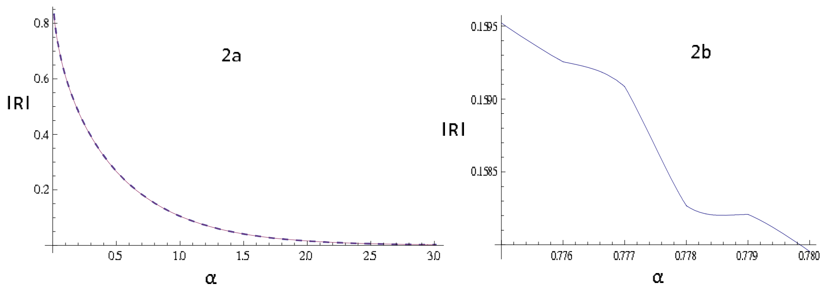

FIGURE 2. (a) Reflection coefficient $|R|$ in the limiting case $M \to 0$ ($\beta = 0.0001$) for immersed barrier; (b) window showing variation in the curve.

Thus, the above relation (3.454) can be expressed in terms of the unknown constants and is given by

$$\begin{aligned}(3.45)\quad \lambda C_1 + \lambda_1 C_2 + \overline{\lambda}_1 C_3 + \left[\frac{i\lambda^2}{Q(\lambda)}\Big(1 - iJ_3(-\lambda)\Big)\right] \\ + \lambda_1^2 J_3(\lambda_1) D_1 + \overline{\lambda}_1^2 J_3(\overline{\lambda}_1) D_2 \\ = A_0 \frac{i\lambda^2}{Q(\lambda)} + \frac{iM(\mu^+ - \mu^-)\lambda^2}{(1+3M\lambda^2)Q(\lambda)} J_3(-\lambda) \\ - \frac{M}{\pi}(\mu^+ - \mu^-)\int_0^\infty \frac{\xi^2}{\Delta(\xi)} J_0(b\xi)\, d\xi \\ - \frac{d}{dy}\Big\{\mathcal{L}^{-1}[U(y)]\Big\}\Big|_{y=0},\end{aligned}$$

with $J_3(x)$ given by relation (3.21).

This completes the determination of the analytical solution for the general mixed boundary value problem posed in Section 2, corresponding to the case of surface piercing wave-maker or barrier. All the unknown constants in the relations (3.29) and (3.30) can be obtained from the expressions (3.31), (3.32), (3.40)–(3.42) and (3.45).

3.3. Numerical discussion. By setting $A_0 = 0$ in the boundary value problem and its solution described in the previous subsections,

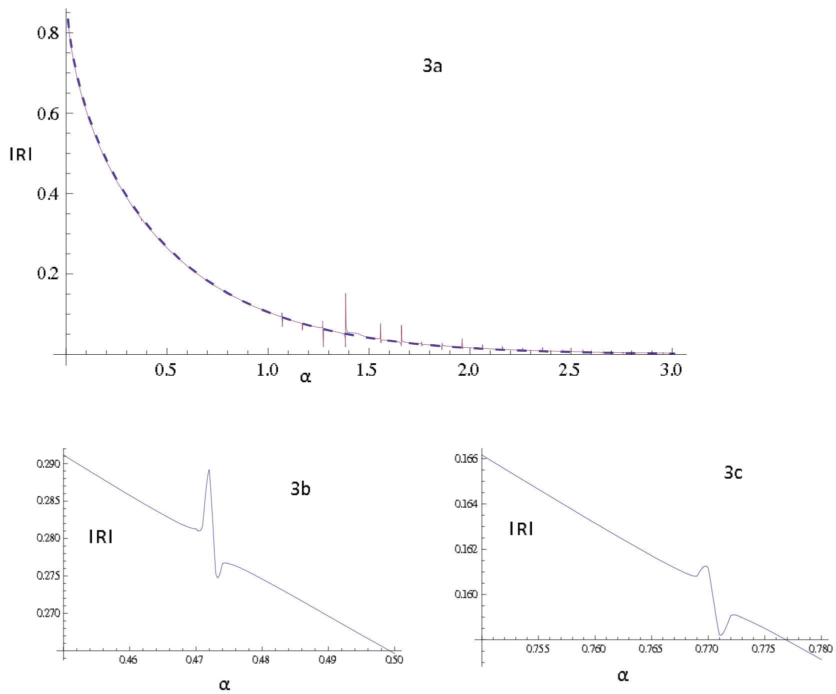

FIGURE 3. (a) Reflection coefficient $|R|$ for immersed barrier with $\beta = 0.001$; (b), (c) window around sharp rises.

