PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Cite this article: Alagappan P, Rajagopal KR, Kannan K. 2018 A damage initiation criterion for a class of viscoelastic solids. *Proc. R. Soc. A* **474**: 20180064.
http://dx.doi.org/10.1098/rspa.2018.0064

Received: 31 January 2018
Accepted: 18 May 2018

Subject Areas:
mechanics

Keywords:
damage, viscoelastic solid, inhomogeneity, compressible solid, second law

Author for correspondence:
K. R. Rajagopal
e-mail: krajagopal@tamu.edu

THE ROYAL SOCIETY
PUBLISHING

A damage initiation criterion for a class of viscoelastic solids

P. Alagappan[1], K. R. Rajagopal[2] and K. Kannan[3]

[1]Department of Civil Engineering, Indian Institute of Technology Madras, Chennai 600036, India
[2]Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123, USA
[3]Department of Mechanical Engineering, Indian Institute of Technology Madras, Chennai 600036, India

KRR, 0000-0002-4636-5131

We extend the methodology introduced for the initiation of damage within the context of a class of elastic solids to a class of viscoelastic solids (Alagappan *et al.* 2016 *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* **472**, 20160231. (doi:10.1098/rspa.2016.0231)). In a departure from studies on damage that consider the body to be homogeneous, with initiation of damage being decided by parameters that are based on a quantity such as the strain, that requires information concerning a special reference configuration, or the use of ad hoc parameters that have no physically meaningful origins, in this study we use a physically relevant parameter that is completely determined in the current deformed state of the body to predict the initiation of damage. Damage is initiated due to the inhomogeneity of the body wherein certain regions in the body are unable to withstand the stresses, strains, etc. The specific inhomogeneity that is considered is the variation of the density in the body. We consider damage within the context of the deformation of two representative viscoelastic solids, a generalization of a model proposed by Gent (1996 *Rubber Chemistry and Technology* **69**, 59–61. (doi:10.5254/1.3538357)) for polymeric solids and a generalization of the Kelvin–Voigt model. We find that the criterion leads to results that are in keeping with the experiments of Gent & Lindley (1959 *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* **249**, 195–205. (doi:10.1098/rspa.1959.0016)).

1. Introduction

Recently, Alagappan *et al.* [1] proposed a possible methodology to identify the initiation of damage in a class of elastic bodies. The criterion that they came up with, which we shall discuss in more detail later, merely identifies the onset of damage and does not apply to the evolution and progression of damage leading ultimately to the failure of the body. By failure we mean the inability of the body to carry out its intended purpose, which from a purely mechanical perspective invariably implies the inability to carry the applied loads. Once a material is damaged, the constitutive relation by means of which its response is specified before damage becomes inoperative and a new constitutive relation needs to be put into place. While a composite constitutive relation that describes the response of the material both before and after the initiation of damage can be specified, some new parameters have to be identified that come into play after the onset of damage that determine how the damage progresses. In the literature pertinent to damage, one often finds the use of 'damage parameters', scalar, vector or tensor in character that evolve as the damage unfolds. In many instances, there is no clear physical meaning given to such 'damage parameters', and when one can in fact attribute physical meaning it usually depends on knowledge and information from some reference configuration, which one is not aware of, in which the body was free of stress or strain. More importantly, the body that is undergoing damage is invariably treated as a homogeneous body and herein lies the main drawback of most approaches to damage.[1] It would be appropriate to point out that the recent study by Puglisi & Saccomandi [7] is relevant to the current work in that in their study of the Gent model, the response exhibits a non-uniformity that occurs in the limit of infinite loading. But such a response is not what we refer to as 'damage'; to us 'damage' is the deterioration in the load-carrying capacity that occurs at a finite load. The study of Puglisi & Saccomandi [7] also provides an explanation for the Mullins effect which is observed in rubber-like materials, but to explain this phenomenon they introduce a 'damage' parameter which satisfies a rate equation and is allowed to take values between 0 and 1, but this is precisely what we want to avoid, the use of a 'damage' parameter without any clear physical underpinning.

Damage invariably occurs as a consequence of the inhomogeneity of the body under consideration, at a location wherein there is some structural defect at the microscopic scale which weakens the body at that particular place. While this microscopic structural defect could take various forms, for certain classes of polymeric materials this might be a consequence of the value of the mass density of the polymer or the density of the cross-links or the density of the elastic strands or a combination of these quantities, which comprises the polymer network. These quantities are related (see Bueche [8] and Langley [9]). In this study, we use the mass density of the polymer as the variable to characterize and diagnose the onset and evolution of damage.

In this paper, we are only concerned with pinpointing the inception of damage in a class of viscoelastic materials. We consider a class of compressible bodies whose material properties change with density. Thus, if the body is inhomogeneous with the density varying over the extent of the body, then the material properties at different locations would be different. As the body undergoes deformation due to the applied forces, its density changes at different locations and we expect the onset of damage to occur at the location wherein the material properties are such that the damage criterion is met. We postulate that the initiation of damage will occur at the location wherein the derivative of the magnitude of the stress with respect to the stretch becomes negative or when one reaches an inflection point for the norm of the stress versus density during loading (see [1] for the rationale for the criterion).

In our previous study, we had looked into the commencement of damage in elastic bodies and compared the predictions of our methodology with the experimental results of Gent & Lindley [10] on rubber specimens. Unlike the idealized inhomogeneity that is assumed in our study, their experimental study concerns a real specimen with a random distribution of inhomogeneities. Thus, one cannot expect perfect quantitative agreement with regard to the initiation of damage in the specimen; at best one can expect reasonable quantitative and good qualitative agreement.

[1]There have been numerous studies devoted to the issue of damage and it would be impossible to discuss even the salient ones in this paper. One of the early studies concerning damage was carried out by Kachanov [2]. Copious references to studies regarding damage can be found in the books by Kachanov [3], Lemaitre [4] and Krajcinovic [5]. Volokh [6] appeals to the notion of a bond energy that needs to be overcome for the initiation of damage.

In our previous study (see [1]), we assumed that rubber could be described by a constitutive relation for elastic bodies. While rubber can and is often assumed to respond like an elastic body in that the dissipation is negligible, natural rubber and especially filled rubbers are viscoelastic. It is the dissipation that is at the heart of phenomena such as the Mullins effect. Thus, it would be reasonable to investigate whether the methodology that we proposed in our earlier study is also applicable to viscoelastic bodies, and in order to determine this we consider the consequences of our methodology when applied to viscoelastic solid models that have been proposed to describe the response of rubber and rubber-like materials. In this study, we evaluate the efficacy of our proposed methodology within the context of two viscoelastic models, one a generalization of a model proposed by Gent [11] and the other a generalization of the compressible Kelvin-Voigt model. Some remarks concerning our generalization of the Gent model is warranted. Gent developed a model that describes elastic response and used it to describe the response of rubber. However, the response of rubber is viscoelastic, and hence it is necessary to generalize his model to describe viscoelastic response. In this paper, the generalization is made in such a manner that the generalization is compatible with the demands of thermodynamics. To start with, Gent's model has a complicated structure, hence it is not a surprise that our generalization leads to a very complicated constitutive relation. Our consideration of a generalization of the compressible Kelvin–Voigt model with a much simpler constitutive structure stems from our wanting to ensure that the initiation of the 'damage' that arises is not due to the complicated structure of the generalization of the Gent model. We found that the generalized Kelvin–Voigt model also qualitatively showed the same sort of behaviour as the generalized Gent model.

Gent & Lindley [10] found that the specimens were damaged between stretch ratios of 4 and 10, depending on the specimen and this is in qualitative agreement with the results established for the above two models in this paper. We emphasize that our study concerns an ideal body with a single inhomogeneity while the experiments pertain to a random distribution of inhomogeneities. Furthermore, we assume a particular value for the density of the inhomogeneity and this may not necessarily reflect the densities associated with any of the inhomogeneities in the experimental specimen.

We consider a thin square plate of length L comprised of a viscoelastic material with a central circular region of radius 0.0001 L wherein the density is significantly smaller than the region external to it. We consider two types of loading, first a ramp (figure 2*a*) and then a two-step ramp (figure 2*b*). On being loaded, the square plate stretches and the density starts to change, the change being much more pronounced in the circular region wherein the initial density was lower. We find, as is to be expected, that damage is initiated in this central circular region of lower density. Our results are in keeping with the results that we obtained for the initiation of damage for elastic bodies and the results are once again in good qualitative agreement with the experimental observations of Gent & Lindley [10].

The parameter that determines damage, namely the density, is specific to the problem under consideration. For other problems, say involving metals, the parameter could be a quantity that is related to the microstructure, for instance the density of dislocations. The important point to be cognizant of is that damage is mostly a consequence of material inhomogeneity and the nature of the inhomogeneity might depend on the class of materials being considered.

