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A Brief Note on a Theorem of Levinson

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Submitted by Norman Levinson

In the context of relativistic addition of velocities, Levinson [1] has clarified a point by proving the following theorem:

Let u, v be two numbers such that

$$w = \frac{u - v}{1 - g(u, v)} \tag{1}$$

where $g(u, v)$ obeys the conditions

$$|u|, \quad |v| < 1 \qquad \text{implies} \qquad |w| < 1 \tag{2}$$

$$g(u, v) = g(-u, -v) = g(v, u) \tag{3}$$

$$g \leqslant 1 \qquad \text{and} \qquad g(0, 0) = 0 \tag{4}$$

$$g(u, 1) = u \qquad \text{and} \qquad g(1, v) = v. \tag{5}$$

Further, let $h(x)$ be any function such that $h(1) = 1$ and $0 \leqslant h(x) < 1$ in $[0, 1)$ and $[1 - h(x)]/(1 - x) \leqslant A$ for some A in the same interval.

Also, any function $F(x, r)$ is defined for $|x| \leqslant 1$ and $0 \leqslant r \leqslant 2$ such that $F(x, r) \geqslant 0$ for $x \geqslant 0$ and $F(x, r) = \min[2, (1/4A^2)]$ for $x < 0$.

Then all functions

$$g(u, v) = uv - uvF(uv, u^2 + v^2)\,[1 - h(u^2)]\,[1 - h(v^2)] \tag{6}$$

satisfy the requirements (2)–(5) and not only $g = uv$ as was supposed by Ramakrishnan [2]. In fact, the class of functions is enlarged even further in an appendix.

This note intends to point out that Levinson's class of functions will pick $g = uv$ as the only choice provided it is suitably extended beyond the square $|u|, |v| = 1$.

For every v in the infinite strips $|v| > 1$, $|u| < 1$ there exists a Levinson function g for which

$$w[u, (1/v)] = \frac{u - (1/v)}{1 - g(u, 1/v)} \neq 0.$$

Therefore we may define the relative velocity in this strip by the identity

$$w(u, v) \equiv \frac{1}{w(u, 1/v)} . \tag{7}$$

This implies that the equivalent condition satisfied by g for every v is

$$\frac{u - v}{1 - g(u, v)} \cdot \frac{u - (1/v)}{1 - g(u, 1/v)} = 1. \tag{8}$$

Using (6) this reduces to

$$\begin{aligned}(v - u) F(uv, u^2 + v^2) [1 - h(v^2)] \\ + \left(\frac{1}{v} - u\right) F\left(\frac{u}{v}, u^2 + \frac{1}{v^2}\right) \left[1 - h\left(\frac{1}{v^2}\right)\right] \\ + uF(uv, u^2 + v^2) [1 - h(v^2)] \\ \times F\left(\frac{u}{v}, u^2 + \frac{1}{v^2}\right) \left[1 - h\left(\frac{1}{v^2}\right)\right] [1 - h(u^2)] = 0\end{aligned} \tag{9}$$

which is true only when $F \equiv 0$.

Next, when $| u |$ is also >1 the relation similar to (7) is

$$w(u, 1/v) = \frac{1}{w(1/u, 1/v)} \tag{10}$$

so that we now have

$$w(u, v) = w\left(\frac{1}{u}, \frac{1}{v}\right) . \tag{11}$$

Equivalently

$$\begin{aligned}-F(uv, u^2 + v^2) [1 - h(u^2)] [1 - h(v^2)] \\ = \frac{1}{uv} F\left(\frac{1}{uv}, \frac{1}{u^2} + \frac{1}{v^2}\right) \left[1 - h\left(\frac{1}{u^2}\right)\right] \left[1 - h\left(\frac{1}{v^2}\right)\right],\end{aligned} \tag{12}$$

which is true only again when $F \equiv 0$. Hence the result.

It is desirable to point out that velocities >1 are not "choatic" [2] at least because phase velocities beyond the resonance frequencies *are* >1. Therefore (7) (and (11)) are certainly "natural completions of the Newtonian definition" [2] of relative velocity. Indeed, they have been derived from first principles and discussed extensively elsewhere [3, 4].

References

1. N. Levinson, *J. Math. Anal. Appl.* **47** (1974), 222–225.
2. A. Ramakrishnan, *J. Math. Anal. Appl.* **42** (1973), 377–380.
3. T. S. Shankara, *Found. Phys.* **4** (1974), 97–104.
4. E. A. Lord and T. S. Shankara, *Proc. Roy. Soc.*, submitted.