we derive a complete analytical solution for the wave radiation, under surface tension, by the partial vertical wave-makers. Numerical computations are carried out in *Mathematica* for the cases when $A_0 = 1$ and $U(y) = 0$, corresponding to the incident water wave scattering by partial vertical barriers. The unknown constants C_1, C_2, C_3, D_1, D_2, R can be determined from the relations (3.6), (3.7), (3.19), (3.22), (3.23) and (3.25) in the case of an immersed vertical barrier and from the relations (3.31), (3.32), (3.40)–(3.42) and (3.45) in the case of a surface piercing barrier.

It is well known that, for water at $25^0 C$, surface tension is $M = 72$ dynes/cm, the density of water is $\rho = 1$ gm/cm^2 and the gravitation constant is $g = 981$ cm/sec^2. As water temperature varies over the range 5^0C–60^0C, surface tension too varies linearly and lies approximately between 60–80 dynes/cm. Reflection coefficients are calculated

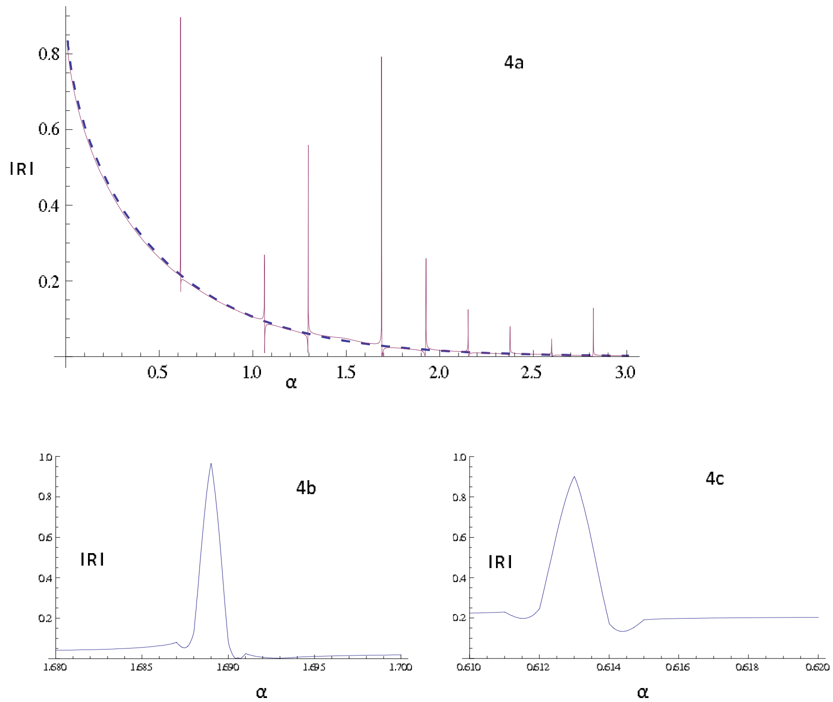

FIGURE 4. (a) Reflection coefficient $|R|$ with $\beta = 0.005$ for immersed barrier; (b), (c) window around sharp rises.

in *Mathematica* for different values of a non-dimensional surface tension parameter $\beta = MK^2$ against a non-dimensional wave parameter $\alpha = Kb$ in both cases of barrier configuration.

The boundary value problem under consideration is an isolated one in the sense that the wave motion is purely because of the surface tension and cannot exist for $M = 0$. But it is expected that calculations for very small β values match with that of the zero surface tension. For an immersed barrier, reflection coefficients in the limiting case, for example, $\beta = 0.00001$, are exact with the analytical result for zero surface tension as given by

