

Universal scaling of spectral fluctuation transitions for interacting chaotic systems

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The spectral properties of interacting strongly chaotic systems are investigated for growing interaction strength. A very sensitive transition from Poisson statistics to that of random matrix theory is found. We introduce a new random matrix ensemble modeling this dynamical symmetry breaking transition which turns out to be universal and depends on a single scaling parameter only. Coupled kicked rotors, a dynamical systems paradigm for such transitions, are compared with this ensemble and excellent agreement is found for the nearest-neighbor-spacing distribution. It turns out that this transition is described quite accurately using perturbation theory.

Quantization of fully chaotic systems is known to lead to the spectral fluctuations of random matrix theory (RMT) [1] and to exhibit energy level repulsion [2]. More generally, even non-integrable models without apparent classical limits, such as spin systems, can also show such features [3, 4]. In contrast, integrable systems generally follow Poisson statistics, which are devoid of level repulsion [5]. It is also well understood that combining spectra of different irreducible representations tends toward Poisson statistics in the limit of superposing many sequences [6, 7]. An instance where this occurs is for the spectra of two separable, but individually chaotic systems. Whereas each subsystem possesses RMT fluctuations, the full spectrum tends to Poisson fluctuations in the large dimensionality limit [8].

This begs the question of what happens to spectral fluctuations if such separable, but quantum chaotic subsystems, interact. There are many motivations for studying such systems. For example, they may be of direct physical interest, such as conduction electrons in chaotic quantum dots interacting through a screened Coulomb potential [9]. Another motivation derives from quantum information theory where the development of entanglement is of particular importance [10]. Two quantum spin chains with RMT spectral fluctuations coupled in a ladder configuration is a many-body situation where such transitions are possible as well.

Typically, the interaction between two subsystems leads to entanglement and paves the way for RMT fluctuations of the combined system. This Poisson-to-RMT transition can be viewed as a dynamical symmetry breaking in analogy to fundamental symmetry breaking; the first exact RMT solution to a symmetry breaking problem involved time-reversal invariance [11]. For modeling a particular dynamical system, it is important to connect dynamical system parameters with the abstract transi-

tion parameter built into an RMT transition ensemble, and to confirm that this correctly describes the dynamical system's statistical properties. For example, partial transport barriers often arise classically and an RMT ensemble can correctly model their effects, but only if they have been accounted for their respective fluxes, relative phase space volumes, and tendency to couple locally [12, 13].

There are a couple of other possible cases of a Poisson-to-RMT transition, a metal-insulator transition where states transition from localized to extended [14], and the perturbation of an integrable dynamical system [12, 15] that renders it chaotic for sufficient perturbation strength. In neither of these possibilities is there a simple globally coupled Poisson-to-RMT ensemble. Various complications, such as the metal-insulator transition, the KAM theorem regarding the survivability of tori, and partial transport barriers, all prevent this.

In this paper we address how strong the interactions must be between the subsystems in order to recover the spectral fluctuations of fully chaotic systems, and how the Poisson-to-RMT transition occurs as the interaction magnitude varies from non-interacting to strongly interacting particles cases. We show that the interplay between the coupling and an effective Planck constant gives rise to a dimensionless scaling transition parameter. This is obtained in much the same way as found in the context of a global symmetry breaking in RMT [12, 16–20]. There, arbitrarily small couplings, but uniform everywhere, lead discontinuously to level repulsion [21]. If scaled properly, the entire transition is universal and predicted well by a perturbation theory.

Interacting systems.— To study the Poisson-to-RMT transition, consider the unitary Floquet operator, or kicked version, of generic bipartite systems described by

$$\mathcal{U} = (U_1 \otimes U_2) U_{12} \quad (1)$$

where the U_j are the subsystem Floquet operators and U_{12} describes the interaction. An example is the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + [V_1(q_1) + V_2(q_2) + bV_{12}(q_1, q_2)] \delta_t \quad (2)$$

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where $\delta_t = \sum_{n=-\infty}^{\infty} \delta(t-n)$ is a periodic train of kicks, and a unit time has been chosen as the kicking period. The propagator connecting states separated by one kick follows from $U_j = \exp[-ip_j^2/(2\hbar)] \exp(-iV_j/\hbar)$ ($j = 1, 2$) and $U_{12} = \exp(-ibV_{12}/\hbar)$. The classical limit is a 4-dimensional symplectic map $(q_j, p_j) \mapsto (q'_j, p'_j) = (q_j + p'_j, p_j - \partial V_j/\partial q_j - b \partial V_{12}/\partial q_j)$, connecting the state of the system immediately prior to consecutive kicks.

