

SYZ DUALITY FOR PARABOLIC HIGGS MODULI SPACES

INDRANIL BISWAS AND A. DEY

ABSTRACT. We prove the SYZ (Strominger-Yau-Zaslow) duality for the moduli space of full flag parabolic Higgs bundles over a compact Riemann surface. In [HT2], the SYZ duality was proved for moduli spaces of Higgs vector bundles over a compact Riemann surface.

1. INTRODUCTION

1.1. Mirror symmetry and SYZ duality. Mirror symmetry was discovered in the late 1980's by physicists studying superconformal field theories. Let X be an n dimensional complex Calabi-Yau manifold with a Ricci-flat Kähler form ω and a nowhere vanishing holomorphic n -form Ω on X . A submanifold $Z \subset X$ of real dimension n is called Lagrangian if $\omega|_Z = 0$; further a Lagrangian submanifold is said to be special if $(\text{Im } \Omega)|_Z = 0$. After simplifying a great deal, mirror symmetry is an one-to-one duality (in an appropriate sense) between two class of objects:

- (1) Pairs of the form (Z, L) , where Z is a holomorphic submanifold of X and L is a holomorphic line subbundle on Z (such a pair is called a holomorphic D -brane).
- (2) Special Lagrangian D -branes, which is a pair $(\widehat{Z}, \widehat{L})$ where \widehat{Z} is a special Lagrangian submanifold of a certain Calabi-Yau manifold \widehat{X} (mirror partner), and \widehat{L} is a flat $U(1)$ line bundle on \widehat{Z} .

Since any point $x \in X$ is a submanifold, it should correspond to a pair (\widehat{Z}, Λ) , where \widehat{Z} is a special Lagrangian submanifold of a fixed Calabi-Yau manifold \widehat{X} and Λ is a flat $U(1)$ -line bundle on \widehat{Z} . By a theorem of McLean, the deformation space for a special Lagrangian manifold \widehat{Z} is unobstructed and is parametrized by $H^1(\widehat{Z}, \mathbb{R})$, hence $\dim H^1(\widehat{Z}, \mathbb{R}) = \dim_{\mathbb{C}}(X) = n$. The moduli space of flat $U(1)$ line bundles is given by the torus $H^1(\widehat{Z}, \mathbb{R})/H^1(\widehat{Z}, \mathbb{Z})$. This gives a hint that a moduli of special Lagrangian submanifolds of a fixed Calabi-Yau manifold should have a n -torus fibration over an affine base of real dimension n .

Motivated by this Strominger-Yau-Zaslow made a conjecture [SYZ].

SYZ Conjecture: If X and \widehat{X} are mirror pair of Calabi-Yau n -folds, then there exist fibrations $f : X \rightarrow B$ and $\widehat{f} : \widehat{X} \rightarrow B$ whose fibers are special Lagrangian such that the general fiber is an n -torus. Furthermore, these fibrations are dual, in the sense that canonically $X_b = H^1(\widehat{X}_b, S^1)$ and $\widehat{X}_b = H^1(X_b, S^1)$, whenever the fibers X_b and \widehat{X}_b are non-singular tori.

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1.2. Reformulation of the SYZ conjecture in terms of unitary gerbes. Hitchin introduced the notion of a flat unitary gerbe (known as \mathcal{B} -fields to physicists) and reformulated the SYZ conjecture in terms of this \mathcal{B} -fields [Hi1]. To make sense one needs a further assumption that the special Lagrangian fibers are linearly equivalent for both X and \widehat{X} . Then by [Hi1, Theorem 3.3], the restriction map for the second cohomology $H^2(X, \mathbb{R}) \rightarrow H^2(X_b, \mathbb{R})$ is zero. This means that the restriction map $H^2(X, S^1) \rightarrow H^2(X_b, S^1)$ is trivial. So the flat unitary gerbe \mathcal{B} has trivial holonomy on each torus fiber (hence trivial). Therefore, one should work with pairs (X_b, T) , where X_b is a special Lagrangian submanifold and T is a flat trivialization of the gerbe \mathcal{B} on Z .

The modified mirror conjecture as proposed by Hitchin, [Hi1], is the following:

Conjecture: The mirror of a Calabi-Yau manifold X with a \mathcal{B} -field is the moduli space of pairs (Z, T) , where Z is a special Lagrangian submanifold of X and T is a flat trivialization of the gerbe \mathcal{B} on Z .

Two Calabi-Yau n -orbifolds M and \widehat{M} , equipped with flat unitary gerbes B and \widehat{B} respectively, are said to be *SYZ mirror partners* if there is an orbifold N of real dimension n and there are smooth surjections

$$\mu : M \rightarrow N, \quad \widehat{\mu} : \widehat{M} \rightarrow N$$

such that for every $x \in N$ which is a regular value of μ and $\widehat{\mu}$, the fibers $L_x := \mu^{-1}(x) \subset M$ and $\widehat{L}_x := \widehat{\mu}^{-1}(x) \subset \widehat{M}$ are special Lagrangian tori which are dual to each other in the sense that there are smooth identifications

$$L_x = \text{Triv}^{U^1}(\widehat{L}_x, \widehat{B}) \quad \text{and} \quad \widehat{L}_x = \text{Triv}^{U^1}(L_x, B)$$

that depend smoothly on x .

1.3. The result of Hausel and Thaddeus. The moduli spaces of Higgs bundles admit natural dual pairs of hyper-Kähler integrable systems [Hi2], [Hi3]. The hyper-Kähler metric and the collection of Poisson-commuting functions determining the integrable system produce a family of special Lagrangian tori on the moduli spaces, which is a key requirement of SYZ conjecture. Moreover, the families of tori on the $\text{SL}(r, \mathbb{C})$ and $\text{PGL}(r, \mathbb{C})$ moduli spaces are dual in the appropriate sense, which is the other requirement of SYZ conjecture.

This work of Hausel and Thaddeus was extended to principal G_2 in [Hi4]. In [DP], this was extended to all semisimple groups. (See related works [FW], [GW] and [Wi].)

In [HT1], Hausel-Thaddeus made an announcement that moduli spaces of $\text{SL}(r, \mathbb{C})$ and $\text{PGL}(r, \mathbb{C})$ parabolic Higgs bundles are mirror partner to each other (in the sense of SYZ) and their stinging E polynomials are same. In [HT2], they gave a proof of this conjecture in non-parabolic case.

Our aim here is to address the case of parabolic vector bundles with complete quasi-parabolic flags. We follow the proof of Hausel-Thaddeus; the key ingredient here is the identification of the parabolic Hitchin fiber as the Prym variety of a certain spectral cover (which was done in [BM], [GL]).

It should be clarified that the Higgs fields that we consider have nilpotent residue. The corresponding moduli space forms a symplectic leaf of the moduli space of Higgs bundles

for which the residue of the Higgs field satisfies the weaker condition that it is only flag preserving.

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2. PRELIMINARIES

2.1. Parabolic Higgs bundles. Let X be an irreducible smooth projective curve over \mathbb{C} of genus g , with $g \geq 2$. Let $D \subset X$ be a nonempty finite subset of n points.