The organization of our paper is as follows. In the next section, we provide a brief introduction to the relevant kinematics. The constitutive relations that we use are introduced in §3, and this is followed by the delineation of the initial-boundary value problem in §4. We devote §5 to a discussion of the results and the final section is devoted to some concluding remarks.

2. Kinematics

Let $\kappa_R(\mathfrak{B})$ and $\kappa_t(\mathfrak{B})$ denote the reference configuration of the body and the configuration of the body $\mathfrak{B}$ at time t, respectively. Let $\mathbf{X}$ and $\mathbf{x}$ denote a typical point belonging to $\kappa_R(\mathfrak{B})$ and $\kappa_t(\mathfrak{B})$, respectively. Let χ_{κ_R} be the one to one mapping that assigns each $\mathbf{X}$ in the reference configuration, $\kappa_R(\mathfrak{B})$, to $\mathbf{x}$ in the current configuration, $\kappa_t(\mathfrak{B})$, i.e. $\mathbf{x} = \chi_{\kappa_R}(\mathbf{X}, t)$.

rspa.royalsocietypublishing.org Proc. R. Soc. A 474: 20180064

We shall assume that χ_{κ_R} is sufficiently smooth so that all the derivatives that are taken make sense. The velocity $\mathbf{v}$, the velocity gradient $\mathbf{L}$ and the deformation gradient $\mathbf{F}_{\kappa_R}$ are defined through

$$\mathbf{v} := \frac{\partial \chi_{\kappa_R}}{\partial t}, \quad \mathbf{L} := \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \quad \text{and} \quad \mathbf{F}_{\kappa_R} := \frac{\partial \chi_{\kappa_R}}{\partial \mathbf{X}}. \tag{2.1}$$

It immediately follows that

$$\mathbf{L} = \dot{\mathbf{F}}_{\kappa_R}\mathbf{F}_{\kappa_R}^{-1}. \tag{2.2}$$

The symmetric part of velocity gradient $\mathbf{D}$ is given by

$$\mathbf{D} := \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \tag{2.3}$$

where $\dot{(.)}$, $(.)^{-1}$ and $(.)^{\mathrm{T}}$ are the material time derivative, inverse and transpose of second-order tensor, respectively. The left and right Cauchy–Green tensors $\mathbf{B}_{\kappa_R}$ and $\mathbf{C}_{\kappa_R}$ are, respectively, defined through

$$\mathbf{B}_{\kappa_R} := \mathbf{F}_{\kappa_R}\mathbf{F}_{\kappa_R}^{\mathrm{T}} \quad \text{and} \quad \mathbf{C}_{\kappa_R} := \mathbf{F}_{\kappa_R}^{\mathrm{T}}\mathbf{F}_{\kappa_R}. \tag{2.4}$$

Let $\kappa_{p(t)}(\mathfrak{B})$ denote the natural configuration[2] corresponding to $\kappa_t(\mathfrak{B})$. We can define $\mathbf{F}_{\kappa_{p(t)}} := \partial\mathbf{x}/\partial\mathbf{X}_P$, where $\mathbf{X}_P \in \kappa_{p(t)}(\mathfrak{B})$ corresponds to the point $\mathbf{X}$. We can define the corresponding Cauchy–Green tensors through

$$\mathbf{B}_{\kappa_{p(t)}} := \mathbf{F}_{\kappa_{p(t)}}\mathbf{F}_{\kappa_{p(t)}}^{\mathrm{T}} \quad \text{and} \quad \mathbf{C}_{\kappa_{p(t)}} := \mathbf{F}_{\kappa_{p(t)}}^{\mathrm{T}}\mathbf{F}_{\kappa_{p(t)}}. \tag{2.5}$$

The unimodular tensors $\bar{\bar{\mathbf{B}}}$ and $\bar{\bar{\mathbf{C}}}$ are defined through

$$\bar{\bar{\mathbf{B}}}_{\kappa_R} := (\det \mathbf{F}_{\kappa_R}^{-2/3})\mathbf{B}_{\kappa_R} \quad \text{and} \quad \bar{\bar{\mathbf{C}}}_{\kappa_R} := (\det \mathbf{F}_{\kappa_R}^{-2/3})\mathbf{C}_{\kappa_R},$$
$$\bar{\bar{\mathbf{B}}}_{\kappa_{p(t)}} := (\det \mathbf{F}_{\kappa_{p(t)}}^{-2/3})\mathbf{B}_{\kappa_{p(t)}} \quad \text{and} \quad \bar{\bar{\mathbf{C}}}_{\kappa_{p(t)}} := (\det \mathbf{F}_{\kappa_{p(t)}}^{-2/3})\mathbf{C}_{\kappa_{p(t)}}$$

and their corresponding first principal invariants of are $I_{1_{\kappa_R}} = \operatorname{tr}\bar{\bar{\mathbf{B}}}_{\kappa_R} = \operatorname{tr}\bar{\bar{\mathbf{C}}}_{\kappa_R}$, $I_{1_{\kappa_{p(t)}}} = \operatorname{tr}\bar{\bar{\mathbf{B}}}_{\kappa_{p(t)}} = \operatorname{tr}\bar{\bar{\mathbf{C}}}_{\kappa_{p(t)}}^{-1}$. The tensor $\mathbf{G}$ is a mapping between appropriate tangent spaces from $\mathbf{X}$ belonging to $\kappa_R(\mathfrak{B})$ to that at $\mathbf{x}$ belonging to $\kappa_{p(t)}(\mathfrak{B})$, i.e. $\mathbf{G} = \mathbf{F}_{\kappa_R \to \kappa_{p(t)}} := \mathbf{F}_{\kappa_{p(t)}}^{-1}\mathbf{F}_{\kappa_R}$. We define the tensors $\mathbf{C}_{\kappa_R \to \kappa_{p(t)}}$, $\mathbf{B}_{\kappa_{p(t)}}$, $\mathbf{L}_{\kappa_{p(t)}}$ and $\mathbf{D}_{\kappa_{p(t)}}$ through

$$\left.\begin{aligned} \mathbf{C}_{\kappa_R \to \kappa_{p(t)}} &:= \mathbf{G}^{\mathrm{T}}\mathbf{G}, \quad \mathbf{B}_{\kappa_{p(t)}} := \mathbf{F}_{\kappa_R}\mathbf{C}_{\kappa_R \to \kappa_{p(t)}}\mathbf{F}_{\kappa_R}^{\mathrm{T}} \\ \text{and} \qquad \mathbf{L}_{\kappa_{p(t)}} &= \dot{\mathbf{G}}\mathbf{G}^{-1} \end{aligned}\right\} \tag{2.6}$$

and

$$\mathbf{D}_{\kappa_{p(t)}} := \tfrac{1}{2}(\mathbf{L}_{\kappa_{p(t)}} + \mathbf{L}_{\kappa_{p(t)}}^{\mathrm{T}}). \tag{2.7}$$

Further, we note that the upper convected Oldroyd derivative [13] of $\mathbf{B}_{\kappa_{p(t)}}$ can be related to $\mathbf{D}_{\kappa_{p(t)}}$ through[3] (see [14])

$$\overset{\triangledown}{\mathbf{B}}_{\kappa_{p(t)}} := \dot{\mathbf{B}}_{\kappa_{p(t)}} - \mathbf{L}\mathbf{B}_{\kappa_{p(t)}} - \mathbf{B}_{\kappa_{p(t)}}\mathbf{L}^{\mathrm{T}} = -2\mathbf{F}_{\kappa_{p(t)}}\mathbf{D}_{\kappa_{p(t)}}\mathbf{F}_{\kappa_{p(t)}}^{\mathrm{T}}. \tag{2.8}$$

3. Constitutive relations

We use the second law of thermodynamics in the form of the reduced energy dissipation equation [15], which is given by

$$\mathbf{T} \cdot \mathbf{D} - \rho\dot{\psi} = \xi \geq 0, \tag{3.1}$$

where $\mathbf{T}$ is the Cauchy stress, ψ is the specific Helmholtz free energy and ξ is the rate of dissipation.

[2]A detailed discussion of what is meant by a natural configuration can be found in Rajagopal [12]. For our purpose, it is sufficient to think of the natural configuration as the configuration that the body would attain when all the external stimuli acting on the body in its current configuration at time t are removed.

[3]Henceforth, we suppress 't' from the subscript.

