Research Article

Santhosh George* and M. Thamban Nair

A derivative-free iterative method for nonlinear ill-posed equations with monotone operators

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Abstract: Recently, Semenova [12] considered a derivative free iterative method for nonlinear ill-posed operator equations with a monotone operator. In this paper, a modified form of Semenova's method is considered providing simple convergence analysis under more realistic nonlinearity assumptions. The paper also provides a stopping rule for the iteration based on an a priori choice of the regularization parameter and also under the adaptive procedure considered by Pereverzev and Schock [11].

Keywords: Iterative method, derivative free method, nonlinear ill-posed equations, Lavrentiev regularization, adaptive method

MSC 2010: 41H25, 65F22, 65J15, 65J22, 47A52

1 Introduction

In this paper, we consider the problem of approximately solving the operator equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to X$ is a monotone operator, i.e.,

$$\langle F(v) - F(w), v - w\rangle \geq 0 \quad \text{for all } v, w \in D(F),$$

which is, in general, nonlinear, X is a real Hilbert space and $y \in X$. We denote the inner product and the corresponding norm in X by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively, and $\overline{B(x, r)}$ denotes the closed ball in X with center $x \in X$ and radius $r > 0$.

It is assumed that (1.1) has a solution, namely $x^\dagger$, and that the data y is known only approximately, say y^δ such that

$$\|y - y^\delta\| \leq \delta$$

for some error level $\delta > 0$. Equation (1.1) is, in general, ill-posed, in the sense that a small perturbation in the data can cause large deviations in the solution. Thus, for obtaining stable approximations for the solution $x^\dagger$ from noisy data y^δ, some regularization method has to be employed. As the operator F is monotone, a regularization method which has been widely used in the literature is the Lavrentiev regularization method (see [6, 8, 14, 15]). In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta, \tag{1.2}$$

where x_0 is an initial guess of $x^\dagger$, say $\|x_0 - x^\dagger\| \leq r_0$ for some $r_0 > 0$.

***Corresponding author: Santhosh George:** Department of Mathematical and Computational Sciences, NIT Karnataka, Surathkal 575 025, India, e-mail: sgeorge@nitk.ac.in
M. Thamban Nair: Department of Mathematics, IIT Madras, Chennai 600 036, India, e-mail: mtnair@iitm.ac.in

It is known that equation (1.2) has a unique solution for every right-hand side provided F is Fréchet differentiable in some open ball centered at $x^\dagger$ contained in $D(F)$ and x_0 belongs to that open ball (cf. [2, Theorem 11.2] or [14, Theorem 1.1]). However, the nonlinearity of the operator involved in (1.2) can cause difficulties in solving them numerically.

Another alternative is to consider iterative procedures along with stopping rules to obtain approximations to $x^\dagger$. Such procedures are available in the literature. However, in all such procedures, the iterations involve the Fréchet derivative of the operator F (see, e.g., [1, 5, 7]). In [12], Semenova considered a derivative-free iterative method,

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - \gamma[F(x^\delta_{n,\alpha}) + \alpha(x^\delta_{n,\alpha} - x_0) - y^\delta] \tag{1.3}$$

for fixed α, δ by assuming that F is Lipschitz continuous with Lipschitz constant R and with γ satisfying

$$0 < \gamma < \min\Big\{\frac{1}{\alpha}, \frac{2\alpha}{\alpha^2 + R^2}\Big\},$$

and also an additional nonlinearity condition (see [12, Assumption 3]) on the Fréchet derivative F' involving the unknown solution $x^\dagger$. The convergence of the iterates in (1.3) to the solution x^δ_α of (1.2) is proved by showing it to be a Cauchy sequence by using contraction mapping arguments.

The purpose of this paper is to consider the iterative procedure (1.3), but with a γ independent of the regularization parameter α and also with a nonlinearity condition on F' which is independent of the unknown solution $x^\dagger$. Further, our convergence analysis is also simpler in the sense that it does not make use of the contraction mapping arguments as in [12].