A *quasi-parabolic structure*, over D , on a holomorphic vector bundle $E \rightarrow X$ is a filtration

$$E_x = E_{x,0} \supset E_{x,1} \supset \cdots \supset E_{x,r_x} \supset E_{x,r_x+1} = \{0\}$$

for each $x \in D$. A *parabolic structure* on E is a quasi-parabolic structure as above together with rational numbers

$$0 \leq \alpha_{x,0} < \alpha_{x,1} < \cdots < \alpha_{x,r_x} < 1,$$

which are called *parabolic weights*. A *parabolic vector bundle* over X of rank r is a holomorphic vector bundle of rank r on X equipped with a quasi-parabolic structure over D together with parabolic weights. The system of parabolic weights $\{(\alpha_{x,0}, \dots, \alpha_{x,r_x})\}_{x \in D}$ will be denoted by α .

For a parabolic vector bundle $E_* = (E, \{E_{x,i}\}, \alpha_*)$, the *parabolic degree* is defined to be

$$\text{par-deg}(E_*) = \text{deg}(E) + \sum_{x \in D} \sum_{i=0}^{r_x} \alpha_{x,i},$$

and the *parabolic slope* is defined to be $\text{par-}\mu(E_*) := \text{par-deg}(E_*)/\text{rk}(E)$. Any holomorphic subbundle F of E has a parabolic structure induced by the parabolic structure on E ; the resulting parabolic vector bundle will be denoted by F_* . A parabolic bundle is said to be *stable* (respectively, *semistable*) if for all holomorphic subbundles $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$,

$$(2.1) \quad \text{par-}\mu(F) < \text{par-}\mu(E) \quad (\text{respectively, } \text{par-}\mu(F) \leq \text{par-}\mu(E)).$$

The moduli space of semistable parabolic vector bundles of rank r and degree d with fixed parabolic data was constructed by Mehta and Seshadri, [MS], using Mumford's Geometric Invariant Theory. This moduli space, which we will denote by \mathcal{PM}_α^d , is smooth for a generic choice of weights α . We recall that given rank and degree, a system of parabolic weights α with multiplication is called *generic* if the semistability condition implies the stability condition.

Consider the determinant morphism

$$(2.2) \quad \det : \mathcal{PM}_\alpha^d \rightarrow \text{Jac}^d(X)$$

that sends a parabolic vector bundle E_* to $\bigwedge^{\text{top}} E$. Choose $\Lambda \in \text{Jac}^d(X)$, and define

$$(2.3) \quad \mathcal{PM}^\Lambda := \det^{-1}(\Lambda).$$

So \mathcal{PM}^Λ is a moduli space of twisted $\mathrm{SL}(r, \mathbb{C})$ -bundles with parabolic structure (see [BLS, Section 2] for twisted $\mathrm{SL}(r, \mathbb{C})$ -bundles). For any other line bundle $\Lambda_1 \in \mathrm{Jac}^d(X)$, the morphism

$$\det^{-1}(\Lambda) \longrightarrow \det^{-1}(\Lambda_1)$$

that sends any parabolic vector bundle E_* to $E_* \otimes \zeta$, where ζ is a fixed r -th root of $\Lambda_1 \otimes \Lambda^{-1}$, is an isomorphism. Thus the isomorphism class of the moduli space \mathcal{PM}^Λ does not depend on the choice of $\Lambda \in \mathrm{Jac}^d(X)$.

The abelian variety $\mathrm{Pic}^0(X) = \mathrm{Jac}^0(X)$ acts on \mathcal{PM}_α^d via

$$(L, E_*) \longmapsto L \otimes E_*.$$

The quotient

$$\widetilde{\mathcal{PM}}_\alpha^d := \mathcal{PM}_\alpha^d / \mathrm{Pic}^0(X),$$

which exist as a projective variety by [Se], is the component of the moduli space of parabolic $\mathrm{PGL}(r, \mathbb{C})$ -bundles corresponding to degree d (the connected components of the moduli space of parabolic $\mathrm{PGL}(r, \mathbb{C})$ -bundles are irreducible).

Let

$$(2.4) \quad \Gamma := \mathrm{Pic}^0(X)[r]$$

be the group of r -torsion points of the Jacobian; it is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{2g}$, in particular, its order is r^{2g} . The action of Γ on \mathcal{PM}_α^d preserves the subvariety \mathcal{PM}^Λ defined in (2.3). Let

$$\widetilde{\mathcal{PM}}^\Lambda := \mathcal{PM}^\Lambda / \Gamma$$

be the quotient; it is a projective variety by [Se].

Let K be the holomorphic cotangent bundle of X . The line bundle $K \otimes \mathcal{O}_X(D)$ will be denoted by $K(D)$. A *parabolic Higgs bundle* is a pair (E_*, Φ) , where E_* is a parabolic vector bundle and

$$\Phi : E \longrightarrow E \otimes K(D)$$

is a homomorphism which is *strongly parabolic*, meaning

$$\Phi(E_{x,i}) \subset E_{x,i+1} \otimes K(D)_x$$

for each point $x \in D$ and $i \in [0, r_x]$.

A parabolic Higgs bundle (E_*, Φ) is called *(semi)-stable* if the slope condition in (2.1) holds whenever Φ preserves F . Let $\mathcal{PM}_{\mathrm{Higgs}}^d$ denote the moduli space of semistable parabolic Higgs bundles of rank r and degree d with the given parabolic data. Let

$$\mathcal{PM}_{\mathrm{Higgs}}^\Lambda \subset \mathcal{PM}_{\mathrm{Higgs}}^d$$

be the subvariety consisting of all (E_*, Φ) such that $\det(E) = \Lambda$ and $\mathrm{trace}(\Phi) = 0$.

For $E_* \in \mathcal{PM}_\alpha^d$,

$$T_{E_*} \mathcal{PM}_\alpha^d = H^1(X, \mathrm{End}(E_*))$$

[Yo1], [Yo2], where $\mathrm{End}(E_*)$ is the sheaf of endomorphisms of the underlying vector bundle E preserving the quasi-parabolic filtrations. Applying the parabolic analog of Serre duality, [Yo1, Section 3],

$$T_{E_*} \mathcal{PM}_\alpha^d = H^0(X, \mathrm{SEnd}(E_*) \otimes K(D))^*,$$

where $\text{SEnd}(E_*) \subset \text{End}(E_*)$ is the strongly parabolic endomorphisms. Hence the total space $T_{E_*}^* \mathcal{P}\mathcal{M}_\alpha^d$ of the cotangent bundle maps to $\mathcal{P}\mathcal{M}_{\text{Higgs}}^d$. This map is an open embedding.

The group Γ acts on $\mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda$ via tensor product (the Higgs field does not change). The quotient $\mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda / \Gamma$ will be denoted by $\widetilde{\mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda}$.

2.2. Parabolic Hitchin system. In this subsection we recall the Hitchin map and the spectral curve for a parabolic Higgs bundle (see [BM], [GL], [LM] for details).

For notational convenience, the line bundle $K^{\otimes a} \otimes \mathcal{O}_X(bD)$ will be denoted by $K^a D^b$.