Perhaps the simplest paradigm for our purposes is that of coupled kicked rotors [22]. They have been realized in experiments on cold atoms [23], can be made strongly chaotic, and the interaction strength continuously varied. The most elementary case is for two interacting rotors with the single particle potentials $V_j = K_j \cos(2\pi q_j)/4\pi^2$ and interaction

$$V_{12} = \frac{1}{4\pi^2} \cos[2\pi(q_1 + q_2)]. \quad (3)$$

The unit periodicity in the angle variables q_j is extended here to the momenta p_j so that the phase space is a 4-dimensional torus. If the kicking strengths, $\{K_1, K_2\}$, here $\{9, 10\}$ respectively, are each chosen sufficiently large, the individual maps are strongly chaotic with a Lyapunov exponent of $\approx \ln(K_j/2)$ [24]. The interaction strength is tuned by the parameter b .

The quantum mechanics on a torus phase space of a single rotor gives rise to a finite Hilbert space of dimension N . The effective Planck constant is $\hbar = 1/N$. Thus U_1 and U_2 are N dimensional unitary operators on their respective spaces \mathcal{H}_1^N and \mathcal{H}_2^N whereas the interaction U_{12} is a N^2 dimensional unitary operator on the tensor product space $\mathcal{H}_1^N \otimes \mathcal{H}_2^N$. The quantized 4-dimensional map is given by Eq. (1). Such coupled quantum maps have been studied in different contexts [25, 26] where more details can be found. Here we are interested in the statistics of the eigenphases φ_n defined by $\mathcal{U}|\psi_n\rangle = e^{i\varphi_n}|\psi_n\rangle$. The boundary conditions are chosen to break both parity and time-reversal symmetries. Note that coupled kicked rotors with a different interaction term have also been studied on the cylinder [27].

RMT transition ensemble.— The first task is to construct the RMT transition ensemble associated with Eq. (1),

$$\mathcal{U}_{\text{RMT}}(\epsilon) = \left(U_{\text{CUE}}^{(1)} \otimes U_{\text{CUE}}^{(2)} \right) U_{12}(\epsilon) \quad (4)$$

where the tensor product is taken of two independently chosen circular unitary ensembles (CUE) of dimension N representing the individual strongly chaotic rotors, and $U_{12}(\epsilon)$ a diagonal unitary matrix in the resulting N^2 dimensional space representing the coupling. Its diagonal elements are taken as $\exp(2\pi i \epsilon \xi_{n_1 n_2})$ where $\xi_{n_1 n_2}$ are independent random variables that are uniform on the interval $(-1/2, 1/2]$, ϵ is a real number, and $1 \leq n_1, n_2 \leq N$ label subsystem bases. Any special structure of the rotor's U_{12} is ignored because it turns out to be irrelevant for this study. A limiting case of this ensemble has been studied previously [10], wherein the entangling power of

$\mathcal{U}_{\text{RMT}}(\epsilon = 1)$ was found analytically. However, its spectral features for general ϵ are not yet explored.

For simplicity, the only statistical quantity considered in this paper is the nearest neighbor spacing (NNS) distribution $P(s)$ of the eigenphases φ_n . For $\epsilon = 0$, \mathcal{U}_{RMT} is the direct (tensor) product of two independent CUE matrices. In this case $P(s)$ has been shown analytically as $N \rightarrow \infty$ to approach the Poissonian result of $\exp(-s)$, where s is the spacing in terms of unit mean [8]. The case $\epsilon = 1$ represents strong coupling and CUE behavior.