In a recent paper, Shubha, George and Jidesh [13] have introduced a derivative-free iterative method which required a simpler nonlinearity condition on F'. However, the convergence analysis in [13] makes use of a condition on the second Fréchet derivative F''. Moreover, the iterative procedure in [13] itself is a bit more cumbersome than (1.3) .

The remainder of the paper is organized as follows: In Section 2, we present the method and its error analysis without specification of any source condition. In Section 3 we consider the error estimates under a general source condition with an appropriate stopping rule involving α and δ. In the final section, Section 4, we present an a priori choice of the parameter α and also an adaptive strategy based on the balancing principle considered in Pereverzev and Schock [11].

2 The method and the convergence analysis

Taking

$$r \geq 2(r_0 + 1) \quad \text{with } r_0 := \|x^\dagger - x_0\|,$$

we assume that the following conditions hold:

(i) $\overline{B(x_0, r)} \subseteq D(F)$,

(ii) F has self-adjoint Fréchet derivatives $F'(x)$ for every $x \in \overline{B(x_0, r)}$,

(iii) there exists $\beta_0 > 0$ such that

$$\|F'(x)\| \leq \beta_0 \quad \text{for all } x \in \overline{B(x_0, r)}.$$

Let $x^\delta_\alpha \in D(F)$ be the unique solution (1.2) for every $\delta \in (0, d]$ and $\alpha \in [\delta, a)$ for some positive constants a, d with $d < a$. In particular, by taking $y^\delta = y$, there is a unique $x_\alpha \in D(F)$ such that

$$F(x_\alpha) + \alpha(x_\alpha - x_0) = y. \tag{2.1}$$

The method: Let $\delta \in (0, d]$ and $\alpha \in [\delta, a)$. As in [12], we consider the sequence $\{x^\delta_{n,\alpha}\}$ defined iteratively by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - \beta[F(x^\delta_{n,\alpha}) + \alpha(x^\delta_{n,\alpha} - x_0) - y^\delta], \tag{2.2}$$

where

$$x^\delta_{0,\alpha} = x_0 \quad \text{and} \quad \beta := \frac{1}{\beta_0 + a}.$$

We observe that if $\{x^\delta_{n,\alpha}\}$ converges as $n \to \infty$, then the limit is x^δ_α, the solution of (1.2).

We shall make use of the following known result (see [14]). For the sake of completion of the exposition, we provide its proof as well.

Lemma 2.1. *Let x_α and x_α^δ be as in equations* (2.1) *and* (1.2)*, respectively. Then,*

$$\|x_\alpha - x^\dagger\|^2 \le \langle x_0 - x^\dagger, x_\alpha - x^\dagger\rangle, \quad \|x_\alpha^\delta - x_\alpha\| \le \frac{\delta}{\alpha}.$$

In particular,

$$\|x^\dagger - x_\alpha^\delta\| \le \|x^\dagger - x_\alpha\| + \frac{\delta}{\alpha}, \quad \|x^\dagger - x_\alpha\| \le \|x^\dagger - x_0\|.$$

Proof. Since $y = F(x^\dagger)$, we have

$$F(x_\alpha) + \alpha(x_\alpha - x_0) = F(x^\dagger),$$

so that

$$F(x_\alpha) - F(x^\dagger) + \alpha(x_\alpha - x_0) = 0,$$

i.e.,

$$F(x_\alpha) - F(x^\dagger) + \alpha(x_\alpha - x^\dagger) = \alpha(x_0 - x^\dagger).$$

Hence,

$$\langle F(x_\alpha) - F(x^\dagger), x_\alpha - x^\dagger\rangle + \alpha\langle x_\alpha - x^\dagger, x_\alpha - x^\dagger\rangle = \alpha\langle x_0 - x^\dagger, x_\alpha - x^\dagger\rangle.$$

By monotonicity of F, we have $\langle F(x_\alpha) - F(x^\dagger), x_\alpha - x^\dagger\rangle \ge 0$. Thus, we obtain,

$$\|x_\alpha - x^\dagger\|^2 \le \langle x_0 - x^\dagger, x_\alpha - x^\dagger\rangle.$$

Next, we observe that

$$F(x_\alpha^\delta) - F(x_\alpha) + \alpha(x_\alpha^\delta - x_\alpha) = y^\delta - y.$$