The parabolic Hitchin space is defined as

$$\mathcal{H} = H^0(K^2 D) \oplus \cdots \oplus H^0(K^r D^{r-1}).$$

The characteristic polynomial or trace map of a Higgs field defines the parabolic Hitchin map

$$(2.5) \quad h : \mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda \longrightarrow \mathcal{H}$$

It is known that this morphism h is proper (see [Yo2]).

Let $Z := \text{Spec Sym}^\bullet(K^{-1} \otimes \mathcal{O}_X(D)^{-1})$ be the total space of the line bundle $K(D)$ which is a quasi-projective surface. Let

$$(2.6) \quad p : Z \longrightarrow X$$

be the natural projection. For $s \in \mathcal{H}$, there exists an algebraic curve, denoted as X_s , in Z which is known as the *spectral curve*. We will very briefly recall it (for details see [BM], [GL]).

For any $(s_1, \dots, s_{r-1}) \in \mathcal{H}$, consider the map S from Z to the total space of the line bundle $K^r D^r$ given by

$$z \longmapsto z^{\otimes r} + z^{\otimes(r-2)} \otimes s_1(p(z)) + \cdots + s_{r-1}(p(z)) \in (K^r D^r)_{p(z)}$$

where $z \in Z$, and p is the projection in (2.6). The inverse image $S^{-1}(0_X) \subset Z$, where $O_X \subset K^r D^r$ is the zero section, is the parabolic spectral curve associated to $s := (s_1, \dots, s_{r-1})$. This parabolic spectral curve will be denoted by X_s . The restriction of p to X_s will again be denoted as p .

Henceforth, we assume that for each point $x \in D$, the quasi-parabolic flag is complete. In other words, $r_x = r - 1$.

There is a Zariski open dense subset $\mathcal{U} \subset \mathcal{H}$ such that for any $s \in \mathcal{U}$ the spectral curve X_s is smooth and connected [GL, Lemma 3.1]; the assumption that the quasi-parabolic flags are complete is needed for this. The fiber over s is isomorphic to

$$\text{Prym}^{d'}(Y_s) = \{L \in \text{Pic}^{d'}(Y_s) \mid \det(\pi_*(L)) = \xi\},$$

where $d' = d + r(r-1)(n+2g-2)/2$ [GL, Lemma 3.2]. This is a Γ -invariant closed subvariety of $\mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda$.

Note that tensoring by a line bundle does not change the characteristic polynomial, hence the Hitchin map h in (2.5) descends down to

$$(2.7) \quad \tilde{h} : \widetilde{\mathcal{P}\mathcal{M}_{\text{Higgs}}^\Lambda} \longrightarrow \mathcal{H}.$$

So h is the composition of \tilde{h} with the quotient map $\mathcal{PM}_{\text{Higgs}}^\Lambda \longrightarrow \widetilde{\mathcal{PM}_{\text{Higgs}}^\Lambda}$.

We will describe the fibers of the Hitchin maps in (2.5) and (2.7). For any $s \in \mathcal{U}$,

- (1) $P^{d'} := h^{-1}(s) = \text{Prym}^{d'}(X_s) = \text{Nm}^{-1}(\mathcal{O}_X(d'x))$, where

$$\text{Nm} : \text{Pic}^d(X_s) \longrightarrow \text{Pic}^d(X)$$

is the norm map defined by $\mathcal{O}_{X_s}(\sum_i d_i x_i) \longmapsto \mathcal{O}_X(\sum_i d_i \pi(x_i))$.

- (2) $\widehat{P}^{d'} := \widehat{h}^{-1}(s) = \text{Prym}^{d'}(X_s)/\Gamma$ (see (2.4) for Γ).

The fiber $P^{d'}$ is a torsor for

$$P^0 := \text{Nm}^{-1}(\mathcal{O}_X),$$

and $\widehat{P}^{d'}$ is a torsor for $\widehat{P}^0 = \text{Prym}^0(X_s)/\Gamma$. Hence $h|_{\mathcal{U}}$ and $\tilde{h}|_{\mathcal{U}}$ can be thought of as P^0 and \widehat{P}^0 torsors respectively over \mathcal{U} .

Let \mathcal{G} be an abelian group-scheme over X , and let \mathcal{A} be a \mathcal{G} -torsor. For any integer n , the \mathcal{G} -torsor on X obtained by extending the structure group of \mathcal{A} using the endomorphism of \mathcal{G} defined by $z \longmapsto z^n$ will be denoted by $(\mathcal{A})^n$.

Lemma 2.1. *For any integer d , we have*

- (1) $P^d \cong (P^1)^d$ as P^0 torsors over \mathcal{U} , and
(2) $\widehat{P}^d \cong (\widehat{P}^1)^d$ as \widehat{P}^0 torsors over \mathcal{U} .

In section 4 we will show that P^d and \widehat{P}^d are mirror partners for a certain choice of a ‘‘B’’ field.

3. PICARD CATEGORY AND GERBES

We briefly recall definition of *sheaf of categories* over a scheme (for details see [DG], [DM], [Gi]). Let $\text{Sch}_{\text{et}}(X)$ denote the category of all étale neighborhoods of a scheme X . A *presheaf of categories* on $\text{Sch}_{\text{et}}(X)$ is a contravariant functor \mathcal{Q} which assigns to every object $U \longrightarrow X$ in $\text{Sch}_{\text{et}}(X)$ a category $\mathcal{Q}(U)$ and to every morphism $f : U_1 \longrightarrow U_2$ in $\text{Sch}_{\text{et}}(X)$ a functor $f_{\mathcal{Q}}^* : \mathcal{Q}(U_2) \longrightarrow \mathcal{Q}(U_1)$. Moreover, for every composition

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3,$$

there is a transformation $f_{\mathcal{Q}}^* \circ g_{\mathcal{Q}}^* \longrightarrow (g \circ f)_{\mathcal{Q}}^*$ satisfying an obvious compatibility relation for three-fold compositions.

A presheaf \mathcal{Q} of categories on $\text{Sch}_{\text{et}}(X)$ is said to be a *sheaf of categories* if the following two axioms hold:

- (1) For $U \longrightarrow X$ in $\text{Sch}_{\text{et}}(X)$ and a pair of objects $C_1, C_2 \in \mathcal{Q}(U)$, the presheaf of sets on $\text{Sch}_{\text{et}}(U)$ that assigns to $f : U' \longrightarrow U$ the set

$$\text{Hom}_{\mathcal{Q}(U')}(f_{\mathcal{Q}}^*(C_1), f_{\mathcal{Q}}^*(C_2))$$

is a sheaf.

- (2) If $f : U' \longrightarrow U$ is a covering, then the category $\mathcal{Q}(U)$ is equivalent to the category of descent data on $\mathcal{Q}(U')$ with respect to f , meaning every descent data on $\mathcal{Q}(U')$ with respect to f is.

A Picard category is a tensor category, in which every object is invertible. A basic example is the category of line bundles over a scheme.

A sheaf of categories \mathcal{P} is said to be a sheaf of Picard categories if for every

$$(U \rightarrow X) \in \text{Sch}_{et}(X),$$

$\mathcal{P}(U)$ is endowed with a structure of a Picard category such that the pull-back functors $f_{\mathcal{P}}^*$ are compatible with the tensor product in an appropriate sense. If \mathcal{P}_1 and \mathcal{P}_2 are two sheaves of Picard categories, one defines (in a straightforward fashion) a tensor functor between them.