Transition of the NNS distribution.— For non-interacting rotors, the Poisson result is also expected to hold because the separability leads to spectral sequences that are composed of superpositions of sequences. For interacting rotors, the relation between ϵ of Eq. (4) and the b of the coupled rotors is needed. As explained below,

$$\epsilon = \sqrt{\frac{3}{8\pi^4}} Nb \quad (5)$$

for small values of ϵ and Nb , and $N \gg 1$. In practice this approximation is good for the entire transition even if N is only moderately large.

Figure 1 shows a comparison of the results for the RMT transition ensemble and two coupled rotors using Eq. (5). This demonstrates that the ensemble captures the transition in statistics from Poisson-like towards global CUE perfectly well. CUE statistics already arise for small values of b , which reflects the sensitivity of the transition.

Universal scaling.— The second task is to identify a universal scaling parameter, if it exists, and test whether it captures the proper transition scale. In other instances of symmetry breaking [12, 16, 18, 19], it emerged using

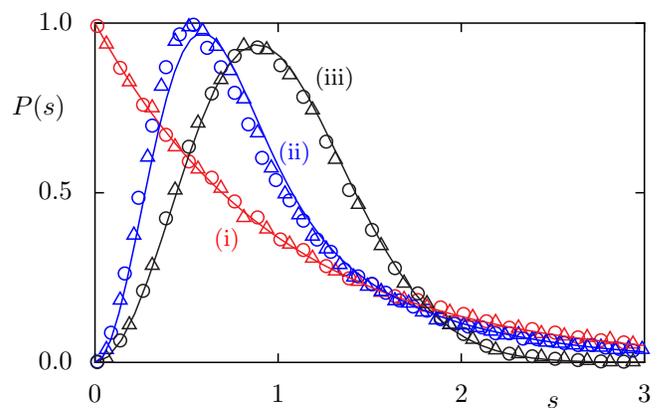


FIG. 1. (Color online) Nearest neighbor level-spacing distribution $P(s)$ for two coupled rotors (circles), and the RMT model (triangles). (i) For $b = 0$, it is Poissonian (red line) as is the $\epsilon = 0$ RMT ensemble. (ii) For $b = 0.0019$, the spacing distribution is intermediate and agrees well with the corresponding RMT ensemble with $\epsilon = 0.012$ and the result of a perturbation theory (blue line) for $\Lambda = 0.1158$. (iii) For $b = 0.008$, the spacing distribution agrees well with both the RMT ensemble ($\epsilon = 0.05$) and the CUE. For all cases the dimensionality of the unitary matrix is $N^2 = 10^4$.

perturbation theory where it was given by $\Lambda = v^2/D^2$, with v^2 the mean square off-diagonal matrix element in the unperturbed system's diagonal representation and D its mean level spacing. If $\Lambda \sim r$, where r is the effective range of the statistics (for the NNS, $r \sim 1$), the transition is nearing completion. It must be emphasized that the interaction is assumed to be generic, has no special symmetries and is entangling. Ideally v^2 is calculated only from those matrix elements that are responsible for the spectral transition, but averaging over a global set of all matrix elements gives the identical result if there is no secular structure, e.g. like bandedness or special correlations [28].

For an N^2 dimensional unitary matrix, necessarily $D = 2\pi/N^2$. That leaves v^2 to calculate. The uncoupled part of the transition ensemble is diagonalized by the direct product of two independent N dimensional unitary matrices, say u and w . An off-diagonal element of the interaction operator in the unperturbed basis is $z_{kl;k'l'} = \sum_{m,n} u_{km} u_{k'm}^* w_{ln} w_{l'n}^* (U_{12})_{mn}$, where $(k,l) \neq (k',l')$, and $(U_{12})_{mn} = \langle mn|U_{12}|mn\rangle$ is a diagonal matrix element due to the interaction. All indices refer to subsystems and therefore lie in $[1, N]$. The off-diagonal elements are restricted to $k \neq k'$ and $l \neq l'$, as the energies of neighboring states after the tensor product must be different in both.