$$R = \frac{K_0(Kb)}{K_0(Kb) + iI_0(Kb)}.$$

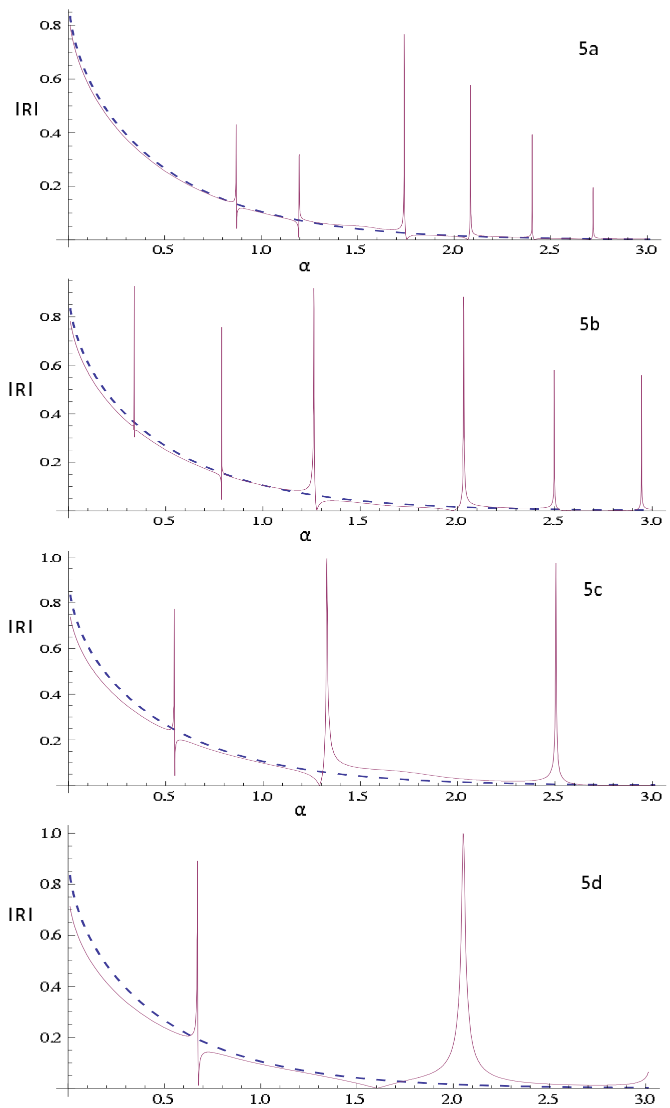

FIGURE 5. (a), (b), (c), (d) Reflection coefficients $|R|$ with $\beta = 0.01$, 0.02, 0.05 and 0.074, respectively, for an immersed barrier.

The biggest value of β used in the computation is the typical one for water at room temperature, that is, 0.074.

The graphical profiles of the reflection coefficient $|R|$, for a submerged barrier, have been plotted in Figures 2–5 for various β values. In Figure 2, reflection coefficients are shown for $\beta = 0.0001$. There is a negligible deviation in $|R|$ from the exact one as shown in the close up curve in sub-figure 1 (b) at around $\alpha = 0.78$, but otherwise it matches exactly. For a slight increase in the value of the tension parameter, i.e.,$\beta = 0.001$ or 0.005, there are spikes at certain frequencies for the values of $|R|$ which are depicted in Figures 3 and 4, along with the typical spikes elaborately shown in the close up curves. These spikes can be attributed to an interplay between the surface tension and the incident wave frequencies. The frequencies at which this happen are called resonant frequencies. However, as β gets higher, the spikes start appearing at lower frequencies, and the gap between their appearance also widens. This can be observed in Figure 5. These graphs show the effect of surface tension on the waves passing through incomplete vertical barriers. Finally, the energy conservation $|R|^2 + |T|^2 = 1$, in this case, has been numerically verified to be exact as expected.

In Figures 6–9, reflection coefficients, for a surface piercing barrier, are plotted for different values of β against the parameter α. As discussed earlier, since the slope of the surface elevation is not continuous across the barrier, one is expected to specify certain edge constants which are assumed to be known. Specifically, they are related to the contact angles of the free surface at the edges (see [**9**]). Also, because of surface tension, work must be done to maintain the contact angles throughout the harmonic time motion near the edges and hence the dissipation of energy. This would imply that, for very small values of β, there is a conservation of energy, and this has been verified numerically. However the reflection curves, with zero or non-zero edge slope difference across the edges, are having few smaller variations like spikes for negligible values of β as well. This may be expected considering the isolated nature of the problem. It has been observed numerically in the limiting case, for example, $\beta = 0.00001$ and $\mu^+ - \mu^- = 0$, reflection coefficients are almost exact with the analytical result

$$R = \frac{\pi I_1(Kb)}{\pi I_1(Kb) + iK_1(Kb)},$$

except at a few places with small spikes.