Hence,

$$\langle F(x_\alpha^\delta) - F(x_\alpha), x_\alpha^\delta - x_\alpha\rangle + \alpha\langle x_\alpha^\delta - x_\alpha, x_\alpha^\delta - x_\alpha\rangle = \langle y^\delta - y, x_\alpha^\delta - x_\alpha\rangle.$$

Again, using the monotonicity of F, we have

$$\alpha\|x_\alpha^\delta - x_\alpha\|^2 \le \langle y^\delta - y, x_\alpha^\delta - x_\alpha\rangle \le \delta\|x_\alpha^\delta - x_\alpha\|.$$

Thus,

$$\|x_\alpha^\delta - x_\alpha\| \le \frac{\delta}{\alpha}.$$

The particular cases are obvious from the previous estimates. □

Remark 2.2. In Section 3, under a source condition on $x^\dagger - x_0$, we obtain an estimate for $\|x_\alpha^\delta - x^\dagger\|$. Then, using an appropriate choice of the parameter $\alpha := \alpha_\delta$, we obtain the convergence $x_{\alpha_\delta}^\delta \to x^\dagger$ as well as an estimate for $\|x_{\alpha_\delta}^\delta - x^\dagger\|$.

Theorem 2.3. *For each $\delta \in (0, d]$ and $\alpha \in [\delta, a)$ the sequence $\{x_{n,\alpha}^\delta\}$ is in $\overline{B(x_0, r)}$ and it converges to x_α^δ as $n \to \infty$. Further,*

$$\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \le \kappa q_\alpha^n,$$

where $q_\alpha := 1 - \beta\alpha$ and $\kappa \ge r_0 + 1$ with $\beta := 1/(\beta_0 + a)$.

Proof. Clearly, $x_{0,\alpha}^\delta = x_0 \in \overline{B(x_0, r)}$. Also, by Lemma 2.1, we have

$$\|x_0 - x_\alpha^\delta\| \le \|x_0 - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\| \le r_0 + \frac{\delta}{\alpha} \le r_0 + 1 < r.$$

Hence, $x_\alpha^\delta \in \overline{B(x_0, r)}$. By the fundamental theorem of integration, we have

$$F(x) - F(u) = \Big[\int_0^1 F'(u + t(x - u))\,dt\Big](x - u)$$

whenever x and u are in a ball contained in $D(F)$. We show iteratively that $x_{n,\alpha}^\delta \in \overline{B(x_0, r)}$, that the operator

$$A_n := \int_0^1 F'(x_\alpha^\delta + t(x_{n,\alpha}^\delta - x_\alpha^\delta))\,dt$$

is a well-defined positive self-adjoint operator and that

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| \le (1-\beta\alpha)\|x_{n,\alpha}^\delta - x_\alpha^\delta\|$$

for $n = 0, 1, 2, \dots$, which will complete the proof since $\|x_0 - x_\alpha^\delta\| < r$.

Formally, we have

$$x_{n+1,\alpha}^\delta - x_\alpha^\delta = x_{n,\alpha}^\delta - x_\alpha^\delta - \beta[F(x_{n,\alpha}^\delta) - F(x_\alpha^\delta) + \alpha(x_{n,\alpha}^\delta - x_\alpha^\delta)].$$

Since

$$F(x_{n,\alpha}^\delta) - F(x_\alpha^\delta) = \int_0^1 F'(x_\alpha^\delta + t(x_{n,\alpha}^\delta - x_\alpha^\delta))(x_{n,\alpha}^\delta - x_\alpha^\delta)\,dt,$$

we have

$$x_{n+1,\alpha}^\delta - x_\alpha^\delta = [I - \beta(A_n + \alpha)](x_{n,\alpha}^\delta - x_\alpha^\delta). \tag{2.3}$$

Now let $n = 0$. We have already seen that $\|x_0 - x_\alpha^\delta\| < r$ so that $x_\alpha^\delta \in \overline{B(x_0, r)}$ and A_0 is a well-defined positive self-adjoint operator with $\|A_0\| \le \beta_0$.