A category Q is said to be a *gerbe over the Picard category* P , if P acts on Q as a tensor category, and for any object $C \in Q$ the functor $P \rightarrow Q$ given by

$$B \in P \implies \text{Action}(P, C) \in Q$$

is an equivalence.

Now, if \mathcal{P} is a sheaf of Picard categories and \mathcal{Q} is another sheaf of categories we say that \mathcal{Q} is a *gerbe over* \mathcal{P} , if the following two conditions hold:

- For every $(U \rightarrow X) \in \text{Sch}_{et}(X)$, $\mathcal{Q}(U)$ has the structure of a gerbe over $\mathcal{P}(U)$. This structure is compatible with the pull-back functors $f_{\mathcal{P}}^*$ and $f_{\mathcal{Q}}^*$.
- There exists a covering $U \rightarrow X$ such that $\mathcal{Q}(U)$ is non-empty.

The basic example of a gerbe over an arbitrary sheaf of Picard categories \mathcal{P} is \mathcal{P} itself; it is called the trivial \mathcal{P} -gerbe.

Let \mathcal{A} be a sheaf of abelian groups over $\text{Sch}_{et}(X)$ which takes values in an abelian group A . For an object $f : U \rightarrow X$ of $\text{Sch}_{et}(X)$, let $\text{Tors}_{\mathcal{A}}(U)$ denote the category of $\mathcal{A}|_U$ -torsors on U . This is a Picard category, and the assignment $U \rightarrow \text{Tors}_{\mathcal{A}}(U)$ defines a sheaf of Picard categories on $\text{Sch}_{et}(X)$ which we will call $\text{Tors}_{\mathcal{A}}$ or \mathcal{A} -torsor. A gerbe over the sheaf of Picard categories \mathcal{A} -torsor will be called an \mathcal{A} -gerbe. Hence an \mathcal{A} -gerbe will be thought of as a torsor over the sheaf of Picard categories \mathcal{A} -torsors.

An *isomorphism* between \mathcal{A} -gerbes is an equivalence of sheaves of categories as torsors over the sheaf of \mathcal{A} -torsors. The isomorphism classes of \mathcal{A} -gerbes are in one-to-one correspondence with $H^2(X, \mathcal{A})$ (see [Gi]).

A *trivialization* of an \mathcal{A} -gerbe is an isomorphism with the trivial gerbe \mathcal{A} -torsor. Two trivializations z, z' are *equivalent* if the automorphism $z' \circ z^{-1}$ is given by tensorization with a trivial \mathcal{A} -torsor. The space of equivalence classes of trivializations of a trivial \mathcal{A} -gerbe B denoted $\text{Triv}^{\mathcal{A}}(X, B)$ is an $H^1(X, \mathcal{A})$ -torsor over a point [Gi].

Remark 3.1. Fix a short exact sequence

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{A}'' \rightarrow \mathcal{A}' \rightarrow 0$$

of sheaves of groups (they need not be abelian) on X , and let $\tau_{\mathcal{A}'}$ be an \mathcal{A}' -torsor over X . We introduce a sheaf of categories $\mathcal{Q} = \mathcal{Q}_{\tau_{\mathcal{A}'}}$ as follows: For $U \in \text{Sch}_{et}(X)$, let $\mathcal{Q}(U)$ be the category of all “liftings” of $\tau_{\mathcal{A}'}|_U$ to an $\mathcal{A}''|_U$ -torsor. It is easy to check that \mathcal{Q} is a \mathcal{A} -gerbe over X .

Remark 3.2. Let \mathcal{Q}_1 be a \mathcal{P}_1 -gerbe over X , and let $\mathbf{a} : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ be a tensor functor of Picard category over X . Then one can construct a canonical induced \mathcal{P}_2 -gerbe \mathcal{Q}_2 over

X with the property that there exists a functor $\mathcal{Q}_1 \rightarrow \mathcal{Q}_2$, compatible with the actions of \mathcal{P}_1 and \mathcal{P}_2 via \mathbf{a} .

Let \mathcal{Q}_1 and \mathcal{Q}_2 be two \mathcal{A} -gerbes over X . Then $\mathcal{Q}_1 \times_X \mathcal{Q}_2$ is an $\mathcal{A} \times_X \mathcal{A}$ -gerbe over X . Consider the multiplication homomorphism $\mathcal{A} \times_X \mathcal{A} \rightarrow \mathcal{A}$. Let $\mathcal{Q}_1 \otimes_{\mathcal{P}} \mathcal{Q}_2$ be the \mathcal{A} -gerbe over X given by $\mathcal{Q}_1 \times_X \mathcal{Q}_2$ using this homomorphism (see Remark 3.2). This \mathcal{A} -gerbe $\mathcal{Q}_1 \otimes_{\mathcal{P}} \mathcal{Q}_2$ is called the *tensor product* of \mathcal{Q}_1 and \mathcal{Q}_2 .

Now consider the inversion homomorphism $\mathcal{A} \rightarrow \mathcal{A}$. The \mathcal{A} -gerbe over X given by \mathcal{Q}_1 using this homomorphism (see Remark 3.2) will be denoted by $(\mathcal{Q}_1)^{-1}$.

Notation. Let \mathcal{Q} be a \mathcal{A} -gerbe over X . For any positive integer n , the n -fold tensor product $\mathcal{Q} \otimes_{\mathcal{P}} \cdots \otimes_{\mathcal{P}} \mathcal{Q}$ will be denoted by $(\mathcal{Q})^d$. For any negative integer n , the n -fold tensor product $\mathcal{Q}^{-1} \otimes_{\mathcal{P}} \cdots \otimes_{\mathcal{P}} \mathcal{Q}^{-1}$ will be denoted by $(\mathcal{Q})^{-d}$.

Remark 3.3. If B is a trivial \mathcal{A} -gerbe over X , then the tensor power B^e for any $e \in \mathbb{Z}$ is also trivial, and, moreover, the set of all trivializations, which is a $H^1(X, \mathcal{A})$ -torsor, has the following identification

$$\mathrm{Triv}^{\mathcal{A}}(X, B^e) = \mathrm{Triv}^{\mathcal{A}}(X, B)^e$$

(it is an identification of $H^1(X, \mathcal{A})$ -torsors).

4. TRIVIALIZATIONS AND B FIELDS

Let $\mathcal{U}(1)$ (respectively, \mathcal{Z}_r) be the sheaf of abelian groups over $\mathrm{Sch}_{et}(\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d)$ which takes values in $U(1)$ (respectively, \mathbb{Z}_r), and let $\mathcal{Tors}_{\mathcal{U}(1)}$ (respectively, $\mathcal{Tors}_{\mathcal{Z}_r}$) be the sheaf of Picard categories over $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d$ (see Section 3 for definition).