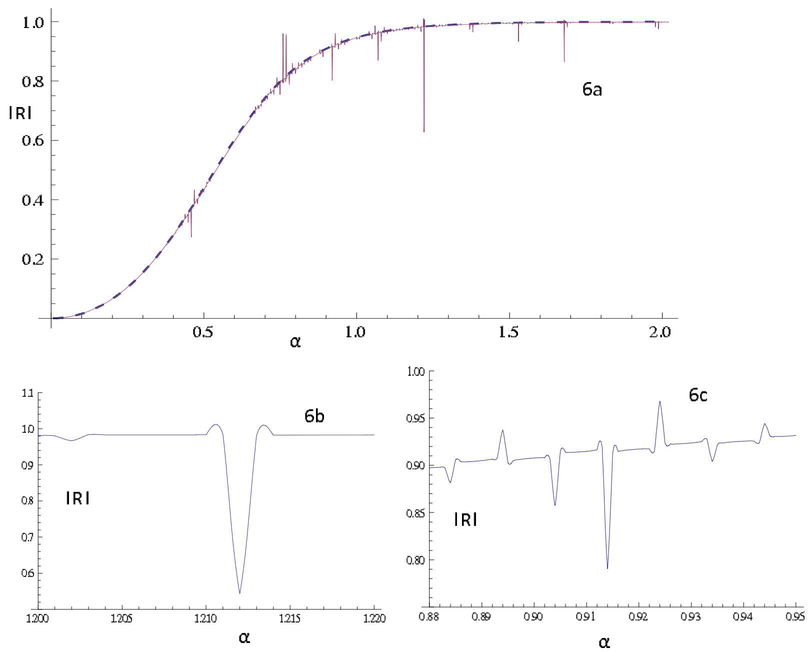

FIGURE 6. (a) Reflection coefficient $|R|$ for the surface piercing barrier with $\beta = 0.00001$ and $\mu^+ - \mu^- = 0.05$; (b), (c) window around sharp variations.

In Figure 6, reflection coefficients are shown for $\beta = 0.00001$ and $\mu^+ - \mu^- = 0.05$. Even here, with β being negligible, energy is conserved but the reflection curve has few considerable spikes otherwise matching with the analytical result. For a slight increase in the value of the tension parameter, i.e.,$\beta = 0.001$ or 0.005, there are many spikes for the values of $|R|$ as shown in Figures 7–9, along with a typical spike curve drawn in close up. That is, at these resonant frequencies, there is a partial or full wave transmission, and clearly it can seen the non-conservation of energy in the sub-figures 7 (d) and 8 (d). Also, as in the case of submerged barrier, the gap between the spikes increases for an increase in the values of the parameter β. Finally, reflection coefficients and corresponding energy curves for β values 0.01, 0.02,0.05, 0.074 are represented in sub-figures 9 (a), (b), (c) and (d), respectively.

3.4. Capillary-gravity wave trapping. It can be observed from the graphs that there is a complete reflection or transmission for certain resonant frequencies, in either of the barrier configurations. At these frequencies, physically, the progressive wave gets trapped on either side of the partial barrier. This phenomenon was also observed in the numerical study (see [**5, 21**]) of gravity waves at the end of a semi-infinitely long channel of finite depth, with either rigid or porous type immersed at the bottom. But it is interesting to see such phenomenon for capillary gravity waves passing through a partial immersed or piercing vertical barrier without any wall being in the background. However, this does not happen in the absence of surface tension. Indeed, the spikes in the reflection curves confirm the partial or complete capillary gravity wave trapping on either side of the non-complete barrier at those frequencies.

