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A Dirac-type characterization of k-chordal graphs

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ABSTRACT

A graph is *k-chordal* if it has no induced cycle of length greater than k. We give a new characterization of k-chordal graphs that generalizes the well-known characterization of chordal graphs by Dirac in terms of simplicial vertices and simplicial orderings.

1. Introduction

The classical *chordal graphs* are those having no induced cycle of length more than 3 (i.e. there are no 'chordless' cycles). Dirac [4] proved that a graph is chordal if and only if every minimal vertex separator is a clique. A vertex is *simplicial* if its neighbourhood is a clique. Every chordal graph has a simplicial vertex and since chordal graphs form a hereditary family, an ordering of vertices referred to as a *perfect vertex elimination ordering* or *simplicial ordering* can be obtained. Also, a graph is chordal if and only if it has a simplicial ordering [5].

For any integer $k \geq 3$, a graph is *k-chordal* if it has no induced cycle of length greater than k. Thus, chordal graphs are precisely the 3-chordal graphs. The problem of determining whether a graph is k-chordal or not is known to be coNP-complete [9]. Hayward [7] showed that k-chordal graphs can be recognized in $O(n^{k+1})$ time, where n is the number of vertices in the input graph. Spinrad [8] improved this bound to $O(n^{k-2}M)$, where M is the time taken to multiply two $n \times n$ matrices. Many interesting properties of k-chordal graphs have been studied in the literature [1–3,7,8]. In particular, Chvátal et al. [3] characterized k-chordal graphs based on the existence of 'simplicial paths'. A chordless path $v_1, \ldots, v_i$ is a *simplicial path* if it does not extend to any chordless path $v_0, v_1, \ldots, v_i, v_{i+1}$. The characterization states that a graph is k-chordal if and only if each of its nonempty induced subgraphs contains a simplicial path with at most $k-2$ vertices. We complement this result by exploring structural characterizations of k-chordal graphs via minimal vertex separators and vertex orderings.

In order to state the characterizations, we need to introduce some terminologies and notations. For a set $A \subsetneq V(G)$, the neighbourhood of A, denoted by $N_G(A)$, is the set of vertices outside A having neighbours in A. The closed neighbourhood of A, denoted by $N_G[A]$, is $A \cup N_G(A)$. Let $G[A]$ denote the subgraph of G induced by A. A vertex separator of G is a set $S \subseteq V(G)$ such that $G - S$ is disconnected. A vertex separator is minimal if no proper subset of it is a vertex separator. The length of a path P, denoted by $\|P\|$, is the number of edges in P. Let $d_G(u, v)$ denote the length of a shortest u, v-path in G; set $d_G(u, v) = \infty$

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if no u, v-path exists. For a graph G, the graph G^k defined on the vertex set $V(G)$ by letting $E(G^k) = \{uv: d_G(u, v) \leq k\}$ is the k^{th} power of G.

A set A of vertices is a *connected non-dominating set* if $G[A]$ is connected and $N_G[A] \neq V(G)$. A connected non-dominating set A is maximal if for each vertex $v \in V(G) \setminus A$, either $G[A \cup \{v\}]$ is disconnected or $N_G[A \cup \{v\}] = V(G)$. For every integer $k \geq 3$, a vertex v is *k-simplicial* in G if

(C1) $N_G(v)$ induces a clique in $(G - v)^{k-2}$
(C2) for each nonadjacent pair x, y in $N_G(v)$, every chordless x, y-path through $G - N_G[v]$ has at most $k - 2$ edges.

Note that neither of the conditions (C1) or (C2) implies the other. For a graph G on n vertices, a vertex ordering $v_1, \ldots, v_n$ is a *k-simplicial ordering* or *k-perfect vertex elimination ordering* if, for each i, v_i is k-simplicial in $G[\{v_i, \ldots, v_n\}]$. We show that a k-chordal graph has a k-simplicial vertex; as a consequence, we obtain a characterization of k-chordal graphs based on k-simplicial orderings. We also generalize the characterization of chordal graphs based on minimal vertex separators to k-chordal graphs. In particular, we prove the following theorem.

Theorem 1. *For a graph G and an integer $k \geq 3$, the following statements are equivalent.*

(i) *G is k-chordal.*
(ii) *G has a k-simplicial ordering.*
(iii) *Every minimal vertex separator S in G is such that, for all nonadjacent $x, y \in S$ and for any two distinct components Q and Q' of $G - S$, every pair P and P' of induced x, y-paths through Q and Q', respectively, satisfies $\|P\| + \|P'\| \leq k$.*

For $k = 3$, our theorem reduces to Dirac's structural results on chordal graphs [4]. We prove the equivalence of (i) and (ii) in Theorem 8 and the equivalence of (ii) and (iii) in Theorem 9. Notation and definitions not given explicitly here can be found in [6,10].

2. Characterization of *k*-chordal graphs

Observation 2. A k-simplicial vertex is also l-simplicial, for every integer $l > k$. Also, for each integer $k \geq 3$, every vertex in a complete graph is k-simplicial.

Observation 3. For an integer $k \geq 3$, every vertex in a k-chordal graph G satisfies (C2).

Observation 4. In a non-complete graph G, for every vertex x that is nonadjacent to at least one other vertex, $\{x\}$ is a connected non-dominating set.

