A Birth and Death Process Related to the Rogers-Ramanujan Continued Fraction

P. R. Parthasarathy, R. B. Lenin

Department of Mathematics
Indian Institute of Technology, Madras
Chennai - 600 036, INDIA

W. Schoutens and W. Van Assche

Katholieke Universiteit Leuven
Celestijnenlaan 200B
B - 3001 Leuven, BELGIUM

Abstract

Time dependent system size probabilities of a birth and death process related to the Rogers-Ramanujan continued fraction are obtained. The range for the parameter in this continued fraction is obtained to ensure the positivity of the recursively defined birth and death rates. The general behavior of the birth and death rates is described and the asymptotic behavior of the transition probabilities and the (quasi) stationary distribution is determined. For the transient case the birth and death process can be seen as a model in queuing theory where the length of the queue encourages customers to join the queue (the more the merrier).

1 Introduction

In this paper we study a birth and death process whose birth and death rates are suggested by the celebrated Rogers-Ramanujan continued fraction [5]. Our objective is to obtain transient (and positive recurrent) solutions of the system size probabilities for this birth and death process. The Rogers-Ramanujan continued fraction is of the form

$$g(q) \;=\; \cfrac{1}{1-\cfrac{q}{1-\cfrac{q^2}{1-\cfrac{q^3}{1-\ldots}}}} \tag{1.1}$$

and has been investigated widely in the literature [3]. It has also the following combinatorial interpretation [15]. An (n, k) fountain is an arrangement of n coins in rows such that there are exactly k coins in the bottom row and such that each coin in a higher row touches exactly two coins in the next lower row. Let $f(n, k)$ denote the number of (n, k) fountains and let $f(n) = \sum_{k=1}^{\infty} f(n, k)$. Then $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, $f(6) = 9$, $f(7) = 15$, and so on. The sequence $f(n)$ has a generating function

$$1 + \sum_{n=1}^{\infty} f(n)x^n \;=\; \cfrac{1}{1-\cfrac{x}{1-\cfrac{x^2}{1-\ldots}}}.$$

Birth and death processes play a fundamental role in the theory and applications that embrace diverse fields, such as queuing and inventory models, quantum optics, chemical kinetics, and population dynamics.

In the study of birth and death processes, the emphasis has been on obtaining steady state solutions, their approximations or bounds. Steady state solutions exist only under convergence conditions, unlike the transient ones. There has been a resurgence of interest in the transient analysis of birth and death processes, throwing fresh light on the structure of the process. The study of the time dependent behavior of birth and death processes involves many intricate and interesting orthogonal polynomials, such as Laguerre, Meixner, Charlier and Chebyshev polynomials [6]. In fact, the three-term recurrence relation lies at the heart of continued fractions, orthogonal polynomials and birth and death processes [4].

When the birth and death rates are constant, the transient solutions involve an infinite series of Bessel functions and their integrals [16, 17]. As the complexity of the system increases, for example, when the rates are state dependent/non-linear, it is impossible to find closed form transient solutions. Due to the difficulties involved in analytical methods, it is pertinent to develop numerical techniques to solve the resulting birth and death equations and gain an insight into the behavior of the various system characteristics such as system size probabilities, expected system size, etc. The Karlin-McGregor representation of the transition probabilities [11], which uses a system of orthogonal polynomials satisfying a three term recurrence relation involving the birth and death rates, is also very useful to understand the asymptotic behavior of the birth and death process.

Approximations employing continued fractions occupy a conspicuous place in mathematical literature due to the interesting convergence properties. Such approximations often provide a good representation for transcendental functions. Their convergence region is usually larger than the convergence region of the classical representation by power series. A systematic study of the theory of continued fractions with emphasis on computations can be found in [10]. Their applications to the study of birth and death processes was initiated by [14]. This method has been applied in queuing problems with state dependent rates [18]. Essential in the continued fraction methodology is that the differential-difference equations satisfied by the system size probabilities are transformed into a set of algebraic linear equations using Laplace transforms. This leads to a continued fraction which can be expressed as a rational function. The solution is obtained by finding the inversion through the properties of tridiagonal matrices. The nature of the transient solutions is illustrated through tables and graphs. We shall show how this continued fraction methodology is related to the Gauss quadrature using the orthogonal polynomials in the Karlin-McGregor representation.

2 Positivity of the rates

We first prove a theorem to determine the range of q in (1.1) so that the associated birth and death rates remain positive.

Theorem 1 *Let λ_n $(n = 0, 1, 2, \ldots)$ be a sequence defined by*

$$\lambda_0 = 1, \quad \lambda_{n-1}(1 - \lambda_n) = q^n, \qquad n \geq 1, \tag{2.1}$$

where $q > 0$. Then there exists a $q_1 \in (0, 1)$ such that for $0 < q < q_1$ we have $\lambda_n > 0$ for all positive integers n.