The required average is that of $|z_{kl;k'l'}|^2$, as u and w are independently chosen according to the Haar measure on $U(N)$, the group of unitary matrices in the subsystem spaces. This is done exactly using known results for the average of $u_{km} u_{k'm'} u_{k'm}^* u_{k'm'}^*$, which is $(\delta_{mm'} - 1/N)/(N^2 - 1)$ when $k \neq k'$ (for example see [29]), and leads to (for any N)

$$\Lambda = \frac{N^6}{4\pi^2(N^2 - 1)^2} \times \left(1 + \left| \frac{1}{N^2} \text{Tr} U_{12} \right|^2 - \frac{1}{N^3} \left[\|U^{(1)}\|^2 + \|U^{(2)}\|^2 \right] \right). \quad (6)$$

Here $U_{ii}^{(1)} = \sum_k (U_{12})_{ik}$, $U_{kk}^{(2)} = \sum_i (U_{12})_{ik}$ are partially traced (still diagonal) interaction operators, which are in general not unitary, and $\|X\|^2 = \text{Tr}(XX^\dagger)$ is the Hilbert-Schmidt norm. Performing the exact ensemble average over $U_{12}(\epsilon)$ gives

$$\Lambda = \frac{N^4}{4\pi^2(N+1)^2} \left[1 - \frac{\sin^2(\pi\epsilon)}{\pi^2\epsilon^2} \right] \approx \frac{N^2\epsilon^2}{12}, \quad (7)$$

where for the last approximation $N \gg 1$ and $\epsilon \ll 1$ have been used. Note that related averages involving Haar measures over product spaces have been obtained recently [30].

It is important to recognize that Eq. (6), in the large- N limit, applies to individual chaotic dynamical systems, and hence the coupled rotors. The eigenbasis of uncoupled rotors behaves like that of a CUE member, and in the large- N limit has the same statistical behavior as the

ensemble. Thus, applying Eq. (6) to the coupled rotors gives

$$\Lambda = \frac{N^2}{4\pi^2} (1 - J_0^2(Nb/2\pi)) \approx \frac{N^4 b^2}{32\pi^4}, \quad (8)$$

where $Nb \ll 1$ in the right form. Using the approximations in Eqs. (7,8) generates Eq. (5). It can be checked that this Λ correctly vanishes if V_{12} is a function of either coordinate alone, and thereby not entangling. However, it does not vanish for general separable potentials, and hence the assumption that *the interaction is entangling* is necessary.

There are two significant consequences of Eqs. (7,8). The first is that there is a universal transition governed by the scaling parameter $\Lambda \propto \epsilon^2 N^2 \propto b^2 N^4$. Indeed, this is well verified in Fig. 2. Varying b and N , but holding $b^2 N^4$ fixed generates precisely the same statistical fluctuations. Given that RMT ensemble statistics match those of the coupled rotors, Fig. 1, the transition of Eq. (4) is the same for fixed $\epsilon^2 N^2$; this was also verified, but left out of Fig. 2 to avoid clutter. Secondly, the transition is very sensitive to the interaction strength. The further a system is in the short wavelength limit, i.e. $\hbar \rightarrow 0$, the smaller the interaction strength needed to drive the transition to completion. This is seen in the small values of b, ϵ chosen in Figs. 1,2. In fact, the transition is discontinuous in the interaction strength as $N \rightarrow \infty$ [21].

For Hamiltonians of the Eq. (2)-type and perturbative interactions (second order in Nb), Eq. (6) simplifies to

$$\Lambda = N^4 b^2 [\langle V_{12}^2 \rangle_{12} + \langle V_{12} \rangle_{12}^2 - \langle \langle V_{12} \rangle_1^2 \rangle_2 - \langle \langle V_{12} \rangle_2^2 \rangle_1]. \quad (9)$$

Here $\langle X \rangle_j = \text{Tr}_j X/N$, $\langle X \rangle_{12} = \text{Tr}_{12} X/N^2$. For the coupled rotors, with the interaction as in Eq. (3), the last three terms vanish in the large N limit as the integral over