There is a universal parabolic Higgs vector bundle over $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d \times X$, because the parabolic flags are complete [BY, p. 465, Proposition 3.2]. Let (\mathcal{E}, Φ) be a Universal parabolic Higgs bundle on $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d \times X$. Restricting \mathcal{E} to $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d \times \{c\}$, where $c \in X \setminus D$ is a fixed point, we get a vector bundle \mathbb{E} on $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d$. Let $\mathbb{P} := P(\mathbb{E})$ be the associated projective bundle on $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d$ parametrizing line in the fibers of \mathbb{E} . From the exact sequence

$$e \rightarrow \mathcal{Z}_r \rightarrow \mathrm{SL}(r, \mathbb{C}) \rightarrow \mathrm{PGL}(r, \mathbb{C}) \rightarrow e$$

it follows that the obstruction to lift the $\mathrm{PGL}(r, \mathbb{C})$ -bundle \mathbb{P} to a $\mathrm{SL}(r, \mathbb{C})$ -bundle gives a class $B \in H^2(\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d, \mathcal{Z}_r)$. This cohomology class B corresponds to the \mathcal{Z}_r -gerbe on $\mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d$ defined by the liftings of \mathbb{P} to a $\mathrm{SL}(r, \mathbb{C})$ bundle (see Remark 3.1).

Lemma 4.1. *The restriction of B to each regular fiber P^d of the Hitchin map h (see (2.5)) is trivial as a \mathcal{Z}_r -gerbe.*

Proof. Let \mathcal{L} be a universal line bundle on $P^d \times X_s$ (see Section 2 for P^d). The projection of the spectral curve X_s to X will be denoted by π . The push-forward $(\mathrm{Id} \times \pi)_* \mathcal{L}$ is a vector bundle which admits a family of parabolic Higgs field inducing the inclusion $P^d \subset \mathcal{P}\mathcal{M}_{\mathrm{Higgs}}^d$. Hence we have $\mathbb{P}((\mathrm{Id} \times \pi)_* \mathcal{L})|_{P^d \times \{c\}} = \mathbb{P}$. Note that $\det((\mathrm{Id} \times \pi)_* \mathcal{L})|_{P^d \times \{c\}}$ is isomorphic to $\xi = \otimes_{y \in \pi^{-1}(c)} \mathcal{L}|_{P^d \times \{y\}}$ (the points of $\pi^{-1}(c)$ are taken with multiplicities).

Consider the above line bundle ξ . Let η be the r -th root of the line bundle ξ^* on P^d , meaning $\eta^r = \xi^*$. Note that since the Néron–Severi class of ξ is divisible by r , such a line bundle η exists. It is easy to see that $(\text{Id} \times \pi)_*(\mathcal{L} \otimes p^*\eta)$, where p is the projection of $P^d \times X_s$ to P^d , is a $\text{SL}(r, \mathbb{C})$ bundle on $P^d \times X$ such that $\mathbb{P}((\text{Id} \times \pi)_*(\mathcal{L} \otimes p^*\eta)) = \mathbb{P}$. Hence B is a trivial \mathcal{Z}_r -gerbe when restricted to P^d . \square

As seen in the proof of Lemma 4.1, a trivialization of B on P^d is equivalent of giving a universal line bundle $\mathcal{L} \rightarrow P^d \times X_s$ such that $\det(\text{Id} \times \pi)_*\mathcal{L}|_{P^d \times \{c\}}$ is trivial on $P^d \times \{c\}$. Hence we have a natural identification of the set of trivializations of B denoted as $\text{Triv}^{\mathcal{Z}_r}(P^d, B)$ with the set of isomorphism classes of such line bundles on $P^d \times X_s$; define

$$T := \{\mathcal{L} \rightarrow P^d \times X_s \mid \mathcal{L} \text{ is universal bundle with } \det(\text{Id} \times \pi)_*(\mathcal{L})|_{P^d \times \{c\}} = \mathcal{O}_{P^d}\}.$$

Note that this T is naturally a $\widehat{P}^0[r]$ -torsor since

$$\det(\text{Id} \times \pi)_*(\mathcal{L} \otimes p^*L) = \det(\text{Id} \times \pi)_*(\mathcal{L}) \otimes L^r.$$

We have the following natural isomorphism

$$H^1(P^d, \mathcal{Z}_r) = H^1(P^0, \mathcal{Z}_r) = \widehat{P}^0[r].$$

In terms of this isomorphism, the $H^1(P^d, \mathcal{Z}_r)$ -torsor $\text{Triv}^{\mathcal{Z}_r}(P^d, B)$ gets identified with the $\widehat{P}^0[r]$ -torsor T . Using this identification, the set of trivialization of B on P^d will be considered as a $H^1(P^d, \mathcal{Z}_r)$ -torsor.

By Remark 3.2, any \mathcal{Z}_r -gerbe extends to a $\mathcal{U}(1)$ -gerbe. Let \mathcal{B} denote the $\mathcal{U}(1)$ -gerbe given by the \mathcal{Z}_r -gerbe B . Since the \mathcal{Z}_r gerbe B on P^d is trivial, the extended $\mathcal{U}(1)$ -gerbe \mathcal{B} on P^d is also trivial. The set of all trivializations of \mathcal{B} on P^d as is denoted by $\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B})$. This $\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B})$ is a $H^1(P^d, \mathcal{U}(1))$ -torsor.

Theorem 4.2. *For any $d, e \in \mathbb{Z}$, there is an isomorphism of \widehat{P}^0 -torsors*

$$\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}^e) \xrightarrow{\sim} \widehat{P}^e.$$

Proof. From Lemma 2.1 and Remark 3.3,

$$\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}^e) \cong (\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}^1))^e \quad \text{and} \quad \widehat{P}^e \cong (\widehat{P}^1)^e.$$

Therefore, it is enough to prove the theorem under the assumption that $e = 1$. So set $e = 1$.

We have a natural identification as extension of scalars,

$$\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}) = \frac{\text{Triv}^{\mathcal{Z}_r}(P^d, B) \times H^1(P^d, \mathcal{U}(1))}{H^1(P^d, \mathcal{Z}_r)}.$$

Under this identification, the above torsor $\text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}^e)$ can be identified set theoretically with \mathfrak{T}_1 defined as follows:

$$\{\mathcal{L} \rightarrow P^d \times X_s \mid \mathcal{L} \text{ is a universal line bundle and } \mathcal{L}|_{P^d \times \{y\}} \in \text{Pic}^0(P^d) \forall y \in X_s\}.$$

We have a natural identification $\text{Pic}^0(P^d) \cong \frac{\text{Pic}^0(\widetilde{J}^0)}{\text{Pic}^0(J^0)} = \frac{\widetilde{J}^0}{J^0}$ [HT2, Lemma 2.2, Lemma 2.3], hence \mathfrak{T}_1 can be identified with $\frac{\widetilde{\mathfrak{T}}}{J^0}$ where $\widetilde{\mathfrak{T}}$ is defined as follows:

$$\{\mathcal{L} \rightarrow \widetilde{J}^d \times X_s \mid \mathcal{L} \text{ is a universal line bundle and } \mathcal{L}|_{\widetilde{J}^d \times \{y\}} \in \text{Pic}^0 \widetilde{J}^d \forall y \in X_s\}.$$

There is a natural isomorphism of J^0 with $\text{Pic}^0(J^0)$ given by the natural theta polarization on J^0 . In terms of this identification, the action of J^0 on \mathfrak{T} corresponds to the action of $\text{Pic}^0(J^0)$ defined by pull-back. Note that \widehat{P}^d can be identified with $\widetilde{J^d}$. So it is enough to show that \mathfrak{T} and $\widetilde{J^1}$ are isomorphic as $\widetilde{J^0}$ -torsors.