The first model we consider is a generalization of a model due to Gent (see [11,16]) whose Helmholtz potential ψ and the rate of dissipation ξ are of the form

$$\psi = -\frac{\mu_1(\mathbf{X})}{2\rho_{\kappa_p}}(I_m^{(1)}(\mathbf{X})-3)\log\left(1-\left(\frac{I_{1\kappa_p}-3}{I_m^{(1)}(\mathbf{X})-3}\right)\right)+\frac{K_1(\mathbf{X})}{2\rho_{\kappa_p}}\left[I_{3\kappa_p}^{\nu_1(\mathbf{X})}+\frac{1}{I_{3\kappa_p}^{\nu_1(\mathbf{X})}}-2\right]$$
$$-\frac{\mu_2(\mathbf{X})}{2\rho_{\kappa_R}}(I_m^{(2)}(\mathbf{X})-3)\log\left(1-\left(\frac{I_{1\kappa_R}-3}{I_m^{(2)}(\mathbf{X})-3}\right)\right)+\frac{K_2(\mathbf{X})}{2\rho_{\kappa_R}}\left[I_{3\kappa_R}^{\nu_2(\mathbf{X})}+\frac{1}{I_{3\kappa_R}^{\nu_2(\mathbf{X})}}-2\right] \tag{3.2}$$

and

$$\xi = \eta_0(\mathbf{X})(\mathbf{D}_{\kappa_p}\cdot\mathbf{D}_{\kappa_p}-\tfrac{1}{3}(\operatorname{tr}\mathbf{D}_{\kappa_p})^2)+\eta_1(\mathbf{X})(\operatorname{tr}\mathbf{D}_{\kappa_p})^2, \tag{3.3}$$

where $\mu_1(\mathbf{X})$, $\mu_2(\mathbf{X})$ and $K_1(\mathbf{X})$, $K_2(\mathbf{X})$ are the shear and bulk moduli, $I_m^{(1)}$, $I_m^{(2)}$ are the stretch limits, ν_1, ν_2 are the volumetric exponents and η_0, η_1 are shear and bulk viscosities and $I_{3_{\kappa_R}}=\det\mathbf{F}_{\kappa_R}$, $I_{3\kappa_p}=\det\mathbf{F}_{\kappa_p}$ are the invariants.

Substituting equations (3.2) in (3.1), we have

$$\left\{\mathbf{T}-\left(\frac{\mu_1(\mathbf{X})}{(\det\mathbf{F}_{\kappa_p})}\frac{1}{(1-((I_{1\kappa_p}-3)/(I_m^{(1)}(\mathbf{X})-3)))}\operatorname{dev}(\bar{\bar{\mathbf{B}}}_{\kappa_p})+\frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{\det\mathbf{F}_{\kappa_p}}\left[I_{3\kappa_p}^{\nu_1(\mathbf{X})}-\frac{1}{I_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]\mathbf{I}\right.\right.$$
$$\left.\left.+\frac{\mu_2(\mathbf{X})}{(\det\mathbf{F}_{\kappa_R})}\frac{1}{(1-((I_{1\kappa_R}-3)/(I_m^{(2)}(\mathbf{X})-3)))}\operatorname{dev}(\bar{\bar{\mathbf{B}}}_{\kappa_R})+\frac{K_2(\mathbf{X})\nu_2(\mathbf{X})}{\det\mathbf{F}_{\kappa_R}}\left[I_{3\kappa_R}^{\nu_2(\mathbf{X})}-\frac{1}{I_{3\kappa_R}^{\nu_2(\mathbf{X})}}\right]\mathbf{I}\right)\right\}\cdot\mathbf{D}$$
$$+\mathbf{A}\cdot\mathbf{D}_{\kappa_p}=\xi(\mathbf{B}_{\kappa_p},\mathbf{D}_{\kappa_p}), \tag{3.4}$$

where dev ($\mathbf{A}$) represents the deviatoric part of a second-order tensor $\mathbf{A}$.

A sufficient condition for the equation (3.4) to be satisfied is

$$\mathbf{T}=\frac{\mu_1(\mathbf{X})}{\det\mathbf{F}_{\kappa_p}}\frac{1}{(1-((I_{1\kappa_p}-3)/(I_m^{(1)}(\mathbf{X})-3)))}\operatorname{dev}(\bar{\bar{\mathbf{B}}}_{\kappa_p})+\frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{\det\mathbf{F}_{\kappa_p}}\left[I_{3\kappa_p}^{\nu_1(\mathbf{X})}-\frac{1}{I_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]\mathbf{I}$$
$$+\frac{\mu_2(\mathbf{X})}{\det\mathbf{F}_{\kappa_R}}\frac{1}{(1-((I_{1\kappa_R}-3)/(I_m^{(2)}(\mathbf{X})-3)))}\operatorname{dev}(\bar{\bar{\mathbf{B}}}_{\kappa_R})+\frac{K_2(\mathbf{X})\nu_2(\mathbf{X})}{\det\mathbf{F}_{\kappa_R}}\left[I_{3\kappa_R}^{\nu_2(\mathbf{X})}-\frac{1}{I_{3\kappa_R}^{\nu_2(\mathbf{X})}}\right]\mathbf{I} \tag{3.5}$$

and

$$\mathbf{A}\cdot\mathbf{D}_{\kappa_p}=\xi(\mathbf{B}_{\kappa_p},\mathbf{D}_{\kappa_p}), \tag{3.6}$$

where

$$\mathbf{A}=\frac{\mu_1(\mathbf{X})}{\det\mathbf{F}_{\kappa_p}}\frac{1}{(1-((I_{1\kappa_p}-3)/(I_m^{(1)}(\mathbf{X})-3)))}\operatorname{dev}(\bar{\bar{\mathbf{B}}}_{\kappa_p})$$
$$+\frac{\mu_1(\mathbf{X})}{2\det\mathbf{F}_{\kappa_p}}(I_m^{(1)}(\mathbf{X})-3)\log\left(1-\left(\frac{I_{1\kappa_p}-3}{I_m^{(1)}(\mathbf{X})-3}\right)\right)\mathbf{I}$$
$$+\frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{2\det\mathbf{F}_{\kappa_p}}\left[I_{3\kappa_p}^{\nu_1(\mathbf{X})}-\frac{1}{I_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]\mathbf{I}-\frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{2\det\mathbf{F}_{\kappa_p}}\left[I_{3\kappa_p}^{\nu_1(\mathbf{X})}+\frac{1}{I_{3\kappa_p}^{\nu_1(\mathbf{X})}}-2\right]\mathbf{I}.$$

The equation (3.6) places restrictions on $\mathbf{B}_{\kappa_p}$ and $\mathbf{D}_{\kappa_p}$. The latter evolves in such a way that ξ is maximized subject to the constraint in equation (3.6). Following Rajagopal & Srinivasa [14], we

rspa.royalsocietypublishing.org Proc. R. Soc. A 474: 20180064

maximize the rate of dissipation in equation (3.3) subject to the constraints in equation (3.6) by varying $\mathbf{D}_{\kappa_p}$, for fixed $\mathbf{B}_{\kappa_p}$. For this, we maximize the auxiliary function $\hat{\xi}$, which is defined by

$$\hat{\xi} = \xi + \lambda(\xi - \mathbf{A} \bullet \mathbf{D}_{\kappa_p}), \tag{3.7}$$

where λ is the Lagrange multiplier. Now, setting

$$\frac{\partial \hat{\xi}}{\partial \mathbf{D}_{\kappa_p}} = 0, \tag{3.8}$$

we get

$$\frac{1+\lambda}{\lambda} = \frac{1}{2}. \tag{3.9}$$

Using equations (3.8) and (3.9) , we have

$$\mathbf{D}_{\kappa_p} + \left(\frac{\eta_1}{\eta_0} - \frac{1}{3}\right)\frac{\mathrm{tr}\,(\mathbf{A})}{3\eta_1}\mathbf{I} = \frac{\mathbf{A}}{\eta_0}. \tag{3.10}$$

We assume the material is isotropic, hence we can choose configurations such that $\mathbf{F}_{\kappa_p} = \mathbf{V}_{\kappa_p}$, where $\mathbf{V}_{\kappa_p}$ is the right stretch tensor in the polar decomposition $\mathbf{F}_{\kappa_p} = \mathbf{V}_{\kappa_p}\mathbf{R}_{\kappa_p}$. Now pre-multiplying and post-multiplying equation (3.10) by $\mathbf{V}_{\kappa_p}$ and using (2.8), we have the following evolution equation:

$$\overset{\triangledown}{\mathbf{B}}_{\kappa_p} + 2\frac{\mathbf{B}_{\kappa_p}}{\eta_0}\left(\mathrm{dev}(\mathbf{A}) + \frac{\eta_0}{9\eta_1}\mathrm{tr}\,(\mathbf{A})\mathbf{I}\right) = \mathbf{0}. \tag{3.11}$$