Next assume that for some $n \ge 0$ we have $x_{n,\alpha}^\delta \in \overline{B(x_0, r)}$ and A_n is a well-defined positive self-adjoint operator with $\|A_n\| \le \beta_0$. Then from (2.3), we have

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| \le \|I - \beta(A_n + \alpha I)\|\|(x_{n,\alpha}^\delta - x_\alpha^\delta)\|.$$

Since A_n is a positive self-adjoint operator, we have (cf. [9])

$$\|I - \beta(A_n + \alpha I)\| = \sup_{\|x\|=1} \big|\langle[(1-\beta\alpha)I - \beta A_n]x, x\rangle\big| = \sup_{\|x\|=1} \big|(1-\beta\alpha) - \beta\langle A_n x, x\rangle\big|,$$

and since $\|A_n\| \le \beta_0$ for all $n \in \mathbb{N}$ and $\beta = 1/(\beta_0 + a)$, we have

$$0 \le \beta\langle A_n x, x\rangle \le \beta\|A_n\| \le \beta\beta_0 < 1 - \beta\alpha$$

for all $\alpha \in (0, a)$. Therefore,

$$\|I - \beta(A_n + \alpha I)\| \le 1 - \beta\alpha.$$

Thus,

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| \le \|I - \beta(A_n + \alpha)\|\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \le (1-\beta\alpha)\|x_{n,\alpha}^\delta - x_\alpha^\delta\|.$$

Hence,

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| \le \|x_0 - x_\alpha^\delta\| \le r_0 + \frac{\delta}{\alpha} \le r_0 + 1$$

and

$$\|x_{n+1,\alpha}^\delta - x_0\| \le \|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_0\| \le 2\|x_0 - x_\alpha^\delta\| \le 2(r_0 + 1) \le r.$$

Thus, $x_{n+1,\alpha}^\delta \in \overline{B(x_0, r)}$. Also, for $0 \le t \le 1$, we have

$$\big\|[x_\alpha^\delta + t(x_{n+1,\alpha}^\delta - x_\alpha^\delta)] - x_0\big\| = \big\|(x_\alpha^\delta - x_0) + t(x_{n+1,\alpha}^\delta - x_\alpha^\delta)\big\| \le 2(r_0 + 1) \le r.$$

Hence, A_{n+1} is a well-defined positive self-adjoint operator with $\|A_{n+1}\| \le \beta_0$. This completes the proof. □

Theorem 2.4. *Let $\delta \in (0, d]$, $\alpha \in (\delta, a]$ with $d < a$. Let x_α^δ and x_α be as in equations* (1.2) *and* (2.1), *respectively, and for each $\delta \in (0, d]$ and $\alpha \in [\delta, a)$ let the sequence $\{x_{n,\alpha}^\delta\}$ be defined as in* (2.2). *Let*

$$n_{\alpha,\delta} := \min\{n \in \mathbb{N} : \alpha q_\alpha^n \le \delta\}.$$

Then,

$$\|x_{n_{\alpha,\delta}+1,\alpha}^\delta - x^\dagger\| = (\kappa + 1)\Big(\|x^\dagger - x_\alpha\| + \frac{\delta}{\alpha}\Big),$$

where $\kappa \ge r_0 + 1$.

Proof. By Lemma 2.1 and Theorem 2.3, we have

$$\|x_{n+1,\alpha}^\delta - x^\dagger\| = \|x_{n_{\alpha,\delta}+1,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x^\dagger\| = \kappa q_\alpha^n + \frac{\delta}{\alpha} + \|x_\alpha - x^\dagger\|.$$

Now for $n = n_{\alpha,\delta}$ we have $q_\alpha^n \le \delta/\alpha$. Thus, we obtain the required estimate in the theorem. □

3 Error bounds under source conditions

For obtaining an estimate for $\|x^\dagger - x_\alpha\|$ we have to impose some nonlinearity conditions on F and assume that $x_0 - x^\dagger$ belongs to some source set. For this purpose, we use the following two assumptions. The first one is a simplified form of the standard nonlinear assumptions in the literature, whereas the second one regarding the source condition is exactly the same considered earlier (cf. [7, 8]). These conditions are also assumed in the paper [13].