Lemma 5. *If A is a maximal connected non-dominating set in a non-complete graph G, then every vertex in $V(G) \setminus N_G[A]$ is adjacent to every vertex in $N_G(A)$.*

Proof. If there exist nonadjacent vertices $x' \in N_G(A)$ and $y' \in V(G) \setminus N_G[A]$, then the set $A' = A \cup \{x'\}$ is a connected non-dominating set in G, contradicting the maximality of A. □

Lemma 6. *Let A be a maximal connected non-dominating set in a non-complete k-chordal graph G, where $k \geq 3$ is an integer. There exists a vertex in $V(G) \setminus N_G[A]$ that is k-simplicial in G.*

Proof. Let $n = |V(G)|$. We prove the claim by induction on n. It is easy to verify the claim when $n \leq 3$. For $n \geq 4$, consider a non-complete k-chordal graph G on n vertices. Let A be a maximal connected non-dominating set in G, and let $B = V(G) \setminus N_G[A]$. If B is a clique, then by Observation 2 every vertex in B in k-simplicial in $G[B]$. Otherwise, by the induction hypothesis, there exists a vertex that is k-simplicial in $G[B]$. Let $b \in B$ be k-simplicial in $G[B]$; we show that b is also k-simplicial in G. From Observation 3, it suffices to show condition (C1). For this purpose, we identify a path of length at most $k - 2$ in $G - v$ between every pair of vertices in $N_G(b)$. Since $b \in B$, every vertex in $N_G(b)$ is either in B or in $N_G(A)$. Let $x, y \in N_G(b)$.

Case 1 ($x, y \in B$): Since b is k-simplicial in $G[B]$, there exists a x, y-path P in $G[B \setminus \{b\}]$ and thereby in $G[V(G) \setminus \{b\}]$ such that $\|P\| \leq k - 2$.

Case 2 ($x \in N_G(A), y \in B$) *or* ($x \in B, y \in N_G(A)$): Since every vertex in $N_G(A)$ is adjacent with every vertex in B by Lemma 5, we can take the required x, y-path to be the edge xy.

Case 3 ($x, y \in N_G(A)$): If $xy \in E(G)$, then the edge xy itself can be thought of as the x, y-path. Suppose $xy \notin E(G)$. Since A is connected and $x, y \in N_G(A)$, there is an x, y-path whose internal vertices all lie in A. Let P be the shortest of all such paths. Clearly, P is present in $G[V(G) \setminus \{b\}]$. We claim that $\|P\| \leq k - 2$. If $\|P\| > k - 2$, then b forms an induced cycle with P having length at least $k + 1$. This contradicts the fact that G is k-chordal. □

Lemma 7. *For an integer $k \geq 3$, every k-chordal graph G has a k-simplicial vertex. Moreover, if G is a non-complete graph, then it has two nonadjacent k-simplicial vertices.*

Proof. If G is a complete graph, then by Observation 2, every vertex in G is k-simplicial. Suppose that G is not a complete graph. Let A be a maximal connected non-dominating set in G, and let $B = V(G) \setminus N_G[A]$. The sets A and B are well-defined by Observation 4. By Lemma 6, there exists a vertex $u \in B$ that is k-simplicial in G. Now, let A' be a maximal connected non-dominating set in G containing u, and let $B' = V(G) \setminus N_G[A']$. The sets A' and B' are well-defined by Observation 4, since no vertex in A is adjacent to u. By Lemma 6, some vertex $v \in B'$ is k-simplicial in G. Thus, u and v are two nonadjacent k-simplicial vertices in G. □

Since k-chordal graphs form a hereditary family, we obtain the following characterization of k-chordal graphs.

Theorem 8. *For an integer $k \geq 3$, a graph G is k-chordal if and only if G has a k-simplicial ordering.*

Proof. ($\Rightarrow$) As every induced subgraph of a k-chordal graph is k-chordal, the proof follows from Lemma 7.

($\Leftarrow$) Consider a k-simplicial ordering $v_1, \ldots, v_n$ of G. If G is not k-chordal, then let C be an induced cycle of length at least $k+1$. Let v_i be the vertex of C with least index in the k-simplicial ordering. Let v_j and v_k be the neighbours of v_i on C; they come later in the ordering. Let $G' = G[\{v_i, \ldots, v_n\}]$. Since v_i is k-simplicial in G', there is no chordless v_j, v_k-path of length more than $k-2$ through $G' - N_{G'}[v]$. However, the v_j, v_k-path along C is such a path, yielding a contradiction. □

We now characterize k-chordal graphs based on minimal vertex separators.

Theorem 9. *Let $k \geq 3$ be an integer. A graph G is k-chordal if and only if for all minimal vertex separators S in G, for all nonadjacent $x, y \in S$ and for any two distinct components Q and Q' of $G - S$, every pair P and P' of induced x, y-paths through Q and Q', respectively, satisfies $\|P\| + \|P'\| \leq k$.*

Proof. ($\Rightarrow$) Assume to the contrary that there exists a minimal vertex separator S in G containing two nonadjacent vertices x and y such that $\|P\| + \|P'\| > k$ for some such paths P and P'. Clearly, $V(P) \cup V(P')$ induces a cycle of length at least $k+1$, contradicting that G is k-chordal.

($\Leftarrow$) Suppose there exists an induced cycle C of length at least $k+1$ in G. Let $C = (x, a, y_1, \ldots, y_{l-2}, x)$, with $l \geq k+1$. Let $b = y_{l-2}$ and $S' = N_G(a)$. Let B be the component of $G - S'$ containing b. Let $S = N_G(B) \cap S'$. Now, S is a minimal separator of G separating a and b. Also, $\{x, y_1\} \subseteq S$ and S contains no vertex from the set $\{y_2, \ldots, y_{l-3}\}$. Letting $z = y_1$, we now have chordless x, z-paths P and P' through Q (component of $G - S$ containing a) and Q' (component of $G - S$ containing b) such that $\|P\| + \|P'\| \geq k+1$, contradicting our hypothesis. Therefore, as claimed, G is k-chordal. □

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