Note that the Cauchy stress vanishes when the deformation gradient $\mathbf{F}_{\kappa_R}$ and $\mathbf{F}_{\kappa_p}$ are $\mathbf{I}$. We non-dimensionalize the constitutive equations using the characteristic length scale 'L' and modulus K_1: $u = L\bar{u}$, $v = L\bar{v}$, $w = L\bar{w}$, $t = t_0\bar{t}$ and $\mathbf{T} = K_1\bar{\mathbf{T}}$. The deformation gradient $\mathbf{F}_{\kappa_p}$ is non-dimensional: but to be consistent in our notation, we denote it with an over bar. The final non-dimensionalized Cauchy stress is given by

$$\begin{aligned}\bar{\mathbf{T}} = {} & \frac{\mu_1(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_p}} \frac{1}{(1 - ((\bar{I}_{1\kappa_p} - 3)/(I_m^{(1)}(\mathbf{X}) - 3)))}\mathrm{dev}(\bar{\mathbf{B}}_{\kappa_p}) + \frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_p}} \\ & \left[\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})} - \frac{1}{\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]\mathbf{I} + \frac{\mu_2(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_R}} \frac{1}{(1 - ((\bar{I}_{1\kappa_R} - 3)/(I_m^{(2)}(\mathbf{X}) - 3)))} \\ & \mathrm{dev}(\bar{\mathbf{B}}_{\kappa_R}) + \frac{K_2(\mathbf{X})\nu_2(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_R}}\left[\bar{I}_{3\kappa_p}^{\nu_2(\mathbf{X})} - \frac{1}{\bar{I}_{3\kappa_R}^{\nu_2(\mathbf{X})}}\right]\mathbf{I}\end{aligned} \tag{3.12}$$

and the non-dimensionalized evolution equation is as follows:

$$\overset{\triangledown}{\bar{\mathbf{B}}}_{\kappa_p} + M\bar{\mathbf{B}}_{\kappa_p}\left(\mathrm{dev}(\bar{\mathbf{A}}) + \frac{\eta_0}{9\eta_1}\mathrm{tr}\,\bar{\mathbf{A}}\right)\mathbf{I} = \mathbf{0}, \tag{3.13}$$

where

$$M = \frac{2K_0 t_0}{\eta_0} \tag{3.14}$$

and

$$\begin{aligned}\bar{\mathbf{A}} = {} & \frac{\mu_1(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_p}} \frac{1}{(1 - ((\bar{I}_{1\kappa_p} - 3)/(I_m^{(1)}(\mathbf{X}) - 3)))}\mathrm{dev}\,(\bar{\mathbf{B}}_{\kappa_p}) + \frac{\mu_1(\mathbf{X})}{2K_0 \det \bar{\mathbf{F}}_{\kappa_p}}(I_m^{(1)}(\mathbf{X}) - 3) \\ & \times \log\left(1 - \left(\frac{\bar{I}_{1\kappa_p} - 3}{I_m^{(1)}(\mathbf{X}) - 3}\right)\right)\mathbf{I} + \frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{2K_0 \det \bar{\mathbf{F}}_{\kappa_p}}\left[\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})} - \frac{1}{\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]\mathbf{I} \\ & - \frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{2K_0 \det \bar{\mathbf{F}}_{\kappa_p}}\left[\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})} + \frac{1}{\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})}} - 2\right]\mathbf{I}.\end{aligned} \tag{3.15}$$

Also, in the above equation (3.12) K_0 is a modulus that is related to the compressibility of the body. The constitutive relations characterizing this model are equations (3.12) and (3.13).

The second model we consider in this study is a generalized form of the Kelvin–Voigt model wherein the Cauchy stress is given by

$$\mathbf{T} = K_0(\mathbf{X})\frac{(\det \mathbf{F} - 1)}{(\det \mathbf{F})^3}\mathbf{I} + \mu_0(\mathbf{X})\frac{1}{\det \mathbf{F}}\,\mathrm{dev}(\bar{\bar{\mathbf{B}}}) + 2\eta_0(\mathbf{X})\mathbf{D},$$

where $\mu_0(\mathbf{X})/\det \mathbf{F}$ and $K_0(\mathbf{X})/\det \mathbf{F}^3$ are the shear and bulk moduli which decrease with increase in det$\mathbf{F}$, and $\eta_0(\mathbf{X})$ is the viscosity.

Similar to the earlier model, non-dimensionalization was carried out using the characteristic scales so that $u = L\bar{u}$, $v = L\bar{v}$, $w = L\bar{w}$, $t = t_0\bar{t}$ and $\mathbf{T} = K_1\bar{\mathbf{T}}$. The non-dimensionalized Cauchy stress tensor is then given by

$$\bar{\mathbf{T}} = \frac{K_0(\bar{\mathbf{X}})}{K_1}\frac{(\det(\bar{\mathbf{F}}) - 1)}{(\det \bar{\mathbf{F}})^3}\mathbf{I} + \frac{\mu_0(\bar{\mathbf{X}})}{K_1}\frac{1}{\det \bar{\mathbf{F}}}\,\mathrm{dev}(\bar{\mathbf{B}}) + 2\frac{\eta_0(\bar{\mathbf{X}})}{t_0 K_1}\bar{\mathbf{D}}. \tag{3.16}$$

4. The initial-boundary value problem studied

We consider a thin square plate of side '2L'. The reference density is constant over the region external to a small circular region of radius 0.0001 L at the centre of the plate wherein the density is significantly less than in the region external to it. As the variation is over a small region compared to the overall dimension of the plate, figure 1 shows only a small portion of the plate. Using symmetry, we consider only one quarter of the plate for analysis.

We assume that, for a thin sheet, the non-dimensionalized displacement components in X, Y and Z are $u(X, Y, t)$, $v(X, Y, t)$ and $w(X, Y, Z, t) = Z\phi(X, Y, t)$, respectively. It has been shown that the Z direction displacement is for all practical purposes linear in Z based on the three-dimensional study done by Alagappan *et al.* [1]. Hence, the deformation gradient is given by

$$\mathbf{F}_{\kappa_R} = \mathbf{I} + \frac{\partial \mathbf{u}(\mathbf{X})}{\partial \mathbf{X}}, \tag{4.1}$$

i.e.

$$\mathbf{F}_{\kappa_R} = \begin{bmatrix} 1 + \dfrac{\partial u}{\partial X} & \dfrac{\partial u}{\partial Y} & 0 \\ \dfrac{\partial v}{\partial X} & 1 + \dfrac{\partial v}{\partial Y} & 0 \\ Z\dfrac{\partial \phi}{\partial X} & Z\dfrac{\partial \phi}{\partial Y} & 1 + \phi \end{bmatrix}. \tag{4.2}$$

The assumption of linear dependence of Z results in the Cauchy stress components, $\bar{T}_{XZ}$ and $\bar{T}_{YZ}$ also being linear in Z (see [1]). Therefore, by selecting an appropriate thickness for the plate, the components $\bar{T}_{XZ}$ and $\bar{T}_{YZ}$ will be negligible and from this perspective we can drop these terms. This results in the deformation gradient being only a function of 'X' and 'Y', i.e.

$$\mathbf{F}_{\kappa_R} = \begin{bmatrix} 1 + \dfrac{\partial u}{\partial X} & \dfrac{\partial u}{\partial Y} & 0 \\ \dfrac{\partial v}{\partial X} & 1 + \dfrac{\partial v}{\partial Y} & 0 \\ 0 & 0 & 1 + \phi \end{bmatrix}. \tag{4.3}$$

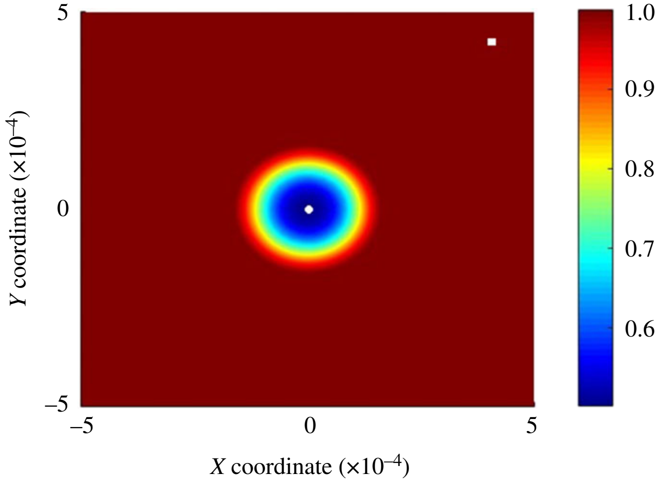

Figure 1. Initial reference density in a square of size 0.001 unit (close-up view). The filled shapes circle and square represents critical points A and B, respectively. (Online version in colour.)