Assumption 3.1. There exists a constant $k_0 \ge 0$ such that for every $x \in \overline{B(x_0, r)}$ and $v \in X$ there exists an element $\Phi(x, x_0, v) \in X$ such that

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v)$$

and

$$\|\Phi(x, x_0, v)\| \le k_0\|v\|\|x - x_0\|$$

for all $x, v \in \overline{B(x_0, r)}$.

Assumption 3.2. There exists a continuous and strictly monotonically increasing function

$$\varphi : (0, a_0] \to (0, \infty)$$

with $a_0 \ge \|F'(x_0)\|$ satisfying the following conditions:

(i) $\lim_{\lambda\to 0} \varphi(\lambda) = 0$,

(ii) $\sup_{\lambda\ge 0} \frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \le \varphi(\alpha)$ for all $\lambda \in (0, a_0]$,

(iii) there exists $v \in X$ with $\|v\| \le 1$ such that

$$x_0 - x^\dagger = \varphi(F'(x_0))v.$$

The proof of the following theorem is given in [13] with less details. For the sake of completeness of the presentation, we give the details of the proof here as well.

Theorem 3.3. *For $\delta \in (0, d]$, $\alpha \in (\delta, a]$ with $d < a$ let x_α^δ and x_α be as in equations* (1.2) *and* (2.1), *respectively, and let Assumptions 3.1 and 3.2 be satisfied. Then,*

$$\|x_\alpha - x^\dagger\| \le k_0 r_0\|x_\alpha - x^\dagger\| + \varphi(\alpha).$$

Proof. Since $F(x_\alpha) + \alpha(x_\alpha - x_0) = y$ and $y = F(x^\dagger)$, we have

$$F(x_\alpha) - F(x^\dagger) + \alpha(x_\alpha - x^\dagger) = \alpha(x_0 - x^\dagger). \tag{3.1}$$

But, by the fundamental theorem of integration,

$$F(x_\alpha) - F(x^\dagger) = M_\alpha(x_\alpha - x^\dagger),$$

where $M_\alpha := \int_0^1 F'(x^\dagger + t(x_\alpha - x^\dagger))\,dt$. Note that, by Lemma 2.1, we have

$$\|x^\dagger + t(x_\alpha - x^\dagger) - x_0\| \le 2r_0 \le r.$$

Thus, $x^\dagger + t(x_\alpha - x^\dagger) \in \overline{B(x_0, r)}$ so that M_α is a well-defined positive self-adjoint operator. Let us rewrite equation (3.1) as

$$F'(x_0)(x_\alpha - x^\dagger) + \alpha(x_\alpha - x^\dagger) = (F'(x_0) - M_\alpha)(x_\alpha - x^\dagger) + \alpha(x_0 - x^\dagger).$$

Then, by Assumption 3.1, we have

$$\begin{aligned} x_\alpha - x^\dagger &= (F'(x_0) + \alpha I)^{-1}[(F'(x_0) - M_\alpha)(x_\alpha - x^\dagger) + \alpha(x_0 - x^\dagger)] \\ &= (F'(x_0) + \alpha I)^{-1}F'(x_0)\int_0^1 \Phi(x^\dagger + t(x_\alpha - x^\dagger), x_0, (x_\alpha - x^\dagger))\,dt + \alpha(F'(x_0) + \alpha I)^{-1}(x_0 - x^\dagger). \end{aligned}$$

Since $F'(x_0)$ is a positive self-adjoint operator, we have

$$\|(F'(x_0) + \alpha I)^{-1}F'(x_0)\| \le 1,$$

and by Assumption 3.2, we have

$$\|\alpha(F'(x_0) + \alpha I)^{-1}(x_0 - x^\dagger)\| = \|\alpha(F'(x_0) + \alpha I)^{-1}\varphi(F'(x_0))v\| \le \|v\| \sup_{\lambda \ge 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le \varphi(\alpha).$$

Also, Assumption 3.1 yields

$$\|\Phi(x^\dagger + t(x_\alpha - x^\dagger), x_0, (x_\alpha - x^\dagger))\| \le k_0\|x_\alpha - x^\dagger\|\|x^\dagger + t(x_\alpha - x^\dagger) - x_0\|.$$