4. Conclusions. A special class of mixed boundary value problems for Laplace's equation with higher order boundary condition has been tackled for an analytical solution. This is associated with the capillary-gravity wave scattering, in deep water, by partial vertical barriers. Using certain mode-coupling relations, the mixed boundary value problem has been reduced to a dual integral equations with trigonometric kernel. These, in turn, are solved completely by the aid of a weakly singular integral equation. The bounded behavior of the solution of the weakly singular integral equation forces a mathematical constraint which finally helps in determining the unknowns associated with the dual integral equations. The graphical profiles of reflection and transmission coefficients have been plotted against a non-dimensional parameter and are found to match with the closed form expressions corresponding to the problem without surface tension. In this study, slopes of the surface elevation on both sides of the barrier are assumed to be known as the edge conditions. Hocking ([**9**]) has given dynamical edge conditions involving contact angles of the surface elevation, but again they are presumed to be known from the experiments. However, we have not utilized these edge conditions in the solution procedure as they too are again assumed to be constants. An extension of the present method of solution, to a mixed boundary value problem for the scattering of surface gravity waves under surface tension by vertical barrier with a finite number of gaps in it, is found to be possible (see [**3**]). Also, the solution method applied to solve these kinds of dual integral equations is useful in other physical situations where they arise naturally.

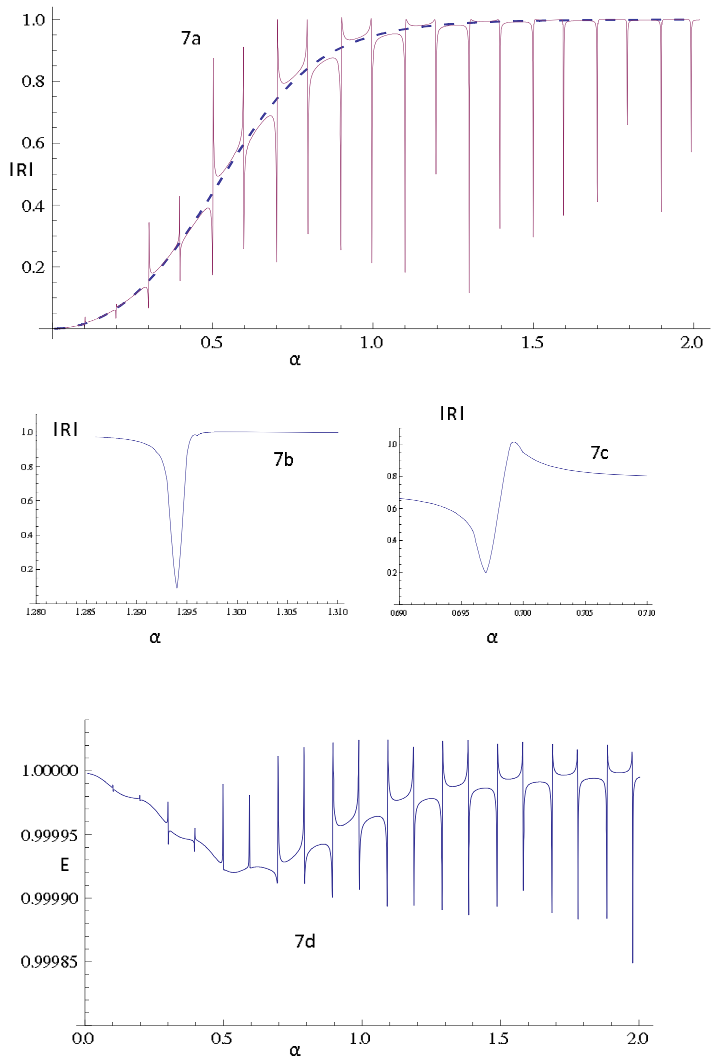

FIGURE 7. (a) Reflection coefficient $|R|$ for the surface piercing barrier with $\beta = 0.001$ and $\mu^{+} - \mu^{-} = 0.05$; (b), (c) window around sharp variations; (d) Total energy E.