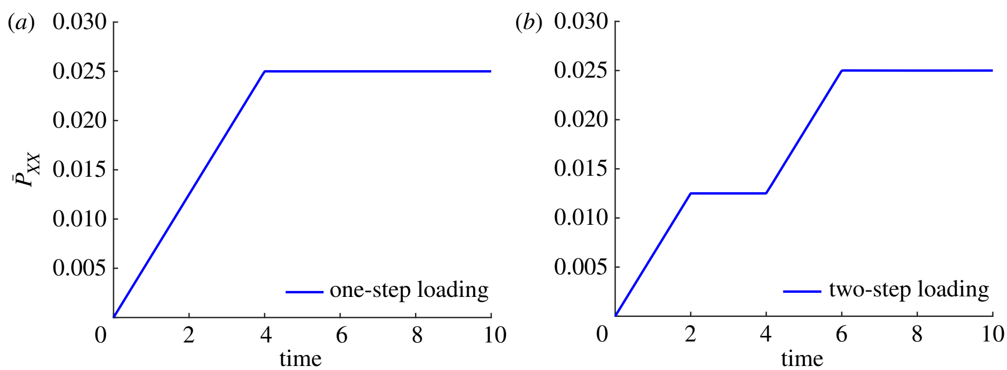

Figure 2. Loadings. (*a*) One-step loading and (*b*) two-step loading. (Online version in colour.)

(a) Loading and boundary condition

The non-dimensionalized length of the plate is taken to be 1. To study the initiation of damage, we consider the plate subjected to bi-axial loading condition. To enforce the boundary conditions, we use the Piola–Kirchhoff stress $\bar{\mathbf{P}}$ which is defined as

$$\bar{\mathbf{P}} = (\det \bar{\mathbf{F}}_{\kappa_R})\bar{\mathbf{T}}\bar{\mathbf{F}}_{\kappa_R}^{-\mathrm{T}}. \tag{4.4}$$

The boundary conditions for the bi-axial loading are given by

$$\bar{P}_{XX}(1, Y, t) = 6.25 \times 10^{-3} t$$
$$\text{and} \quad \bar{P}_{YY}(X, 1, t) = 6.25 \times 10^{-3} t$$

and the symmetry conditions are

$$u(0, Y, t) = 0 \quad \forall\, 0 \le Y \le 1 \quad \forall\, t > 0$$
$$\text{and} \quad v(X, 0, t) = 0 \quad \forall\, 0 \le X \le 1 \quad \forall\, t > 0.$$

rspa.royalsocietypublishing.org Proc. R. Soc. A 474: 20180064

The response of the viscoelastic plate is studied also by using one- and two-step loading. The loading condition for one-step loading is

$$\bar{P}_{XX}(1, Y, t) = \text{see figure 2}a \quad \forall\, t > 0$$
$$\text{and} \quad \bar{P}_{YY}(X, 1, t) = \text{see figure 2}a \quad \forall\, t > 0$$

and the two-step loading is given by

$$\bar{P}_{XX}(1, Y, t) = \text{see figure 2}b \quad \forall\, t > 0$$
$$\text{and} \quad \bar{P}_{YY}(X, 1, t) = \text{see figure 2}b \quad \forall\, t > 0$$

and the symmetry conditions for one- and two-step loading condition are

$$u(0, Y, t) = 0 \quad \forall\, 0 \le Y \le 1 \quad \forall\, t > 0$$
$$\text{and} \quad v(X, 0, t) = 0 \quad \forall\, 0 \le X \le 1 \quad \forall\, t > 0.$$

In this study, we shall ignore inertial effects. The non-dimensional governing equilibrium equation in the absence of body forces is given by

$$\operatorname{Div} \bar{\mathbf{P}} = \mathbf{0}, \tag{4.5}$$

where $\bar{\mathbf{P}}$ is the first Piola–Kirchhoff stress tensor. The relation between the Cauchy stress and first Piola–Kirchhoff tensor is given by

$$\bar{\mathbf{P}} = (\det \bar{\mathbf{F}}_{\kappa_R})\bar{\mathbf{T}}\bar{\mathbf{F}}_{\kappa_R}^{-\mathrm{T}}. \tag{4.6}$$

Here, we adopt the Lagrangian formulation as it is numerically convenient and, in the case of large deformations, it does not involve the use of an evolving geometry to study the governing equations.

(b) Generalized Gent model

As $\mathbf{F}_{\kappa_R}$ is only a function of the coordinates X and Y, the traction-free condition on the top and the bottom of the sheet reduces to $\bar{P}_{ZZ} = 0$.

It follows from the form of $\mathbf{F}_{\kappa_R}$ that the matrix associated with $\mathbf{B}_{\kappa_p}$ is given by

$$\mathbf{B}_{\kappa_p} = \begin{bmatrix} b_1 & b_4 & 0 \\ b_4 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}. \tag{4.7}$$

Using equation (3.12) and relation (4.6), the Piola–Kirchhoff stress components for the generalized Gent model are given by

$$\begin{aligned}
\bar{P}_{XX} = &-\left(\left(\alpha_2\left(\left(\frac{\partial u}{\partial X}+1\right)\frac{\partial v}{\partial X}+\left(\frac{\partial v}{\partial Y}+1\right)\frac{\partial u}{\partial Y}\right)+\alpha_1 b_4\right)\frac{\partial u}{\partial Y}\bar{I}_{3\kappa_R}\right)\Big/ \\
&\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right) \\
&-\left(\left(\frac{\partial v}{\partial Y}+1\right)\left(\alpha_2\left(\frac{\partial v}{\partial X}^{2/3}-\frac{(2(\partial u/\partial Y^2))}{3}+(w+1)^{2/3}\right.\right.\right. \\
&\left.-\frac{(2(\partial u/\partial X+1)^2)}{3}+\left(\frac{\partial v}{\partial Y}+1\right)^{2/3}\right)-\alpha_3 \\
&\left.\left.+\alpha_1\left(\frac{b_2}{3}-\frac{(2b_1)}{3}+\frac{b_3}{3}\right)\right)\bar{I}_{3\kappa_R}\right)\Big/\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right),
\end{aligned}$$

$$\bar{P}_{YY} = -\left(\left(\alpha_2\left(\left(\frac{\partial u}{\partial X}+1\right)\frac{\partial v}{\partial X}+\left(\frac{\partial v}{\partial Y}+1\right)\frac{\partial u}{\partial Y}\right)+\alpha_1 b_4\right)\frac{\partial v}{\partial X}\bar{I}_{3\kappa_R}\right)\Big/$$
$$\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right)$$
$$-\left(\left(\frac{\partial u}{\partial X}+1\right)\left(\alpha_2\left(\frac{\partial u}{\partial Y}^{2/3}-\frac{(2(\partial v/\partial X^2))}{3}+(w+1)^{2/3}\right.\right.\right.$$
$$\left.+\left(\frac{\partial u}{\partial X}+1\right)^{2/3}-\frac{(2(\partial v/\partial Y+1)^2)}{3}\right)-\alpha_3+\alpha_1$$
$$\left.\left(\frac{b_1}{3}-\frac{(2b_2)}{3}+\frac{b_3}{3}\right)\right)\bar{I}_{3\kappa_R}\right)\Big/\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right),$$

$$\bar{P}_{XY} = \left(\left(\alpha_2\left(\frac{\partial v}{\partial X}^{2/3}-\frac{(2(\partial u/\partial Y^2))}{3}+(w+1)^{2/3}-\left(2\left(\frac{\partial u}{\partial X}+1\right)^2\right)\Big/3+\left(\frac{\partial v}{\partial Y}+1\right)^{2/3}\right)\right.\right.$$
$$\left.\left.-\alpha_3+\ \alpha_1\left(\frac{b_2}{3}-\frac{(2b_1)}{3}+\frac{b_3}{3}\right)\right)\frac{\partial v}{\partial X}\bar{I}_{3\kappa_R}\right)\Big/\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right)$$
$$+\left(\left(\frac{\partial u}{\partial X}+1\right)\left(\alpha_2\left(\left(\frac{\partial u}{\partial X}+1\right)\frac{\partial v}{\partial X}+\left(\frac{\partial v}{\partial Y}+1\right)\frac{\partial u}{\partial Y}\right)+\alpha_1 b_4\right)\bar{I}_{3\kappa_R}\right)\Big/$$
$$\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}l+\frac{\partial v}{\partial Y}+1\right)$$