Since $\|x_\alpha - x^\dagger\| \le \|x_0 - x^\dagger\|$, we have

$$\begin{aligned} \|x^\dagger + t(x_\alpha - x^\dagger) - x_0\| &\le \|(x^\dagger - x_0) + t[(x_\alpha - x_0) + (x_0 - x^\dagger)]\| \\ &\le \|(1 - t)(x^\dagger - x_0) + t(x_\alpha - x_0)\| \\ &\le \|x^\dagger - x_0\|. \end{aligned}$$

Thus, we obtain

$$\|x_\alpha - x^\dagger\| \le k_0 r_0\|x_\alpha - x^\dagger\| + \varphi(\alpha).$$ □

Now Theorems 2.4 and 3.3 lead to the following theorem.

Theorem 3.4. *Let $\delta \in (0, d]$, $\alpha \in (\delta, a]$ with $d < a$, and let Assumptions 3.1 and 3.2 be satisfied. Assume further that $q := k_0 r_0 < 1$. Let*

$$n_{\alpha,\delta} := \min\{n \in \mathbb{N} : \alpha q_\alpha^n \le \delta\}. \tag{3.2}$$

Then,

$$\|x^\delta_{n_{\alpha,\delta}+1,\alpha} - x^\dagger\| = \tilde{C}\Big(\varphi(\alpha) + \frac{\delta}{\alpha}\Big), \tag{3.3}$$

where $\tilde{C} = \min\{\kappa + 1, \frac{1}{1-q}\}$ with $\kappa \ge r_0 + 1$.

In order to choose the regularization parameter α, one may use an a priori choice by requiring

$$\alpha\varphi(\alpha) = \delta$$

or the adaptive method considered in [11] by Pereverzev and Schock which has been further investigated in various papers including [4, 10].

4 Parameter choice and stopping rules

4.1 A priori rule

Note that, given δ, the quantity $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (3.3) is least if α satisfies the equation

$$\alpha\varphi(\alpha) = \delta.$$

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$, $0 < \lambda \leq a_0$, we have

$$\alpha\varphi(\alpha) = \psi(\varphi(\alpha)),$$

so that $\alpha_\delta := \varphi^{-1}(\psi^{-1}(\delta))$ satisfies

$$\alpha_\delta\varphi(\alpha_\delta) = \delta$$

and

$$\frac{\delta}{\alpha_\delta} = \varphi(\alpha_\delta) = \psi^{-1}(\delta).$$

Hence from Theorem 3.4 we obtain the following theorem.

Theorem 4.1. *Let $\delta \in (0, d]$, $\alpha \in (\delta, a]$ with $d < a$, and let Assumptions 3.1 and 3.2 be satisfied. Assume further that $q := k_0 r_0 < 1$. Let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$, where $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a_0$ and*

$$n_\delta := \min\{n \in \mathbb{N} : \alpha_\delta q^n_{\alpha_\delta} \leq \delta\}.$$

Then

$$\|x^\delta_{n_\delta,\alpha_\delta} - x^\dagger\| \leq 2\tilde{C}\psi^{-1}(\delta),$$

where $\bar{C} = \min\{\kappa + 1, \frac{1}{1-q}\}$ with $\kappa \geq r_0 + 1$.

4.2 Adaptive scheme and stopping rule

We modify the procedure of Pereverzev and Schock [11] slightly to suit the present context as has been done in [3, 4, 10, 13].

Let $\alpha_0 > \delta$ and for $i = 1, \dots, N$ let $\alpha_i := \mu^i\alpha_0$ with $\mu > 1$. Then we have $\alpha_0 < \alpha_1 < \dots < \alpha_N$. For given α, δ let $n_{\alpha,\delta}$ be as in (3.2). For notational convenience, let us denote

$$n_i := n_{\alpha_i,\delta}, \quad x^\delta_i := x^\delta_{n_i,\alpha_i}, \qquad i = 0, 1, \dots, N.$$

We assume that

$$\Big\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\Big\} \neq \emptyset \quad \text{and} \quad \max\Big\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\Big\} < N.$$

Now, we can state the following known result (see [4]) on the choice of regularization parameter.