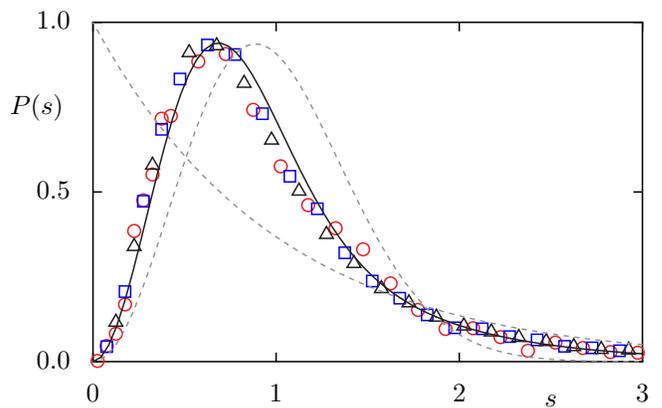


FIG. 2. (Color online) Universal scaling of the level-spacing distribution $P(s)$ for two coupled rotors. Three results (triangles, squares, circles) are shown for $(b, N) = (0.0025, 100)$, $(0.01, 50)$, and $(0.04, 25)$. They all correspond to the same transition parameter $\Lambda \approx 0.2$. The solid curve comes from a perturbation theory described in the text and the dashed curves are the Poisson and CUE distribution, respectively.

q_1 or q_2 is zero. Under such conditions, $\Lambda \approx N^4 b^2 \langle V_{12}^2 \rangle$, where $\langle X \rangle = \int_0^1 \int_0^1 dq_1 dq_2 X(q_1, q_2)$ and X is a classical function of the two coordinates. The interaction is assumed to be diagonal in the position basis for the continuum approximation.

Perturbation theory.— A perturbative theory can be derived based on previously developed techniques [16, 31]. The first step is to consider a very large dimensional case and the perturbation expansion for the eigenvalues to second order. The first order term is diagonal and shifts the energies around randomly, and thus does not alter the Poisson nature of the spectral statistics. Amongst the second order terms, terms connecting lower energies push a level up and those above push down. Overall, a level's motion fluctuates, but remains in the same neighborhood. However, when calculating the NNS spacing, there is a common term, which enters opposite in sign for the two levels, and always pushes them apart. To a rough approximation, the bulk of the perturbation terms keep the mean spacing constant, but the exceptional term introduces level repulsion.

Thus if ΔE_i^0 is the spacing between two neighboring levels at $(i, i+1)$ before perturbation, it becomes $\Delta E_i \approx \Delta E_i^0 + 2|V_{i,i+1}|^2/\Delta E_i^0$, ignoring the effect of other levels. Here $V_{i,i+1}$ is an interaction matrix element, whose real and imaginary parts are complex Gaussian random variables with a scale determined by the variance v^2 introduced above. This is appropriate given the normal fluctuations of the real and imaginary parts of the GUE matrix elements. Scaling the spacings and matrix elements, $r = \Delta E_i/D$, $s_0 = \Delta E_i^0/D$ and $|V_{i,i+1}|^2 = v^2 w$, introduces Λ as above, $r \approx s_0 + 2\Lambda w/s_0$, with w and s_0 distributed exponentially, the latter due to the Poissonian nature of the unperturbed spectrum. Close lying or nearly degenerate levels are frequent for an unperturbed Poissonian spectrum. The resulting divergences are regu-

larized using degenerate perturbation theory, which gives

$$\rho(r) = \int_0^\infty ds_0 dw \exp(-s_0 - w) \delta\left(r - \sqrt{s_0^2 + 4\Lambda w}\right) \quad (10)$$

The result of integration is

$$\rho(r) = \frac{r}{\sqrt{\Lambda}} \left[e^{-r} D_+ \left(\frac{r - 2\Lambda}{2\sqrt{\Lambda}} \right) + e^{-\frac{r^2}{4\Lambda}} D_+ \left(\sqrt{\Lambda} \right) \right] \quad (11)$$

where $D_+(x)$ is the Dawson function $e^{-x^2} \int_0^x dt e^{t^2}$. The effect of all the other levels is accounted for by recompressing the spectrum to unit mean spacing using