$$\bar{P}_{YX} = \left(\left(\alpha_2\left(\frac{\partial u}{\partial Y}^{2/3}-\frac{(2\partial v/\partial X^2)}{3}+(w+1)^{2/3}+\left(\frac{\partial u}{\partial X}+1\right)^{2/3}-\frac{(2(\partial v/\partial Y+1)^2)}{3}\right)-\alpha_3\right.\right.$$
$$\left.\left.+\alpha_1\left(\frac{b_1}{3}-\frac{(2b_2)}{3}+\frac{b_3}{3}\right)\right)\frac{\partial u}{\partial Y}\bar{I}_{3\kappa_R}\right)\Big/\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}+1\right)$$
$$+\left(\left(\frac{\partial v}{\partial Y}+1\right)\left(\alpha_2\left(\left(\frac{\partial u}{\partial X}+1\right)\frac{\partial v}{\partial X}+\left(\frac{\partial v}{\partial Y}+1\right)\frac{\partial u}{\partial Y}\right)+\alpha_1 b_4\right)\bar{I}_{3\kappa_R}\right)\Big/$$
$$\left(\frac{\partial u}{\partial X}\frac{\partial v}{\partial Y}-\frac{\partial u}{\partial Y}\frac{\partial v}{\partial X}+\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Y}+1\right),$$

$$\bar{P}_{ZZ} = -\left(\left(\alpha_2\left(\frac{\partial u}{\partial Y}^{2/3}+\frac{\partial v}{\partial X}^{2/3}-\frac{(2(w+1)^2)}{3}+\left(\frac{\partial u}{\partial X}+1\right)^{2/3}+\left(\frac{\partial v}{\partial Y}+1\right)^{2/3}\right)-\alpha_3\right.\right.$$
$$\left.\left.+\ \alpha_1\left(\frac{b_1}{3}+\frac{b_2}{3}-\frac{(2b_3)}{3}\right)\right)\bar{I}_{3\kappa_R}\right)\Big/(w+1),$$

where

$$\alpha_1 = \frac{\mu_1(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_p}}\frac{1}{(1-((\bar{I}_{1\kappa_p}-3)/I_m^{(1)}(\mathbf{X})-3))},$$

$$\alpha_2 = \frac{\mu_2(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_R}}\frac{1}{(1-((\bar{I}_{1\kappa_R}-3)/I_m^{(2)}(\mathbf{X})-3))}$$

$$\text{and}\quad \alpha_3 = \frac{K_1(\mathbf{X})\nu_1(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_p}}\left[\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})}-\frac{1}{\bar{I}_{3\kappa_p}^{\nu_1(\mathbf{X})}}\right]+\frac{K_2(\mathbf{X})\nu_2(\mathbf{X})}{K_0 \det \bar{\mathbf{F}}_{\kappa_R}}\left[\bar{I}_{3\kappa_p}^{\nu_2(\mathbf{X})}-\frac{1}{\bar{I}_{3\kappa_R}^{\nu_2(\mathbf{X})}}\right].$$

The material parameters for the above model are provided in table 1.

(c) Generalized Kelvin–Voigt model

Equation (3.16) and relation (4.6) are used to obtain the Piola–Kirchhoff stress components, in a procedure similar to that used for determining these components for the generalized Gent model.

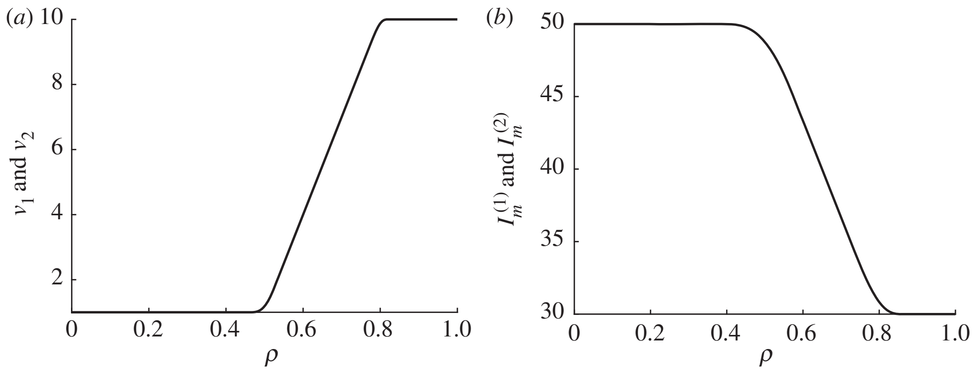

Figure 3. Variation of material parameters with density. (*a*) Variation of ν_1 and ν_2 and (*b*) variation of $I_m^{(1)}$ and $I_m^{(2)}$.

Table 1. Material parameters for generalized Gent model.

parameter	value	parameter	value
μ_1	$10^2\rho_0$	μ_2	$10^2\rho_0$
η_0	$100\ \mu_1$	η_1	$30\ \eta_0$
K_1, K_2	fac $*\ \mu_1$	fac	$100\left(1-\exp\left(\frac{-8\rho_0}{1000}\right)\right)$
K_0	10^4	t_0	80
ν_1, ν_2	smoothed piecewise linear function figure 3*a*		
$I_m^{(1)}$, $I_m^{(2)}$	smoothed piecewise linear function figure 3*b*		

Table 2. Material parameters for the Kelvin–Voigt model.

parameters	functions
μ_0	$10^2\rho_0$
K_0	$100\mu_0\left(1-\exp\left(-\frac{8\rho_0}{1000}\right)\right)$
K_1	10^4
η_0	$100\mu_0$

The material parameters for the compressible Kelvin–Voigt model are documented in table 2.

5. On the criterion for initiation of damage

Initially, as the stretch increases, the norm of the stress increases monotonically. However, after the body attains a certain stretch, we find that the stress required for continued stretching starts to decrease. The state of stress at which the stress starts to decrease with increasing stretch is the location where the norm of the stress has an inflection point with respect to density. The reason for such a change in the response of the body is, we conjecture, due to the initiation of damage which leads to the need for less norm of the stress to effect the elongation. Based on this idea, Alagappan *et al.* [1,17] hypothesized that damage is initiated when the derivative of the norm of the stress with respect to the stretch becomes zero and then starts to become negative. In this

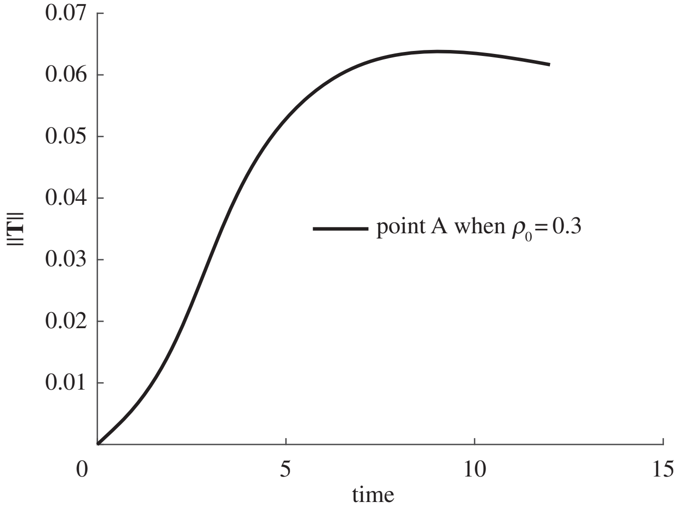

Figure 4. Variation of the norm of the stress tensor with time.

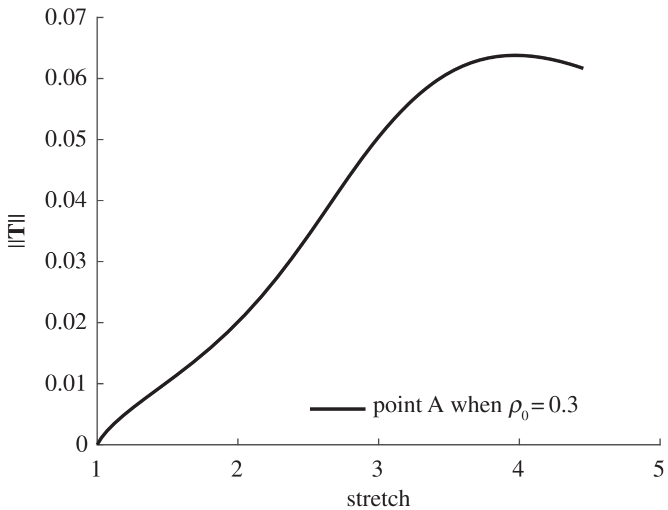

Figure 5. Variation of the norm of the stress tensor with stretch.

study, we find that the same criterion leads to the prediction of the initiation of damage as in the earlier studies by Alagappan *et al.* [1,17] for compressible elastic bodies.