Theorem 4.2. *Let*

$$k := \max\Big\{i : \|x^\delta_i - x^\delta_j\| \leq 4\tilde{C}\frac{\delta}{\alpha_j},\ j = 0, 1, 2, \dots, i\Big\},$$

with $\bar{C} = \min\{\kappa + 1, \frac{1}{1-q}\}$. Then,

$$\|x^\dagger - x^\delta_k\| \leq 6\bar{C}\mu\psi^{-1}(\delta).$$

As per Theorem 4.2 , the choice of the regularization parameter involves the following steps:

(1) Set $i = 0$.
(2) Choose $n_i := \min\{n : \alpha_i q^n_{\alpha_i} \leq \delta\}$.
(3) Solve $x^\delta_i := x^\delta_{n_i,\alpha_i}$ by using iteration (2.2).

(4) If

$$\|x_i^\delta - x_j^\delta\| > 4\bar{C}\frac{\delta}{\alpha_j}, \quad j < i,$$

then take $k = i - 1$ and return x_k.

(5) Else set $i = i + 1$ and go to (2).

Next, we present an academic example which satisfies Assumption 3.1.

Example 4.3. Consider a nonlinear operator equation $F : L^2[0,1] \to L^2[0,1]$ defined by $F(u) := (\arctan(u))^3$. Since $u \to (\arctan(u))^3$ is increasing on $\mathbb{R}$, we have

$$\langle (\arctan(u))^3 - (\arctan(v))^3, u - v\rangle \geq 0 \quad \text{for all } u, v \in L^2[0,1],$$

i.e., F is monotone. The Fréchet derivative of F is

$$F'(u)w = \frac{3(\arctan(u))^2}{1+u^2}w.$$

If $u(x)$ vanishes on a set of positive Lebesgue measure, then $F'(u)$ is not boundedly invertible. If $u \in C[0,1]$ vanishes even at one point x_0, then $F'(u)$ is not boundedly invertible in $L^2[0,1]$.

Note that

$$F'(u)w = F'(u_0)G(u, u_0)w,$$

where

$$G(u, u_0) = \frac{1+u_0^2}{1+u^2}\frac{(\arctan(u))^2}{(\arctan(u_0))^2}.$$

Further, for $u_0 \neq 0$, we have

$$[F'(u) - F'(u_0)]w = F'(u_0)[G(u, u_0) - I]w$$

and

$$\begin{aligned}
\|[G(u,u_0) - I]w\| &\leq \left\| \frac{(1+u_0^2)\big((\arctan(u))^2 - (\arctan(u_0))^2\big) - (\arctan(u_0))^2(u^2 - u_0^2)}{(1+u^2)(\arctan(u))^2} w \right\| \\
&\leq \frac{1 + \|u_0\|^2}{(1+\|u\|^2)\|(\arctan(u_0))^2\|}\|(\arctan(u))^2 - (\arctan(u_0))^2\|\|w\| + \frac{1}{1+\|u\|^2}\|u^2 - u_0^2\|\|w\| \\
&\leq \frac{1 + \|u_0\|^2}{(1+\|u\|^2)\|(\arctan(u_0))^2\|}\|\arctan(u) + \arctan(u_0)\| \\
&\qquad \times \|\arctan(u) - \arctan(u_0)\|\|w\| + \frac{1}{1+\|u\|^2}\|u + u_0\|\|u - u_0\|\|w\| \\
&\leq \left[\frac{2\max\{\|\arctan(u)\|, \|\arctan(u_0)\|\}}{\|(\arctan(u_0))^2\|} + 2\max\{\|\hat{u}\|, \|u_0\|\}\right]\|w\|\|\hat{u} - u_0\|.
\end{aligned}$$

The last but one step follows from the inequality

$$\|\arctan(u) - \arctan(u_0)\| \leq \frac{\|u\| - \|u_0\|}{1 + \|u_0\|^2}.$$

Thus, Assumption 3.1 is satisfied.

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