Proof: We will write $\lambda_n = R_{n+1}/R_n$, where $R_0 = 1$ and $R_1 = 1$. The recurrence then becomes

$$\frac{R_n}{R_{n-1}} \frac{R_n - R_{n+1}}{R_n} = q^n,$$

which can be written as

$$R_{n+1} = R_n - q^n R_{n-1}, \qquad n > 0.$$

We will show that $R_n > 0$ for $0 < q < q_1$, where $q_1 \in (0, 1)$ can be determined by studying the zeros of an entire function related to the Rogers-Ramanujan identities.

Consider the polynomials $P_n(x)$ given by the recurrence relation

$$P_{n+1}(x) = xP_n(x) - q^n P_{n-1}(x), \qquad n \geq 0, \tag{2.2}$$

with initial conditions $P_0 = 1$ and $P_{-1} = 0$. Observe that $R_n = P_n(1)$. The polynomials $P_n(x)$ are orthogonal polynomials with respect to a distribution function ψ on the real line which, for $0 < q < 1$, has a support that consists of the discrete set $\{\pm x_k,\ k = 1, 2, 3, \ldots\}$, where $x_1 > x_2 > x_3 > \cdots$ and $\lim_{n\to\infty} x_n = 0$. They are a special case of polynomials $U_n(x; a, b)$ studied by Al-Salam and Ismail [1], namely $P_n(x) = U_n(x; 0, q)$, but also of polynomials $u_n(x; q; a, b)$ studied by Van Assche [22], namely $P_n(x) = u_n(x; q^{1/2}; 0, q)$. See also Koelink and Van Assche [13, p. 499–501] where the polynomials $P_n(x)$ are considered explicitly. The support of the distribution ψ is $\{\pm x_k,\ k = 1, 2, 3, \ldots\}$, where $\pm x_k = \pm 1/z_k$, and z_k are the zeros of the entire function

$$F(z; q) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k} q^{k^2}}{(q; q)_k} \tag{2.3}$$

(see, e.g., [1, §4] and [13, Thm. 5.5 and Prop. 5.8]). Observe that $F(i; q)$ and $F(iq^{1/2}; q)$ can be evaluated using the Rogers-Ramanujan identities [3]. This entire function only has real zeros, which are symmetric around the origin and simple (see, e.g., [1, Lemma 4.3] and [13, Thm. 5.3]). Observe that the smallest interval containing the support of the orthogonality measure for the polynomials $P_n(x)$ is the interval $[-x_1, x_1] = [-1/z_1, 1/z_1]$, where z_1 is the first positive zero of F. It is well known that all the zeros of orthogonal polynomials are in the smallest open interval containing the support of the orthogonality measure, hence all the zeros of $P_n(x)$ are inside $(-x_1, x_1)$. Therefore if $x \geq x_1$ we will have $P_n(x) > 0$. In particular, if $1 \geq x_1$ then $R_n > 0$ for all n. It is not so difficult, e.g., by using the Hellmann-Feynman theorem [9], to show that $x_1(q)$ is an increasing function of the parameter q and that $x_1(0) = 0$ and $x_1(1) = 2$. In fact, the largest zero $x_{n,n}(q)$ of $P_n(x)$ increases with q because it is the largest eigenvalue of the tridiagonal matrix

$$\begin{pmatrix} 0 & q^{1/2} & 0 & 0 & & \cdots & 0 \\ q^{1/2} & 0 & q & 0 & & \cdots & 0 \\ 0 & q & 0 & q^{3/2} & & \cdots & 0 \\ 0 & 0 & q^{3/2} & \ddots & \ddots & & 0 \\ \vdots & \vdots & & \ddots & & & q^{(n-1)/2} \\ 0 & 0 & 0 & 0 & & q^{(n-1)/2} & 0 \end{pmatrix}$$

to which we can apply the Hellmann-Feynman theorem, and $x_1(q) = \lim_{n\to\infty} x_{n,n}(q)$. Hence there exists a q_1 such that $x_1(q_1) = 1$ and for $0 \leq q \leq q_1$ one then has $x_1(q) \leq 1$, which shows that for $0 \leq q \leq q_1$ we have $R_n > 0$ for all n, and thus also $\lambda_n > 0$ for all n. The number q_1 is therefore the solution of the equation

$$z_1(q_1) = 1, \tag{2.4}$$

where $z_1(q)$ is the first positive zero of the entire function $F(z; q)$.