$$\langle r \rangle = 1 + \Lambda \left[\frac{2}{\sqrt{\pi}} D_+ \left(\sqrt{\Lambda} \right) - e^{-\Lambda} \text{Ei}(\Lambda) \right] \quad (12)$$

where $\text{Ei}(x)$ is the exponential integral. Thus, the rescaled NNS distribution is $P(s) = \langle r \rangle \rho(s \langle r \rangle)$. Also, the variance of the NNS is

$$\sigma^2 = \frac{2 + 4\Lambda}{\langle r \rangle^2} - 1, \quad (13)$$

and is a useful parameter to monitor the whole transition. Figure 2 compares the NNS obtained using Eqs. (11) and (12) with those of the coupled rotors, and is an excellent approximation. Figure 3 compares the variance across the whole transition. While the perturbation theory result is slightly better than a 2×2 model results [32] for $\Lambda \lesssim 1.0$, the latter by design does better for larger Λ . A more complete theory for the NNS covering the whole transition is left for future work.

Summary and outlook.— In summary, this paper addresses an important, rather general question of what happens when two chaotic systems, or systems with random matrix fluctuations interact. The resulting entanglement leads to a universal transition and rapid recovery of global RMT fluctuations. The transition is very well captured by a natural random matrix ensemble, Eq. (4). A universal scaling parameter is derived in terms of the interaction strength and separately the abstract coupling parameter of the RMT ensemble, thus establishing a quantitative relationship between the two objects. Finally, perturbation theory with a simplified global Poisson plus RMT ensemble reproduces the NNS transition in fluctuations very well. In spite of its application to a particular dynamical system, the coupled kicked rotors, the methods and results are of a general nature, and can be expected to hold wherever there is a global coupling of two subsystems each exhibiting RMT fluctuations.

Various extensions of this work are of significant interest. One rather immediate generalization is to subsystems with different dimensions, say a bipartite system with differing numbers of spins in the subsystems. In many of the relations, the substitution $N^2 \rightarrow N_1 N_2$ suffices to capture the new dependences. A much more involved change is to a greater number of subsystems. If the number of subsystems is L , each with a dimensionality N , a new expression for the transition parameter

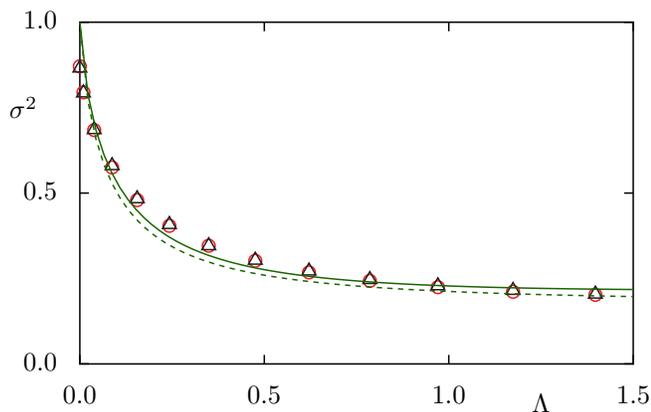


FIG. 3. (Color online) Variance of the nearest neighbor spacing as function of the transition parameter. Shown are results for the coupled rotors (circles) and the RMT model (triangles), in both cases for $N = 50$. The perturbation theory result is shown as solid line, Eq. (13), while the dashed line refers to a 2×2 transition ensemble result [32].

equation in Eq. (6) is possible. There, the new RMT model analogous to Eq. (4) would have L CUE matrices in a tensor product and a global diagonal coupling of the same kind. That leads to a generalization of Eq. (7): $\Lambda = N^{2L}(1 - \text{sinc}^2(\pi\epsilon))/[4\pi^2(N+1)^L] \approx \epsilon^2 N^L/12$. Thus for large N , it is expected that the recovery of RMT fluctuations is much faster with increasing L . However, this

analysis assumes essentially that $U_L(\epsilon)$ is an L -body operator. If it is restricted to a lower rank form, say a 2-body operator, more sophisticated models have to be separately considered. Preliminary results have also been obtained regarding the statistics of eigenfunctions, and entanglement in the eigenstates which develop solely due to the interactions. They are beyond the scope of this paper and left for future publication.

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