The idea is to give two surjective set theoretic maps f_1 and f_2 from X_s to these two torsors:

$$\begin{array}{ccc} & X_s & \\ f_1 \swarrow & & \searrow f_2 \\ \widetilde{J^1} & & \mathfrak{T} \end{array}$$

such that,

$$(4.1) \quad f_1(y') - f_1(y) = f_2(y) - f_2(y').$$

In view of this equality and the fact that both are $\widetilde{J^0}$ -torsors, the identification of $f_1(y)$ with $f_2(y)$ gives the required isomorphism between $\widetilde{J^1}$ and \mathfrak{T} as $\widetilde{J^0}$ -torsors.

Now we will construct f_1 and f_2 . The map f_1 is the Abel-Jacobi map which takes any $y \in X_s$ to the line bundle $\mathcal{O}_{X_s}(y)$ (which is an element of $\widetilde{J^1}$). The map f_2 sends any y to the unique universal line bundle \mathcal{L} on $P^d \times X_s$ satisfying the condition that $\mathcal{L}|_{P^d \times \{y\}} = \mathcal{O}_{P^d}$. To show that (4.1) holds, we need the following:

$$(4.2) \quad (\mathcal{L} \otimes p^*(\mathcal{O}_{X_s}(-y')))|_{P^d \times \{y\}} = f_1(y) - f_1(y') \quad \text{for any } y' \in X_s.$$

Now, (4.2) follows from two facts: Firstly, any universal bundle on $P^d \times X_s$ is of the form $p_2^*(L_0) \otimes F^*\mathcal{P}$, where p_2 is the projection to X_s , $L_0 \in \widetilde{J^d}$ is a fixed line bundle, \mathcal{P} is the universal line bundle on $\text{Pic}^0(\widetilde{J^0}) \times \widetilde{J^0}$ ($= \widetilde{J^0} \times \widetilde{J^0}$), and

$$F: \widetilde{J^d} \times X_s \longrightarrow \widetilde{J^0} \times \widetilde{J^0}$$

is defined by $(L, y) \mapsto (L \otimes L_0^{-1}, f_1(y) - f_1(y'))$.

Secondly, the involution of $\widetilde{J^0} \times \widetilde{J^0}$ exchanging the two factors takes the universal line bundle on $\widetilde{J^0} \times \widetilde{J^0} = \text{Pic}^0(\widetilde{J^0}) \times \widetilde{J^0}$ to its dual. \square

Let

$$(4.3) \quad \widetilde{\Gamma} = \bigsqcup_{\gamma \in \Gamma} L_\gamma - \{0\}$$

be the disjoint union of the total spaces of the nonzero vectors of the line bundles L_γ . This has the structure of a group scheme over X whose fiber at $x \in X$ is an abelian extension

$$(4.4) \quad 1 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{\Gamma}_c \longrightarrow \Gamma \longrightarrow 0.$$

The group Γ acts on $\mathcal{PM}_{\text{Higgs}}^d$; the action of any $L \in \Gamma$ sends any (E_*, ϕ) to $(E_* \otimes L, \phi \otimes \text{Id}_L)$; since the parabolic structure of L is trivial, we may use the notation of the usual tensor product (note that $\phi \otimes \text{Id}_L$ is a Higgs field on $E_* \otimes L$ in a natural way). Since the quasi-parabolic flags are complete, there exists an universal parabolic Higgs bundle (\mathcal{E}, Φ) on $\mathcal{PM}_{\text{Higgs}}^d \times X$ [BY]. In particular, \mathcal{E} is a universal vector bundle on $\mathcal{PM}_{\text{Higgs}}^d \times X$ and

$\Phi \in H^0(\mathbb{E}nd(\mathcal{E}) \otimes p_X^* K(D))$, where p_X is the projection of $\mathcal{P}\mathcal{M}_{\text{Higgs}}^d \times X$ to X . Consider the projective bundle $P(\mathcal{E})$ parametrizing the lines in the fibers of \mathcal{E} . The group Γ acts on $P(\mathcal{E})$; the action of any $L \in \Gamma$ sends any $((E_*, \phi), \xi)$ to $((E_* \otimes L, \phi \otimes \text{Id}_L), \xi \otimes L_y)$, where $\xi \in (E_*)_y$. Fix a point $c \in X$. Restricting $P(\mathcal{E})$ to $\mathcal{P}\mathcal{M}_{\text{Higgs}}^d \times \{c\}$ we get a Γ -equivariant projective bundle \mathbb{P} on $\mathcal{P}\mathcal{M}_{\text{Higgs}}^d$. The obstruction class to lift the Γ -equivariant $\text{PGL}(r, \mathbb{C})$ -bundle \mathbb{P} into a Γ -equivariant $\text{SL}(r, \mathbb{C})$ -bundle gives a nontrivial Γ -equivariant gerbe $B \in H_{\Gamma}^2(\mathcal{P}\mathcal{M}_{\text{Higgs}}^d, \mathcal{Z}_r)$. Let \tilde{B} be the \mathcal{Z}_r -gerbe $\widetilde{\mathcal{P}\mathcal{M}_{\text{Higgs}}^d}$ forgetting the Γ -equivariant structure on B .

The following technical lemma which will be used in proving Lemma 4.4.

Lemma 4.3 ([HT2, Lemma 3.3]). *Let $\mathcal{L} \rightarrow J^0 \times X$ be the universal line bundle which is trivial on $J^0 \times \{c\}$. Then there is an action over X of $\tilde{\Gamma}$ (constructed in (4.3)) on the total space of \mathcal{L} , lifting the action of Γ on J^0 by translation, so that the scalars \mathbb{C}^* act with weight one on the fibers.*

Proposition 4.4. *The restriction of \tilde{B} to each regular fiber \tilde{P}^d of the Hitchin map is trivial as a \mathcal{Z}_r -gerbe.*

Proof. The statement that the restriction of \tilde{B} to each regular fiber \tilde{P}^d of the Hitchin map is trivial as a \mathcal{Z}_r -gerbe is equivalent to the statement that the projective bundle $\mathbb{P}|_{P^d}$ is the projectivization of a Γ -equivariant vector bundle on P^d . We have already seen in Lemma 4.1 that B is a trivial gerbe on P^d . So $\mathbb{P}|_{P^d}$ is the projectivization of a vector bundle V on P^d . Therefore, the only thing to check is that the vector bundle V can be chosen to be Γ equivariant.

Recall how we got hold of a vector bundle V on P^d ; it came from a universal line bundle \mathcal{L} on $P^d \times X_s$. Hence by Lemma 4.3, giving a Γ -equivariant vector bundle V on P^d is equivalent of giving a universal line bundle \mathcal{L} over $P^d \times X_s$ on which $\tilde{\Gamma}$ acts such that the scalars $\mathbb{C}^* \subset \tilde{\Gamma}_c$ (see (4.4)) act with weight one and $\det(\pi_*(\tilde{L}|_{P^d \times \{c\}}))$ is in $\text{Pic}_{\Gamma}^0(P^d)$. The rest of the proof will be devoted in showing the existence of such a universal line bundle \mathcal{L} on $P^d \times X_s$.