Next we will show that for $q > q_1$ the quantity λ_n will be negative for certain n. In this case we have $1 < x_1$. In an interval between two consecutive points of the support of the orthogonality measure, the polynomial $P_n(x)$ can have at most 1 zero. Furthermore, every point in the support of the orthogonality measure is a limit point of zeros of the orthogonal polynomials. This means that the kth largest zero $x_{k,n}$ of $P_n(x)$ has the property that $\lim_{n\to\infty} x_{k,n} = x_k$. Furthermore, the zeros of P_n and P_{n-1} interlace. If 1 is a zero of $P_n(x)$ then obviously $R_n = 0$ and $\lambda_{n-1} = 0$, so we can assume that 1 is not a zero of any $P_n(x)$. Then the number $N(n)$ of zeros of $P_n(x)$ to the right of 1 is an increasing function of n and this number can only increase by one when n is increased by one. Furthermore $N(\infty) := \lim_{n\to\infty} N(n)$ is the number of points x_k to the right of 1. Let n_1 be the smallest n such that $N(n+1) = N(\infty)$, then $N(n_1) = N(\infty) - 1$ and $N(n_1 + 1) = N(\infty)$. But then $\operatorname{sign} P_{n_1}(1) = -\operatorname{sign} P_{n_1+1}(1)$ so that $\lambda_{n_1} < 0$. ■

The number q_1, which is given by equation (2.4), is the largest number that makes $\{q^n, n = 1, 2, 3, \ldots\}$ a chain sequence, and the sequence $\{\lambda_n,\ n = 0, 1, 2, \ldots\}$ is the minimal parameter sequence (see [6]). A well known sufficient condition that this geometric sequence is a chain sequence is that $q \le 1/2$ [6, p. 97], so that $q_1 \ge 1/2$. On the other hand, since $\lambda_0 = 1$, we can easily find $\lambda_1 = 1 - q$ and $\lambda_2 = (1 - q - q^2)/(1 - q)$ and this is positive for $q \in (0, 1)$ when $q < (-1 + \sqrt{5})/2 = 0.61803....$ This gives the bounds $0.5 \le q_1 < 0.61803....$ Solving the equation (2.4) numerically gives up to 50 decimals

$$q_1 = 0.57614876914275660229786857371993878235472466311897...$$

Another interpretation of q_1 is that it is the first positive zero of the function $F(1; q)$, as a function of q. Observe that this number also appeared in [15] in the description of the asymptotic behavior of the number $f(n)$ of (n, k) fountains of n coins.

We can give an explicit formula for the λ_n satisfying the recursion (2.1) because we have an explicit representation of the polynomials $P_n(x)$ satisfying (2.2). Indeed we have

$$\lambda_n = R_{n+1}/R_n, \qquad R_{n+1} = \sum_{j\ge 0} \begin{bmatrix} n+1-j \\ j \end{bmatrix} (-1)^j q^{j^2},$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_j (q;q)_{n-j}}, & \text{if } 0 \le j \le n, \\ 0 & \text{if } j < 0 \text{ or } j > n, \end{cases}$$

is the usual q-binomial coefficient. To show this, we observe that

$$\begin{aligned} R_{n+1} - R_n &= \sum_{j\ge 0} \left(\begin{bmatrix} n+1-j \\ j \end{bmatrix} - \begin{bmatrix} n-j \\ j \end{bmatrix} \right) (-1)^j q^{j^2} \\ &= \sum_{j\ge 0} \begin{bmatrix} n-j \\ j-1 \end{bmatrix} q^{n+1-2j} (-1)^j q^{j^2} \\ &= -q^n \sum_{k\ge 0} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (-1)^k q^{k^2} \qquad (k = j-1) \\ &= -q^n R_{n-1}, \end{aligned}$$

so that R_n satisfies the recurrence relation (2.2) with $x = 1$. It is easy to check that $R_0 = 1$ and $R_1 = 1$, so that $\lambda_n = R_{n+1}/R_n$.

3 Birth and death processes

A birth and death process is a Markov process in which a population, initially of size m, changes to size n after time t by births and deaths. We assume that in an interval $(t, t + \delta t)$ each individual in the population has a probability $\lambda_n \delta t + o(\delta t)$ of giving birth to a new individual and a probability $\mu_n \delta t + o(\delta t)$ of dying. The parameters λ_n, μ_n are respectively called the birth rate and death rate when the population has size n, and we denote by $p_{m,n}(t)$ the probability that the population has size n at time t when initially at time $t = 0$ the size of the population was m. By considering $p_{m,n}(t + \delta t)$ in terms of $p_{m,n-1}(t)$, $p_{m,n}(t)$ and $p_{m,n+1}(t)$, the following set of differential-difference equations, called 'Chapman-Kolmogorov equations' may be obtained:

$$\begin{aligned} p'_{m,0}(t) &= -\lambda_0 p_{m,0}(t) + \mu_1 p_{m,1}(t) \\ p'_{m,n}(t) &= \lambda_{n-1} p_{m,n-1}(t) - (\lambda_n + \mu_n) p_{m,n}(t) + \mu_{n+1} p_{m,n+1}(t) \qquad (3.1) \end{aligned}$$

for $n = 1, 2, \ldots$ and we are interested in finding the solution for which $0 \le p_{m,n}(t) \le 1$ and $\sum_{n=0}^{\infty} p_{m,n}(t) = 1$, subject to the initial condition $p_{m,n}(0) = \delta_{n,m}$ (Kronecker delta) for $m \in \{0, 1, \ldots\}$. Also, $\mu_0 = 0$, $\lambda_n > 0$ and $\mu_n > 0$ for $n = 1, 2, \ldots$.

We construct a birth and death process related to the Rogers-Ramanujan continued fraction (1.1). Specifically, we consider a birth and death process with rates satisfying the conditions

$$\begin{aligned} \lambda_n + \mu_n &= 1 \\ \lambda_{n-1}\mu_n &= q^n \qquad n = 1, 2, \ldots, \end{aligned} \tag{3.2}$$

that is

$$\lambda_{n-1}(1 - \lambda_n) = q^n \qquad n = 1, 2, \ldots \tag{3.3}$$

with $\mu_0 = 0$ and $\lambda_0 = 1$. By the theorem in the previous section the λ_n remain positive when $0 < q \le q_1 = 0.57614\ldots$.

In 1957, Reuter [19] and Karlin and McGregor [11] gave a very elegant way to write the transition probabilities $p_{m,n}(t)$ in terms of a sequence of polynomials Q_n, satisfying the recursion relation

$$-xQ_n(x) = \lambda_n Q_{n+1}(x) - (\lambda_n + \mu_n)Q_n(x) + \mu_n Q_{n-1}(x),$$

with initial values $Q_{-1} = 0$ and $Q_0 = 1$, as

$$p_{m,n}(t) = \pi_n \int_0^{\infty} e^{-xt} Q_m(x) Q_n(x)\, d\phi(x), \tag{3.4}$$

where ϕ is a distribution function with respect to which the polynomials Q_n are orthogonal, and the *potential coefficients* π_n are given by

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}.$$

The function ϕ is known as a spectral distribution (or spectral measure). In our case (3.2) holds and after some calculation we find

$$Q_n(x) = \frac{P_n(1-x)}{P_n(1)},$$

where P_n are the orthogonal polynomials satisfying recurrence relation (2.2). Therefore the Q_n are orthogonal on the set $[1-x_1, 1+x_1]$, where $x_1 = 1/z_1$ and z_1 is the first positive zero of the function $F(z;q)$ given in (2.3). The spectral ϕ is related to the orthogonality measure ψ of the polynomials P_n by $\phi(x) = -\psi(1-x)$ and has mass points at $1 \pm x_k$, where $x_k = 1/z_k$ and z_k are the zeros of the entire function $F(z;q)$.

4 Continued fraction methodology

We take the initial population $m = 0$ and write $p_{0,n}(t) = p_n(t)$. Taking the Laplace transform of (3.1) and denoting

$$\begin{aligned} f_n(s) &= (-1)^n \left(\prod_{i=1}^{n} \mu_i \right) \int_0^{\infty} e^{-st} p_n(t)\, dt, \qquad n = 1, 2, \ldots \\ f_0(s) &= \int_0^{\infty} e^{-st} p_0(t)\, dt, \end{aligned}$$

we get, after some simplification,

$$\frac{f_r(s)}{f_{r-1}(s)} = \frac{-\lambda_{r-1}\mu_r}{s+\lambda_r+\mu_r+\frac{f_{r+1}(s)}{f_r(s)}}, \qquad r=1,2,\ldots$$

This results in

$$f_0(s) = \cfrac{1}{s+\lambda_0-\cfrac{\lambda_0\mu_1}{s+\lambda_1+\mu_1-\cfrac{\lambda_1\mu_2}{s+\lambda_2+\mu_2-\ldots}}}.$$