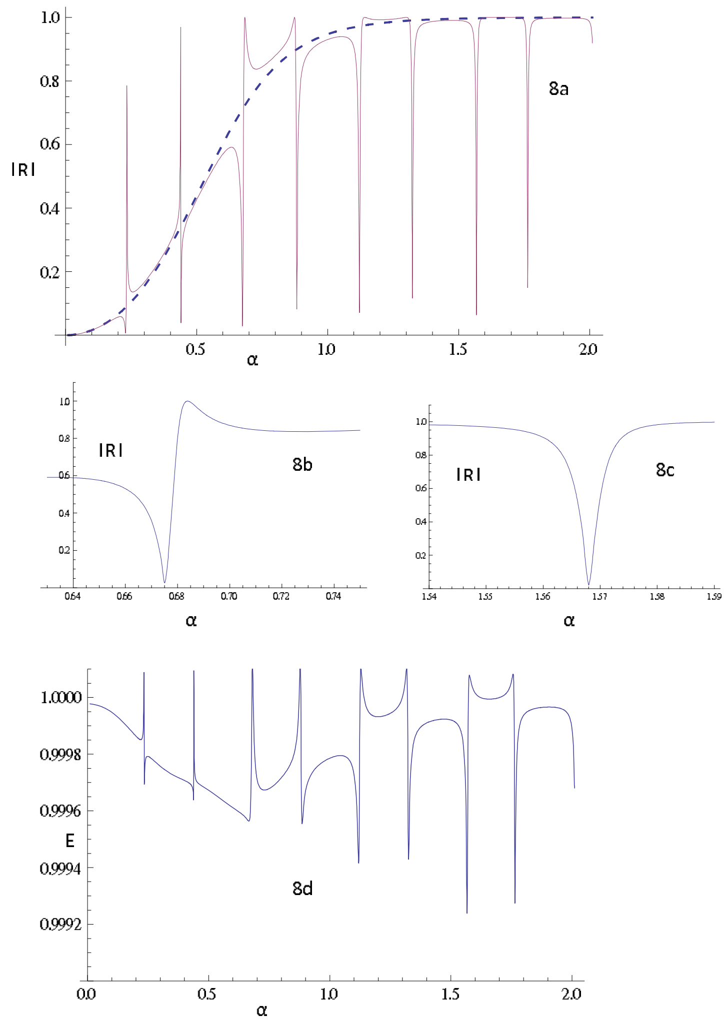

FIGURE 8. (a) Reflection coefficient $|R|$ for the surface piercing barrier with $\beta = 0.005$ and $\mu^{+} - \mu^{-} = 0.05$; (b), (c) window around sharp variations; (d) Total energy E.