Let \tilde{L} be any universal bundle on $\tilde{J}^d \times X_s$ and \mathcal{L} be the Poincaré universal bundle on $J^0 \times X$ rigidified using c . We have following natural projection maps,

$$(4.5) \quad \begin{array}{ccccc} & & P^d \times J^0 \times X_s & \xrightarrow{\tilde{p}_{12}} & \frac{P^d \times J^0}{\Gamma} \times X_s \\ & \swarrow p_{12} & \downarrow p_{23} & \searrow p_{13} & \\ & P^d \times J^0 & J^0 \times X_s & & P^d \times X_s \\ & & \downarrow \text{Id} \times \pi & & \\ & & J^0 \times X & & \end{array}$$

The action of the group Γ on $P^d \times J^0$ is given by

$$\gamma \cdot (L, M) \mapsto (L \otimes L_{\gamma}, M \otimes L_{\gamma}).$$

By [HT2, Lemma 2.2], we have an identification $\frac{P^d \times J^0}{\Gamma} = \tilde{J}^d$. The pulled back line bundle $\tilde{p}_{12}^*(\tilde{L})$ has a natural Γ action; this action can be extended to an action of $\tilde{\Gamma}$ on $\tilde{p}_{12}^*(\tilde{L})$

by making \mathbb{C}^* act trivially. Define

$$\mathcal{M} := \widetilde{p}_{12}^*(\widetilde{\mathcal{L}}) \otimes p_{12}^*(\widetilde{\mathcal{L}}^{-1}) \otimes p_{23}^*(\text{Id} \times \pi)^*\mathcal{L}.$$

Note that for any x, y and z in P^d, J^0 and X_s respectively, we have

$$\mathcal{M}|_{\{x\} \times J^0 \times X_s} = \mathcal{M}|_{P^d \times \{y\} \times X_s} = \mathcal{M}|_{P^d \times J^0 \times \{z\}} = \mathcal{O}.$$

Hence by the theorem of the cube (see [Mu, p. 87, Theorem]), the line bundle \mathcal{M} is trivial, equivalently, there is a $\widetilde{\Gamma}$ -equivariant isomorphism

$$(4.6) \quad \widetilde{p}_{12}^*\widetilde{\mathcal{L}} = p_{12}^*(\widetilde{\mathcal{L}}) \otimes p_{23}^*(\text{Id} \times \pi)^*(\mathcal{L}^{-1}).$$

Since the universal Poincaré line bundle \mathcal{L} is trivial on $J^0 \times \{c\}$, by Lemma 4.3, the group scheme $\widetilde{\Gamma}$ acts on $p_{23}^*(\text{id} \times \pi)^*(\mathcal{L}^{-1})$ with the scalars \mathbb{C}^* acting by weight -1 . From the isomorphism in (4.6) we get that $\widetilde{\Gamma}$ acts on $p_{12}^*(\widetilde{\mathcal{L}})$ with the scalars \mathbb{C}^* acting by weight one. Restricting $p_{12}^*(\widetilde{\mathcal{L}})$ to any base point of J^0 we get a $\widetilde{\Gamma}$ -action on $\widetilde{\mathcal{L}}$; this produces a Γ -action on $\widetilde{\mathcal{L}}$ which we wanted. So we can make $\widetilde{\mathcal{L}}$ in the proof of Lemma 4.1 to be a Γ -equivariant line bundle. \square

The set of all Γ -equivariant trivializations $\text{Triv}^{\mathcal{Z}_r}(\widehat{P}^d, \widetilde{\mathcal{B}})$ can be identified with the set \widetilde{T} defined by

$$\{\widetilde{\mathcal{L}} \longrightarrow P^d \times X_s \mid \widetilde{\mathcal{L}} \text{ is a } \widetilde{\Gamma}\text{-equivariant such that } \det(\text{Id} \times \pi)_*(\widetilde{\mathcal{L}})|_{P^d \times \{c\}} = \mathcal{O}_{P^d}\}.$$

Note that in \widetilde{T} we are interested only those $\widetilde{\Gamma}$ -actions on $\widetilde{\mathcal{L}}$ such that the action of \mathbb{C}^* is of weight one, because by Lemma 4.3 this will ensure that $\widetilde{\mathcal{L}}|_{P^d \times \{y\}}$ is a Γ -equivariant line bundle on P^d . It is easy to see that \widetilde{T} is a $\text{Pic}_\Gamma^0(P^d)[r]$ -torsor. Note that we have the following identifications

$$\text{Pic}_\Gamma^0(P^d)[r] = \text{Pic}^0(P^d/\Gamma)[r] = \text{Pic}^0(P^0/\Gamma)[r] = P^0[r],$$

where the last equality follows from the fact that the dual of P^0 , namely $H^1(P^0, \mathcal{U}(1))$, is P^0/Γ [HT2, Lemma 2.3]. Let $\widetilde{\mathcal{B}}$ denote the extended $\mathcal{U}(1)$ -gerbe given by the \mathcal{Z}_r -gerbe (see Remark 3.2) which is also trivial over \widehat{P}^d .

Theorem 4.5. *For any $d, e \in \mathbb{Z}$, there is a smooth isomorphism of P^0 -torsors*

$$\text{Triv}^{\mathcal{U}(1)}(\widehat{P}^d, \widehat{\mathcal{B}}^e) \cong P^e.$$

Proof. As in the proof of Theorem 4.2 we can assume $e = 1$. The set of all trivialization of $\widetilde{\mathcal{B}}$, which is denoted as $\text{Triv}^{\mathcal{U}(1)}(\widehat{P}^d, \widetilde{\mathcal{B}})$ is a $H^1(P^d/\Gamma, \mathcal{U}(1)) (= P^0)$ -torsor. This $\text{Triv}^{\mathcal{U}(1)}(\widehat{P}^d, \widetilde{\mathcal{B}})$ can be identified with $\widetilde{\mathfrak{T}}$ defined by

$$\{\widetilde{\mathcal{L}} \longrightarrow P^d \times X_s \mid \widetilde{\mathcal{L}} \text{ is a } \widetilde{\Gamma}\text{-equivariant such that } \det(\text{Id} \times \pi)_*(\widetilde{\mathcal{L}})|_{P^d \times \{c\}} \in \text{Pic}_\Gamma^0(P^d)\}.$$

Note that in $\widetilde{\mathfrak{T}}$ we are interested in only those $\widetilde{\Gamma}$ action on $\widetilde{\mathcal{L}}$ such that the action of \mathbb{C}^* is of weight one so that $\widetilde{\mathcal{L}}|_{P^d \times \{y\}}$ is a Γ -equivariant line bundle on P^d . Note that the difference between \widetilde{T} and $\widetilde{\mathfrak{T}}$ is that in \widetilde{T} we require the determinant of $\widetilde{\mathcal{L}}$ is trivial, while in $\widetilde{\mathfrak{T}}$ we require the determinant to be a Γ -equivariant line bundle on P^d of degree zero.

In the rest of the proof we will show that the P^0 -torsor $\widetilde{\mathfrak{T}}$ is isomorphic to the P^0 -torsor P^1 .