From (3.2) we have

$$f_0(s) = \cfrac{1}{s+1-\cfrac{q}{s+1-\cfrac{q^2}{s+1-\cfrac{q^3}{s+1-\ldots}}}} \tag{4.1}$$

which is the Rogers-Ramanujan continued fraction (1.1) when $s=0$, hence justifying the fact that the birth and death process under consideration is related to the Rogers-Ramanujan continued fraction (1.1). We consider the Nth convergent of (4.1), and denote it by

$$\frac{A_N(s)}{B_N(s)}, \tag{4.2}$$

where A_N and B_N are defined recursively by

$$\begin{aligned}
A_1(s) &= 1 \\
A_2(s) &= s+\lambda_1+\mu_1 = s+1 \\
A_n(s) &= (s+\lambda_{n-1}+\mu_{n-1})A_{n-1}(s) - \lambda_{n-2}\mu_{n-1}A_{n-2}(s) \\
&= (s+1)A_{n-1}(s) - q^{n-1}A_{n-2}(s) \qquad n=3,4,\ldots
\end{aligned}$$

and

$$\begin{aligned}
B_0(s) &= 1 \\
B_1(s) &= s+\lambda_0 = s+1 \\
B_n(s) &= (s+\lambda_{n-1}+\mu_{n-1})B_{n-1}(s) - \lambda_{n-2}\mu_{n-1}B_{n-2}(s) \\
&= (s+1)B_{n-1}(s) - q^{n-1}B_{n-2}(s), \qquad n=2,3,\ldots
\end{aligned} \tag{4.3}$$

The polynomials $B_n(s)$ in (4.3) can also be written as a tridiagonal determinant as follows:

$$B_n(s) = \begin{vmatrix}
s+\lambda_0 & 1 & 0 & . & . & . \\
\lambda_0\mu_1 & s+\lambda_1+\mu_1 & 1 & . & . & . \\
0 & \lambda_1\mu_2 & s+\lambda_2+\mu_2 & . & . & . \\
. & . & . & . & . & 1 \\
. & . & . & . & \lambda_{n-2}\mu_{n-1} & s+\lambda_{n-1}+\mu_{n-1}
\end{vmatrix}$$

of order $n \times n$. Observe that the polynomials B_n in (4.3) are related to the polynomials P_n satisfying (2.2) in the proof of the Theorem, by $B_n(s) = P_n(s+1)$. Similarly, the numerator polynomials are also related to the polynomials P_n in (2.2) by $A_n(s) = q^{(n-1)/2} P_{n-1}(q^{-1/2}(s+1))$. The polynomials A_n are also known as associated polynomials.

To analyze (4.2), the roots of B_n are to be computed. We observe that $B_n(s)$ is zero when $-s$ is an eigenvalue of the tridiagonal matrix

$$F_n = \begin{bmatrix} \lambda_0 & 1 & 0 & . & . & . \\ \lambda_0\mu_1 & \lambda_1+\mu_1 & 1 & . & . & . \\ 0 & \lambda_1\mu_2 & \lambda_2+\mu_2 & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & \lambda_{n-2}\mu_{n-1} & \lambda_{n-1}+\mu_{n-1} \end{bmatrix}$$

of order $n \times n$. This matrix can be transformed into a real symmetric positive definite tridiagonal matrix with non-zero subdiagonal elements, and therefore the eigenvalues are real and distinct [23]. Thus the continued fraction given in (4.1) converges in the s-plane cut from 0 to ∞ along the negative real axis. Suppose $-s_1^N, -s_2^N, \ldots, -s_N^N$ are the roots of $B_N(s)$. Then (4.2) can be expressed by partial fractions as

$$f_0(s) \cong \sum_{j=1}^{N} \frac{A_N(-s_j^N)}{(s+s_j^N)\prod_{i=1,i\neq j}^{N}(s_i^N - s_j^N)}.$$

Inverting the Laplace transform gives

$$p_0(t) \cong \sum_{j=1}^{N} \frac{A_N(-s_j^N)}{\prod_{i=1,i\neq j}^{N}(s_i^N - s_j^N)} e^{-s_j^N t}.$$

Murphy and O'Donohoe [14] have discussed the method of finding $p_{m,r}(t)$ with the initial number m in the system. Specifically,

$$p_{m,r}(t) \cong \sum_{j=1}^{k} H_j^r e^{-s_j^k t}, \quad r = 0, 1, \ldots \tag{4.4}$$

where

$$H_j^r = \frac{B_r(-s_j^k) B_m(-s_j^k) A_k(-s_j^k)}{\prod_{i=0}^{m-1}\lambda_i \prod_{i=1}^{r}\mu_i \prod_{i=1,i\neq j}^{k}(s_i^k - s_j^k)}$$

and

$$k = \begin{cases} m+N, & \text{for } r \le m \\ r+N, & \text{for } r \ge m \;. \end{cases}$$

Computing eigenvalues involves errors to some degree. Analysis of errors and numerical examples can be obtained from [14].