6. Results and discussion

In this study, two major types of loading conditions, namely ramp (linear with respect to time) and step (one-step and two-step inputs), and two material models, namely the generalized Gent and the generalized Kelvin–Voigt model, are considered. The final non-dimensional equations were solved using COMSOL Multiphysics® that uses the finite-element technique to solve the system of partial differential equations. The mesh is finer adjacent to the inhomogeneity where there is a large variation in the density in the reference configuration and the mesh is coarser away from the inhomogeneity. Triangular elements were used and their number was fixed to be 8834 based

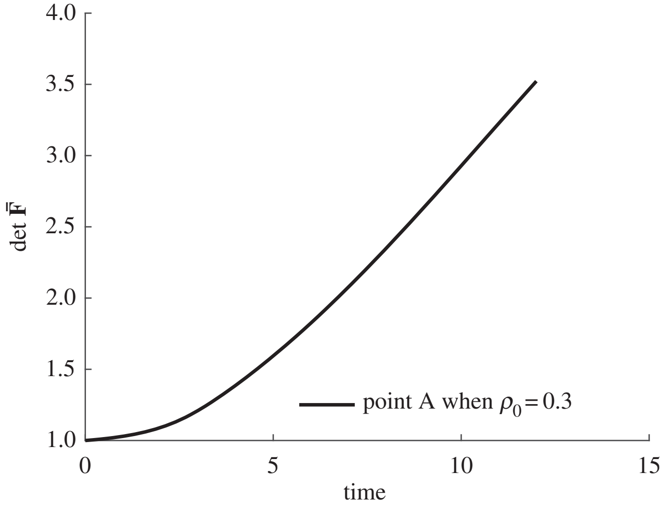

Figure 6. Variation of the determinant of $\bar{\mathbf{F}}$ with time.

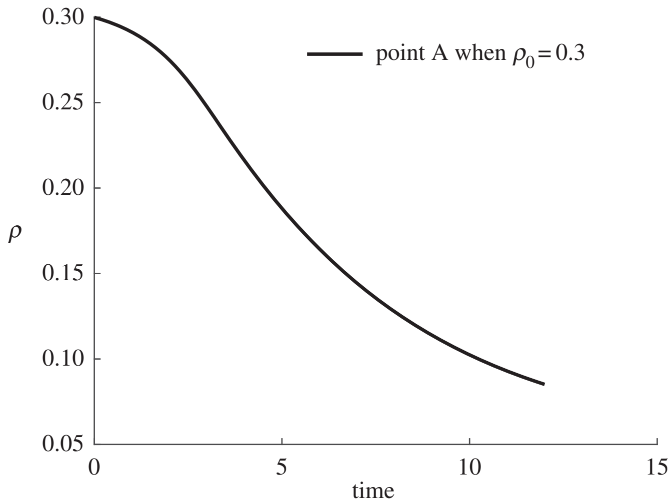

Figure 7. Variation of the density with time.

on the mesh independence study. The non-dimensional mesh size used in this study corresponds to 4×10^{-5} in the finer region and 0.035 in the coarser region. The figures 4–16 display the results for these various cases.

7. Generalized Gent model subject to ramp loading

Figures 4–7 show the response at point A (where the density is initially the lowest) of the generalized Gent model subject to ramp loading. It can be observed that the initiation of damage i.e. when the derivative of norm of Cauchy stress with respect to the stretch attaining the value zero, occurs after approximately after 9 units of time at a stretch of approximately 4 which is perfectly in keeping with experimental results(figures 4 and 5). The corresponding value of det $\bar{\mathbf{F}}$ and current density are 2.6 and 0.115, respectively (figures 6 and 7). Moreover, the reduction of density at the initiation of damage is 61.67%.

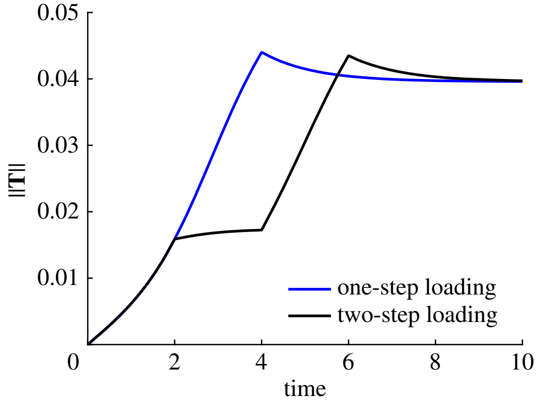

Figure 8. Variation of the norm of the stress tensor with time at point A. (Online version in colour.)

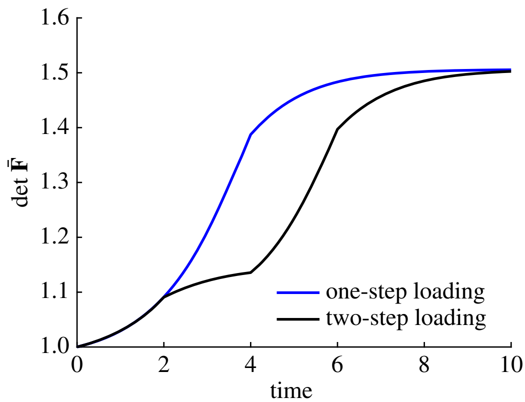

Figure 9. Variation of the determinant of $\bar{\mathbf{F}}$ with time at point A. (Online version in colour.)

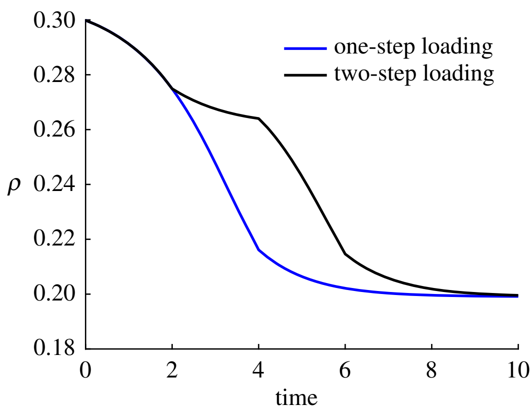

Figure 10. Variation of the density with time at point A. (Online version in colour.)

8. Generalized Gent model subject to one- and two-step loading

The figures 8–10 portray the response of the generalized Gent model subject to one-step and two-step loading. Around 4 and 6 units of time, one- and two-step loadings reach the maximum nominal stress, respectively. But at point A the latter showed a lower value for the norm for

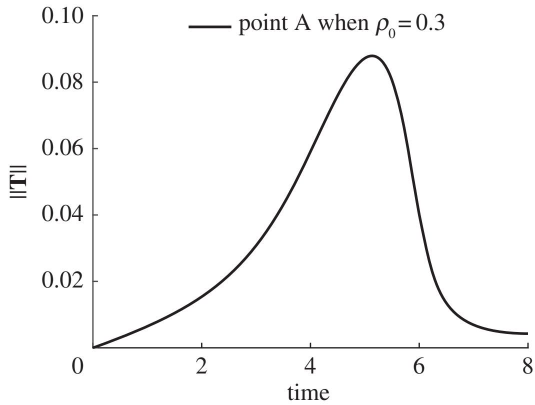

Figure 11. Variation of the norm of the stress tensor with time.

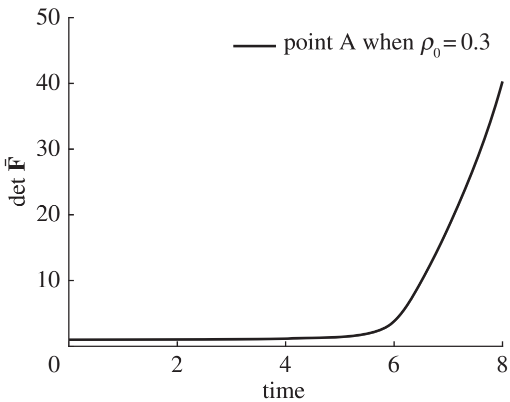

Figure 12. Variation of the determinant of $\bar{\mathbf{F}}$ with time.

the Cauchy stress that is imperceptible in the figure, the difference in the value being 1.1%. The percentage difference in the det $\bar{\mathbf{F}}$ and current density is 5.4 at 6 units of time (see figures 9 and 10). As the loading is held constant, det $\bar{\mathbf{F}}$ and the current density for both the loading conditions are 1.5 and 0.2, respectively.

9. Generalized Kelvin–Voigt model subject to ramp loading

The generalized Kelvin–Voigt model also exhibited behaviour similar to that displayed by the generalized Gent model when subjected to ramp loading, except for the magnitude of the stress (figures 11–13). At approximately 5.15 units of time, the derivative of the norm of the Cauchy stress with respect to stretch is zero and the corresponding det$\bar{\mathbf{F}}$ and current density are 1.5 and 0.2, respectively (figures 12 and 13). The reduction in density at 5.15 units of time is 33.33%.