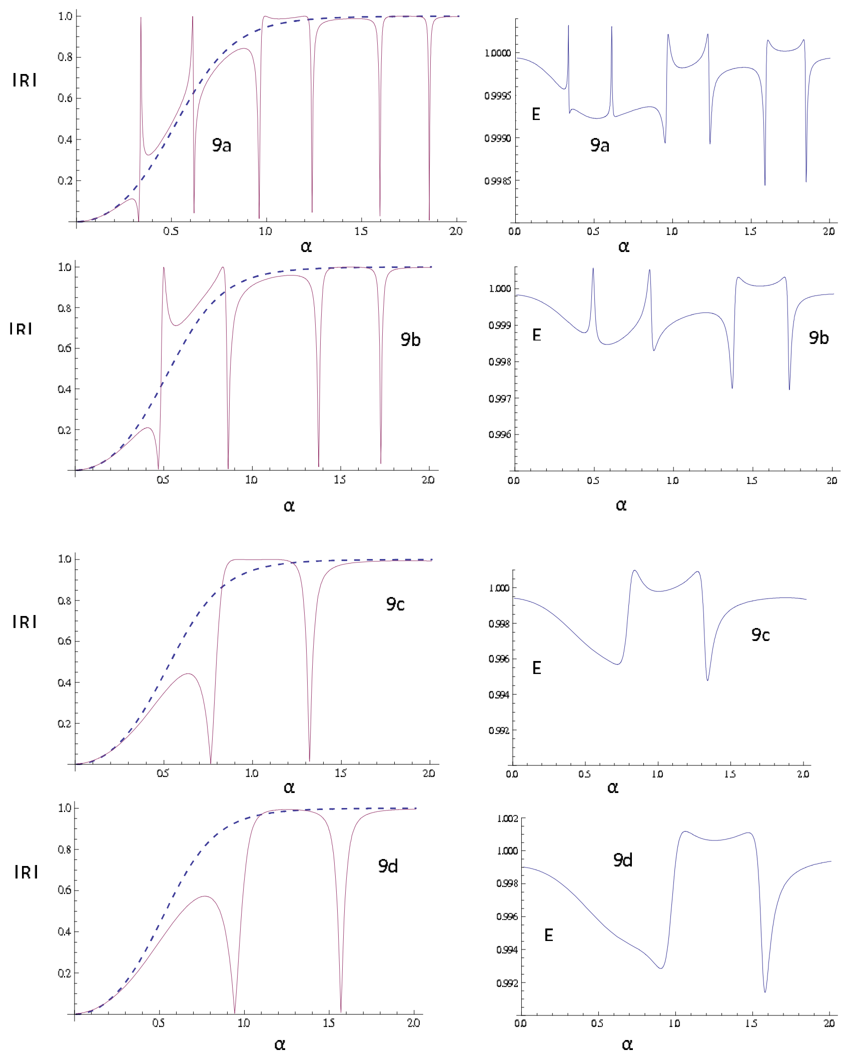

FIGURE 9. (a), (b), (c), (d) reflection coefficient $|R|$ and total energy E for $\beta = 0.01$, 0.02. 0.05, 0.074, respectively, in the surface piercing barrier case with $\mu^+ - \mu^- = 0.05$.

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REFERENCES

1. A. Chakrabarti, S. Banerjea, B.N. Mandal and T. Sahoo, *A unified approach to problems of scattering of surface water waves by vertical barriers*, J. Austral. Math. Soc. **39** (1997), 93–103.

2. A. Chakrabarti and S.R. Manam, *Solution of the logarithmic singular integral equation*, Appl. Math. Lett. **16** (2003), 369–373.

3. A. Chakrabarti, S.R. Manam and S. Banerjea, *Scattering of surface water waves involving a vertical barrier with a gap*, J. Engineer. Math. **45** (2003), 183–194.

4. A. Chakrabarti and B.N. Mandal, *Solution of some dual integral equations*, ZAMM **78** (1998), 141–144.

5. A.T. Chwang and Z.N. Dong, *Wave trapping due to a porous plate*, in Proc. 15th ONR Symposium on Naval Hydrodynamics, 1984, 407–414.

6. R. Estrada and R.P. Kanwal, *Integral equations with logarithmic kernels*, IMA J. Appl. Math. **43** (1989), 133–155.

7. D.V. Evans, *The influence of surface tension on the reflection of water waves by a plane vertical barrier*, Proc. Camb. Phil. Soc. **64** (1968), 795–810.

8. I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Academic Press, London, 1980.

9. L.M. Hocking, *Waves produced by a vertically oscillating plate*, J. Fluid Mech. **179** (1987), 267–281.

10. S.R. Manam, J. Bhattacharjee and T. Sahoo, *Expansion formulae in wave structure interaction problems*, Proc. Roy. Soc. Lond. **462** (2006), 263–287.

11. B.N. Mandal and A. Chakrabarti, *Water wave scattering by barriers*, WIT Press, Southapmton, 2000.

12. C.C. Mei, *The applied dynamics of ocean surface waves*, World Scientific, Singapore, 1989.

13. P.F. Rhodes-Robinson, *On the forced surface waves due to a vertical wave-maker in the presence of surface tension*, Proc. Camb. Phil. Soc. **70** (1971), 323–337.

14. ———, *On forced three-dimensional surface waves in a channel in the presence of surface tension*, Proc. Camb. Phil. Soc. (1974), **75**, 405–426.

15. ———, *The effect of surface tension on the progressive waves due to incomplete vertical wave-makers in water of infinite depth*, Proc. Roy. Soc. Lond. **435** (1991), 293–319.

16. I.N. Sneddon, *The use of integral transforms*, MacGraw Hill, New York, 1972.

17. J.J. Stoker, *Surface waves in water of variable depth*, Quart. Appl. Math. **5** (1947), 1–54.

18. F. Ursell, *The effect of a fixed vertical barrier on surface waves in deep water*, Proc. Camb. Phil. Soc. **43** (1947), 374–382.

19. J.V. Wehausen and E. Laitone, *Surface waves, Handbuch der physik*, Vol. 9, Springer Verlag, 1960.

20. W.E. Williams, *A note on scattering of water waves by a vertical barriers*, Proc. Camb. Phil. Soc. **62** (1966), 507–509.

21. T.L. Yip, T. Sahoo, and Allen T. Chwang, *Trapping of surface waves by porous and flexible structures*, Wave Motion **35** (2002), 41–54.

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