Let us make a connection between this methodology and the Karlin-McGregor representation. If we use the representation

$$p_{0,n}(t) := p_n(t) = \pi_n \int_0^\infty e^{-xt} Q_n(x)\, d\phi(x),$$

Figure 1: Time-dependent system size probabilities with initial condition $m = 2$, parameter $q = 0.3$, and continued fraction approximation at level $N = 50$

then its Laplace transform becomes

$$f_n(s) = (-1)^n \left(\prod_{i=0}^{n-1} \lambda_i \right) \int_0^\infty \frac{Q_n(x)}{x+s} \, d\phi(x), \tag{4.5}$$

and

$$f_0(s) = \int_0^\infty \frac{1}{x+s} \, d\phi(x). \tag{4.6}$$

The latter is known as the Stieltjes transform of the spectral distribution ϕ. It is well known that a Stieltjes transform can be expanded into a continued fraction, which for f_0 gives (4.1). The Nth convergent of a continued fraction is given as the ratio $-p_{N-1}^{(1)}(-s)/p_N(-s)$, where p_n are the monic orthogonal polynomials for the spectral distribution ϕ, and $p_{n-1}^{(1)}$ are the monic associated polynomials. As we have seen before, the monic orthogonal polynomials are $p_n(x) = (-1)^n P_n(1-x) = (-1)^n B_n(-x)$ and the monic associated polynomials are $p_{n-1}^{(1)}(x) = (-1)^{n-1} q^{(n-1)/2} P_{n-1}(q^{-1/2}(1-s)) = (-1)^{n-1} A_n(-x)$. Expanding $p_{N-1}^{(1)}(x)/p_N(x)$ into partial fractions gives

$$\frac{p_{N-1}^{(1)}(x)}{p_N(x)} = \sum_{j=1}^{N} \frac{\lambda_{N,j}}{x - x_{j,N}},$$

where $x_{j,N}$ are the zeros of p_N and $\lambda_{N,j}$ are known as the Christoffel numbers. This sum is in fact the Gauss quadrature formula with the nodes $x_{k,N}$ for the integral $-f_0(-x)$ in

(4.6), so taking the inverse Laplace transform gives the approximation

$$\sum_{j=1}^{N} \lambda_{N,j} e^{-x_{k,N}t},$$

which is the Gauss quadrature formula applied to $p_0(t)$. Reasoning in the same way, we can therefore apply the Gauss quadrature formula to the integral (3.4) defining $p_{m,n}(t)$, giving

$$\pi_n \sum_{j=1}^{N} \lambda_{N,j} Q_m(x_{j,N}) Q_n(x_{j,N}) e^{-x_{j,N}t}$$

as an approximation, where N should be chosen larger than n and m. This corresponds to the approximation given in (4.4). The error analysis then corresponds to the error analysis of Gauss quadrature.

In Figure 1, time-dependent system size probabilities are plotted with the assumption that there are two units in the system initially ($m = 2$) and we took $q = 0.3$. The truncation for the continued fraction approximation is $N = 50$. In Figure 2, time-dependent probability curves corresponding to the state 0 are plotted for different values of q with continued fraction approximations at level $N = 100$ and initial condition $m = 2$.

Figure 2: Time-dependent probability curves corresponding to the state 0 for different values of q, initial condition $m = 2$, and continued fraction approximation at level $N = 100$

5 Potential coefficients and birth-death polynomials

We start with calculating the coefficients π_n of this birth-death process. We find easily

$$\begin{aligned}
\pi_n &= \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n} \\
&= \frac{\lambda_0^2\lambda_1^2\ldots\lambda_{n-1}^2}{q^1q^2\ldots q^n} \\
&= \frac{R_n^2}{q^{n(n+1)/2}} \\
&= \frac{P_n(1)^2}{q^{n(n+1)/2}}.
\end{aligned} \tag{5.1}$$

The birth-death polynomials are determined uniquely by the recurrence relation

$$-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x), \quad n \geq 0$$

together with $Q_{-1} = 0$ and $Q_0 = 1$. Recall that

$$Q_n(x) = \frac{P_n(1-x)}{P_n(1)}$$

and that they are orthogonal on $[1-x_1, 1+x_1] = [1-1/z_1, 1+1/z_1]$, where z_1 is the first positive zero of $F(z;q)$ given in (2.3). Using Anderson's notation [2] we consider

$$A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n}, \quad B = \sum_{n=0}^{\infty} \pi_n, \quad C = \sum_{n=0}^{\infty} \frac{1}{\lambda_n\pi_n}\sum_{i=1}^{n} \pi_i.$$

The process will be non-explosive if and only if $C = \infty$, in which case there is a unique birth-and-death process with the given transition rates. This process is recurrent if and only if $A = \infty$, and then positive recurrent if and only if $B < \infty$. In our case the λ_n are bounded from above, and since the series C represents the expected passage time of the process from 0 to ∞, it therefore follows that $C = \infty$. We conclude that the birth-death process is non-explosive.