First note that P^1 sits naturally inside \widetilde{J}^1 as a P^0 -torsor.

In the proof of Theorem 4.2 we have seen that the set of all universal line bundles $\widetilde{\mathcal{L}} \rightarrow \widetilde{J}^d \times X_s$ with $\widetilde{\mathcal{L}}|_{\widetilde{J}^d \times \{c\}} \in \text{Pic}^0(\widetilde{J}^d)$ is isomorphic to \widetilde{J}^1 as a \widetilde{J}^0 -torsor. We will use this fact to give an inclusion of $\widetilde{\mathfrak{X}}$ into \widetilde{J}^1 compatible with respect to $P^0 \subset \widetilde{J}^0$. Send any $\widetilde{\mathcal{L}} \in \widetilde{\mathfrak{X}}$ to $p_{13}^*(\widetilde{\mathcal{L}}) \otimes p_{23}^*(\text{Id} \times \pi)^*(\mathcal{L})^{-1}$. Since \mathbb{C}^* acts on $p_{13}^*(\widetilde{\mathcal{L}})$ and $p_{23}^*(\text{id} \times \pi)^*(\mathcal{L})^{-1}$ by weights $+1$ and -1 respectively, the above tensor product is a Γ -equivariant line bundle with scalars acting trivially, or in other words, it is a Γ -equivariant line bundle on $P^d \times J^0 \times X_s$. Hence $p_{13}^*(\widetilde{\mathcal{L}}) \otimes p_{23}^*(\text{Id} \times \pi)^*(\mathcal{L})^{-1}$ descends down to

$$\frac{P^d \times J^0}{\Gamma} \times X_s = \widetilde{J}^d \times X_s.$$

One can check easily that $p_{13}^*(\widetilde{\mathcal{L}}) \otimes p_{23}^*(\text{Id} \times \pi)^*(\mathcal{L})^{-1}|_{\widetilde{J}^d \times \{c\}} \in \text{Pic}^0(\widetilde{J}^d)$, and the resulting map

$$\widetilde{\mathfrak{X}} \rightarrow \widetilde{J}^1, \quad \widetilde{\mathcal{L}} \mapsto p_{13}^*(\widetilde{\mathcal{L}}) \otimes p_{23}^*(\text{Id} \times \pi)^*(\mathcal{L})^{-1}$$

is injective.

So $\widetilde{\mathfrak{T}}$ and P^1 are now both P^0 -subtorsors of \widetilde{J}^1 . The quotient by either is the constant torsor J^0 (the Jacobian of C). Therefore, the image of one in the quotient by the other gives a morphism from the base U to J^0 . But U is a Zariski open set in an affine space, so its only morphisms to an abelian variety are the constant ones. Indeed, any nonconstant morphism from a Zariski open subset of \mathbb{A}^1 to an abelian variety extends to a nonconstant morphism from $\mathbb{P}_{\mathbb{C}}^1$ to the abelian variety. But there is no such map. \square

From Theorem 4.2 and Theorem 4.5 we conclude that over a generic open subset $U \subset \mathcal{H}$ the Hitchin fibers are SYZ mirror partners in the sense of Hitchin.

REFERENCES

- [BLS] A. Beauville, Y. Laszlo and C. Sorger: The Picard group of the moduli of G -bundles on a curve, *Compos. Math.* **112** (1998), 183–216.
- [BM] I. Biswas and A. Mukherjee: Symplectic structures on moduli spaces of parabolic Higgs bundles and Hilbert scheme, *Comm. Math. Phys.* **240** (2003), 149–159.
- [BY] H. U. Boden and K. Yokogawa: Rationality of moduli spaces of parabolic bundles, *Jour. London Math. Soc.* **59** (1999), 461–478.
- [DM] P. Deligne and J. S. Milne: Tannakian categories, in: *Hodge Cycles, Motives, and Shimura varieties*, 101–229, Lect. Notes Math. **900**, Springer-Verlag, Berlin-New York, 1982.
- [DG] R. Y. Donagi and D. Gaitsgory: The gerbe of Higgs bundles, *Transform. Groups* **7** (2002), 109–153.
- [DP] R. Y. Donagi and T. Pantev: Langlands duality for Hitchin systems, *Inv. Math.* (2012), DOI: 10.1007/s00222-012-0373-8.
- [FW] E. Frenkel and E. Witten: Geometric endoscopy and mirror symmetry, *Commun. Number Theory Phys.* **2** (2008), 113–283.
- [Gi] J. Giraud: *Cohomologie non abélienne*, Springer-Verlag, Berlin-New York, 1971.
- [GL] T. L. Gómez and M. Logares: A Torelli theorem for the moduli space of parabolic Higgs bundles, *Adv. Geom.* **11** (2011), 429–444.
- [GW] S. Gukov and E. Witten: Gauge theory, ramification, and the geometric Langlands program, in: *Current developments in mathematics, 2006*, 35–180, Int. Press, Somerville, MA, 2008.
- [HT1] T. Hausel and M. Thaddeus: Examples of mirror partners arising from integrable systems, *C. R. Acad. Sci. Paris Math.* **333** (2001), 313–318.

- [HT2] T. Hausel and M. Thaddeus: Mirror symmetry, Langlands duality, and the Hitchin system, *Invent. Math.* **153** (2003), 197–229.
- [Hi1] N. J. Hitchin: Lectures on special Lagrangian submanifolds, in: *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, Cambridge, MA (1999)*, 151–182, AMS/IP Stud. Adv. Math., 23, Amer. Math. Soc., Providence, RI, 2001.
- [Hi2] N. J. Hitchin: The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (1987), 59–126.
- [Hi3] N. J. Hitchin: Stable bundles and integrable systems, *Duke Math. Jour.* **54** (1987), 9–114.
- [Hi4] N. J. Hitchin: Langlands duality and G_2 spectral curves, *Quart. Jour. Math.* **58** (2007), 319–344.
- [LM] M. Logares and J. Martens: Moduli of parabolic Higgs bundles and Atiyah algebroids, *Jour. Reine Angew. Math.* **649** (2010), 89–116.
- [MS] V. Mehta and C. Seshadri: Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.
- [Mu] D. Mumford: *Abelian varieties*, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition, Tata Institute of Fundamental Research Studies in Mathematics, 5, Hindustan Book Agency, New Delhi, 2008.
- [Se] C. S. Seshadri: Quotient space by an abelian variety, *Math. Ann.* **152** (1963), 185–194.
- [SYZ] A. Strominger, S.-T. Yau and E. Zaslow: Mirror symmetry is T-duality, *Nuclear Phys. B* **479** (1996), 243–259.
- [Wi] E. Witten: Mirror symmetry, Hitchin’s equations, and Langlands duality, in: *The many facets of geometry*, 113–128, Oxford Univ. Press, Oxford, 2010.
- [Yo1] K. Yokogawa: Infinitesimal deformation of parabolic Higgs sheaves, *Inter. Jour. Math.* **6** (1995), 125–148.
- [Yo2] K. Yokogawa: Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves, *Jour. Math. Kyoto Univ.* **33** (1993), 451–504.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

E-mail address: `indranil@math.tifr.res.in`

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, MADRAS, CHENNAI 600036, INDIA

E-mail address: `arijitdey@gmail.com`