10. Generalized Kelvin–Voigt subject to one- and two-step loading

The response of the generalized Kelvin–Voigt model subjected to one- and two-step loading are portrayed in figures 14–16. As the nominal load is held constant, there is initially an increase in the norm of the Cauchy stress after which it reaches a constant value irrespective of the

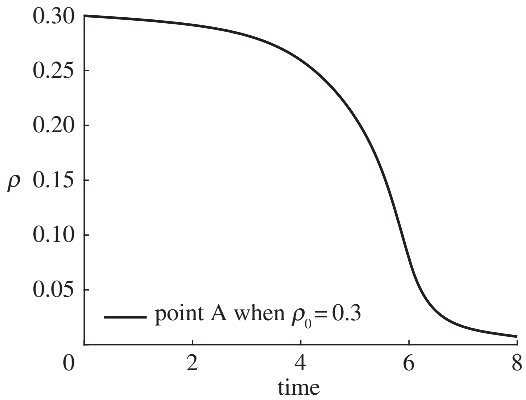

Figure 13. Variation of the density with time.

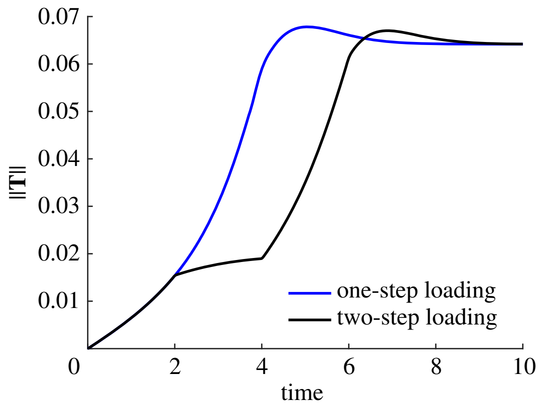

Figure 14. Variation of the norm of the stress tensor with time at point A. (Online version in colour.)

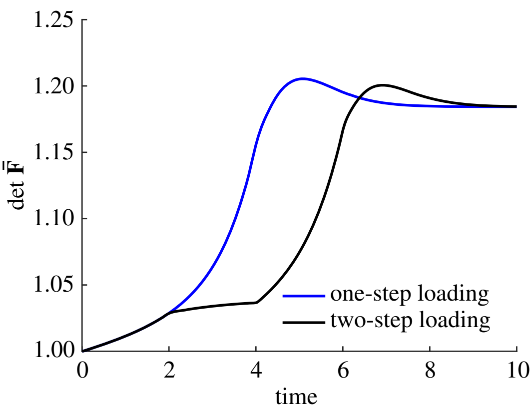

Figure 15. Variation of the determinant of $\bar{\mathbf{F}}$ with time at point A. (Online version in colour.)

type of loading. This variation, after the load is held constant, also resulted in an increase in $\det \bar{\mathbf{F}}$ and decrease in the current density. The difference between the maximum values for the $\det \bar{\mathbf{F}}$ for the one- and two-step loading is 0.4%. The long-term response when the nominal

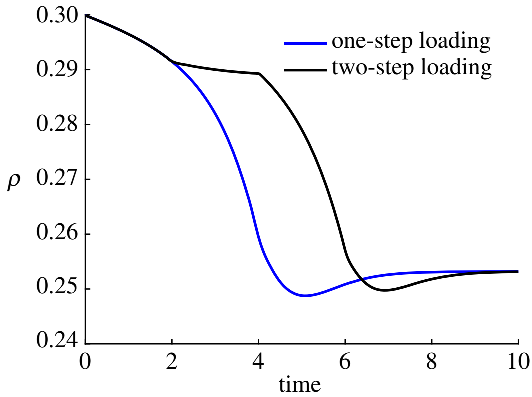

Figure 16. Variation of the density with time at point A. (Online version in colour.)

load is held constant is similar to that for the generalized Gent model. At 10 units of time, $\det \bar{\mathbf{F}}$ and current density are 1.185 and 0.253, respectively, which is the same for both the loading conditions.

11. Conclusion

We have extended our earlier studies regarding the development of a methodology for the initiation of damage for compressible elastic bodies, to the class of compressible viscoelastic bodies. Recognizing that damage is triggered by inhomogeneity and that the initiation is best characterized by means of physically meaningful quantities that are completely determined in the current configuration of the body, we carry out our study based on the density being the physical parameter that determines the onset of damage. We assume that damage starts when the derivative of the magnitude of the Cauchy stress with respect to the stretch becomes negative. At such a state, the body's load-carrying capacity starts to decrease, the signal that the body is starting to get damaged. Our results are in keeping with our earlier study on elastic bodies (see Alagappan *et al.* [1]) in that the stretch ratio at which damage is initiated in the two studies are close. The previous study of Alagappan *et al.* [1] used a model proposed by Gent [11] to describe elastic bodies, and the damage initiation criterion that they proposed agreed well with the experiments of Gent & Lindley [10]. However, modelling rubber as an elastic body is an idealization as it is well known that its response is viscoelastic. In view of this fact, the results established in this work can also be compared against the experiments of Gent [11] on rubber, and the results provide good agreement with the experiments. It is possible to get even better agreement by fine-tuning the model, but our interest is in showing that the damage initiation criterion based on a physically meaningful parameter rather than ad hoc damage parameters is worthwhile and useful.

Data accessibility. This work does not have any experimental data.

Authors' contributions. P.A. was involved in the development of the governing equations and carried out all the numerical computations. K.R.R. suggested the problem and the theoretical framework within which to study it and supervised the solution of the problem. K.K. helped in the formulation of the boundary value problems and the development of the governing equations. All the authors were involved in the write-up and presentation of the problem.

Competing interests. None.

Funding. K.R.R. thanks the National Science Foundation and the Office of Naval Research for support of this work.

Acknowledgements. K.R.R. thanks the National Science Foundation and the Office of Naval Research for support of this work.

References

1. Alagappan P, Kannan K, Rajagopal KR. 2016 On a possible methodology for identifying the initiation of damage of a class of polymeric materials. *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* **472**, 20160231. (doi:10.1098/rspa.2016.0231)
2. Kachanov LM. 1958 On the time to failure under creep conditions. *Izv Akad Nauk SSSR Otd Tekh Nauk* **8**, 26–31.
3. Kachanov LM. 1986 *Introduction to continuum damage mechanics*, vol. 10. Dordrecht, The Netherlands: Martinus Nijhoff Publishers.
4. Lemaitre J. 1992 *A course on damage mechanics*. Berlin, Germany: Springer-Verlag.
5. Krajcinovic D. 1996 *Damage mechanics*. North Holland Series in Applied Mathematics and Mechanics, vol. 41. Amsterdam, The Netherlands: Elsevier.
6. Volokh K. 2013 Review of the energy limiters approach to modeling failure of rubber. *Rubber Chem. Technol.* **86**, 470–487. (doi:10.5254/rct.13.87948)
7. Puglisi G, Saccomandi G. 2015 The Gent model for rubber-like materials: an appraisal for an ingenious and simple idea. *Int. J. Non. Linear Mech.* **68**, 17–24. (doi:10.1016/j.ijnonlinmec.2014.05.007)
8. Bueche A. 1958 The ultimate properties of simple elastomers. *Rubber Chem. Technol.* **31**, 19–26. (doi:10.5254/1.3542258)
9. Langley NR. 1968 Elastically effective strand density in polymer networks. *Macromolecules* **1**, 348–352. (doi:10.1021/ma60004a015)
10. Gent AN, Lindley PB. 1959 Internal rupture of bonded rubber cylinders in tension. *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* **249**, 195–205. (doi:10.1098/rspa.1959.0016)
11. Gent AN. 1996 A new constitutive relation for rubber. *Rubber Chem. Technol.* **69**, 59–61. (doi:10.5254/1.3538357)
12. Rajagopal KR. 1995 *Multiple configurations in continuum mechanics*. Report 6, Pittsburgh, PA: Institute for Computational and Applied Mechanics, University of Pittsburgh.
13. Oldroyd JG. 1950 On the formulation of rheological equations of state. *Proc. R. Soc. Lond. A: Math. Phys. Eng. Sci.* **200**, 523–541. (doi:10.1098/rspa.1950.0035)
14. Rajagopal KR, Srinivasa AR. 2000 A thermodynamic frame work for rate type fluid models. *J. Nonnewton. Fluid Mech.* **88**, 207–227. (doi:10.1016/S0377-0257(99)00023-3)
15. Rajagopal KR, Srinivasa AR. 1998 Mechanics of the inelastic behavior of materials. Part II: Inelastic response. *Int. J. Plast.* **14**, 969–995. (doi:10.1016/S0749-6419(98)00041-2)
16. Alagappan P, Rajagopal KR, Kannan K. 2016 Deformations of infinite slabs of non-linear viscoelastic solids containing an elliptic hole. *Meccanica* **51**, 3067–3080. (doi:10.1007/s11012-016-0539-3)
17. Alagappan P, Rajagopal KR, Kannan K. 2017 Initiation of damage in a class of polymeric materials embedded with multiple localized regions of lower density. *Math. Mech. Solids* **23**, 865–878. (doi:10.1177/1081286517692392)