5.1 Transient case: $0 < q < q_1$

The following theorem explains why one could call this process *the more-the-merrier process.*

Theorem 2 *If* $0 < q < q_1$ *then the above defined birth-death process is transient and*

$$\lim_{n\to\infty} \lambda_n = 1, \qquad \lim_{n\to\infty} \mu_n \to 0.$$

Proof: According to Proposition 5.8 in [13]

$$\lim_{n\to\infty} x^n P_n(1/x) = F(x;q),$$

and 1 is not a zero of $F(z;q)$, since $q < q_1$. Therefore

$$\lim_{n\to\infty} P_n(1) = F(1;q) \neq 0,$$

from which it follows that $\lambda_n = P_{n+1}(1)/P_n(1)$ converges to 1. Since $\lambda_n + \mu_n = 1$, it then also follows that μ_n converges to 0.

Next, since $\pi_n = P_n^2(1)/q^{n(n+1)/2}$, it follows that $\pi_n \to \infty$ as $n \to \infty$, hence $B = \infty$ and positive recurrence is impossible. Finally, since

$$0 < c = \inf\{P_n(1), n \geq 0\} < \infty,$$

we have

$$A = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{P_{n+1}(1)P_n(1)} \leq (1/c^2) \sum_{n=0}^{\infty} q^{n(n+1)/2} < \infty,$$

so null-recurrence is also impossible. ■

Remark: Using (2.1) we can also easily see that λ_n is going geometrically fast to 1, i.e.,

$$\lambda_n - 1 = \frac{q^n}{\lambda_{n-1}} \sim q^n.$$

Consider a queue with n people. If the queue length increases, then in certain occassions people may be encouraged to join the queue thinking that a large queue indicates there may actually be something interesting, whereas people in the queue will have the tendency to stay in the queue. This would require λ_n to be increasing and μ_n to be decreasing, which is precisely the case with the process under consideration. The more people in the queue, the merrier.

For an irreducible transient process one can study the quasi-stationary behavior by the probabilities

$$r_{i,j}(t) = \Pr(X(t) = j | X(0) = i, t < S),$$

where S is the last exit-time from 0 before the process drifts to infinity, and

$$q_{i,j}(t,s) = \Pr(X(t) = j | X(0) = i, t + s < S).$$

In [12] and [20] it was found that

$$r_j = \lim_{t\to\infty} r_{i,j}(t) = \frac{\pi_j Q_j(\gamma)}{\sum_{n=0}^{\infty} \pi_n Q_n(\gamma)},$$

where γ is the infimum of the support of the spectral measure ϕ, and

$$q_j = \lim_{t\to\infty} \lim_{s\to\infty} q_{i,j}(t,s) = \frac{\pi_j Q_j^2(\gamma)}{\sum_{n=0}^{\infty} \pi_n Q_n^2(\gamma)}.$$

For the process under consideration this means

$$r_j = \frac{P_j(1)P_j(x_1)q^{-j(j+1)/4}}{\sum_{n=0}^{\infty} P_n(1)P_n(x_1)q^{-n(n+1)/4}}, \tag{5.2}$$

and

$$q_j = \frac{P_j^2(x_1)q^{-j(j+1)/2}}{\sum_{n=0}^{\infty} P_n^2(x_1)q^{-n(n+1)/2}}. \tag{5.3}$$

5.2 Positive recurrent case: $q = q_1$

Theorem 3 *For $q = q_1$ the birth-death process is positive recurrent and*

$$\lim_{n\to\infty} \lambda_n \to 0, \qquad \lim_{n\to\infty} \mu_n \to 1.$$

Proof: We have

$$P_{n+1}(1) - P_n(1) = -q^n P_{n-1}(1), \quad n \geq 0.$$

Summing from 0 to m gives

$$P_{m+1}(1) - 1 = -\sum_{n=0}^{m} q^n P_{n-1}(1). \tag{5.4}$$

Recall that $F(q_1) = 0$, hence letting $m \to \infty$ we get

$$1 = \sum_{n=0}^{\infty} q^n P_{n-1}(1).$$

Substituting this in (5.4) we get

$$P_{m+1}(1) = \sum_{n=m+1}^{\infty} q^n P_{n-1}(1),$$

or equivalently

$$\frac{P_{m+1}(1)}{P_m(1)} - q^m = \sum_{n=m+2}^{\infty} q^n \frac{P_{n-1}(1)}{P_m(1)} > 0. \tag{5.5}$$

Observe that for $m \leq n-2$

$$\frac{P_{n-1}(1)}{P_m(1)} = \frac{P_{n-1}(1)}{P_{n-2}(1)} \frac{P_{n-2}(1)}{P_{n-3}(1)} \cdots \frac{P_{m+1}(1)}{P_m(1)} = \lambda_{n-2}\lambda_{n-3}\ldots\lambda_m \leq 1,$$

hence

$$0 < \lambda_m - q^m \leq \sum_{n=m+2}^{\infty} q^n = \frac{q^{m+2}}{1-q}.$$

Letting $m \to \infty$ gives $\lambda_m \to 0$, and since $\lambda_m + \mu_m = 1$ this also gives $\mu_m \to 1$.

Consider the polynomials $p_n(x) = P_n(x)/q^{n(n+1)/4}$, then these polynomials satisfy the recurrence

$$xp_n(x) = q^{(n+1)/2} p_{n+1}(x) + q^{n/2} p_{n-1}(x),$$

with initial conditions $p_{-1} = 0$ and $p_0 = 1$. This means that the p_n, $(n = 0, 1, 2\ldots)$ are the *orthonormal* polynomials for the distribution ψ. Furthermore, for $q = q_1$ the point 1 is a mass point of the spectral distribution ψ, since $F(1; q_1) = 0$. But for each mass point x_k (and $-x_k$) of the spectral distribution ψ one has

$$\sum_{n=0}^{\infty} p_n^2(x_k) = \frac{1}{\psi(x_k+) - \psi(x_k-)} < \infty,$$

hence in particular for $x_1 = 1$ we have, using (5.1),

$$B = \sum_{n=0}^{\infty} \pi_n = \sum_{n=0}^{\infty} p_n^2(1) < \infty,$$

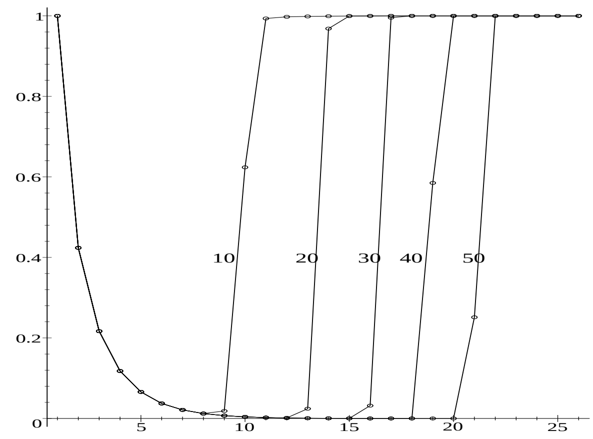

Figure 3: The birth rates λ_n when q approximates q_1 up to 10, 20, 30, 40, and 50 decimals

which shows that the process is positive recurrent. The positive recurrence thus follows from the fact that the spectral distribution $\phi(x) = -\psi(1-x)$ has a mass point at $0 = 1-x_1$. Observe that $\lambda_n \le 1$ implies

$$A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \ge \sum_{n=0}^{\infty} \frac{1}{\pi_n} = \infty$$

since we already have $\sum_{n=0}^{\infty} \pi_n < \infty$. ■

Observe that only at $q = q_1$ we have positive recurrence and $\lambda_n \to 0$. Whenever $q < q_1$ we have a transient process and $\lambda_n \to 1$. The closer one gets to q_1 the longer it takes for the λ_n to jump to 1. Hence taking q closer to q_1 will increase the *critical population* in the queue at which point the queue length starts to encourage people to join the queue. See Figure 3 for the birth rates corresponding to q values that approximate q_1 with 10 to 50 decimals.

Setting $\xi_k = 1 - x_k$ and $\nu_k = 1 + x_k$ and with $\xi_1 = 0$, we thus have for the transition probabilities

$$\begin{aligned} p_{i,j}(t) &= \pi_j \left(\sum_{k=1}^{\infty} e^{-t\xi_k} Q_i(\xi_k) Q_j(\xi_k) \phi(\{\xi_k\}) + \sum_{k=1}^{\infty} e^{-t\nu_k} Q_i(\nu_k) Q_j(\nu_k) \phi(\{\nu_k\}) \right) \\ &= \pi_j \phi(\{0\}) + O(e^{-t\xi_2}) \end{aligned}$$

which again shows the positive recurrence, but which also shows that the process is exponentially ergodic. The stationary distribution is given by

$$p_j = \lim_{t\to\infty} p_{i,j}(t) = \frac{\pi_j}{\sum_{n=0}^{\infty} \pi_n} = \frac{P_j^2(1)/q^{j(j+1)/2}}{\sum_{n=0}^{\infty} P_n^2(1)/q^{n(n+1)/2}}.$$

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