

Sharp inflaton potentials and bi-spectra: Effects of smoothening the discontinuity

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Abstract. Sharp shapes in the inflaton potentials often lead to short departures from slow roll which, in turn, result in deviations from scale invariance in the scalar power spectrum. Typically, in such situations, the scalar power spectrum exhibits a burst of features associated with modes that leave the Hubble radius either immediately before or during the epoch of fast roll. Moreover, one also finds that the power spectrum turns scale invariant at smaller scales corresponding to modes that leave the Hubble radius at later stages, when slow roll has been restored. In other words, the imprints of brief departures from slow roll, arising out of sharp shapes in the inflaton potential, are usually of a finite width in the scalar power spectrum. Intuitively, one may imagine that the scalar bi-spectrum too may exhibit a similar behavior, i.e. a restoration of scale invariance at small scales, when slow roll has been reestablished. However, in the case of the Starobinsky model (viz. the model described by a linear inflaton potential with a sudden change in its slope) involving the canonical scalar field, it has been found that, a rather sharp, though short, departure from slow roll can leave a lasting and significant imprint on the bi-spectrum. The bi-spectrum in this case is found to grow linearly with the wavenumber at small scales, a behavior which is clearly unphysical. In this work, we study the effects of smoothening the discontinuity in the Starobinsky model on the scalar bi-spectrum. Focusing on the equilateral limit, we analytically show that, for smoother potentials, the bi-spectrum indeed turns scale invariant at suitably large wavenumbers. We also confirm the analytical results numerically using our newly developed code BINGO. We conclude with a few comments on certain related points.

Keywords: Cosmic Inflation, Cosmic Microwave Background, Non-Gaussianities

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1 Inflationary models, discontinuities and the scalar bi-spectrum

The inflationary scenario is a very efficient paradigm to resolve the puzzles of the standard cosmological model and to simultaneously describe the origin of perturbations in the early universe [1–21]. Even the simplest of models lead to a sufficiently long duration of inflation that is required to overcome the horizon problem. Moreover, many of these models permit inflation of the slow roll type, which generates a nearly scale invariant primordial power spectrum that is remarkably consistent with the observations of the anisotropies in the Cosmic Microwave Background (CMB) and other cosmological data [22–28].

While attempting to identify the correct inflationary scenario, apart from the power spectrum, the non-Gaussianities and, in particular, the scalar bi-spectrum, also play a significant role. Indeed, the recent Planck data have shown that the non-Gaussianities are consistent with zero, with the three parameters that characterize the scalar bi-spectrum constrained to be: $f_{\text{NL}}^{\text{loc}} = 2.7 \pm 5.8$, $f_{\text{NL}}^{\text{eq}} = -42 \pm 75$ and $f_{\text{NL}}^{\text{ortho}} = -25 \pm 39$ [28]. These constraints imply that the correct model of inflation cannot deviate too much from the standard single field inflation of the slow roll type, involving the canonical kinetic term. In other words, inflation seems to be a non-trivial (i.e. $n_s \neq 1$, where n_s denotes the scalar spectral index), but ‘non-exotic’ (viz. $f_{\text{NL}} \simeq 0$) mechanism [29–31]. On the theoretical front, the most complete formalism to calculate the three-point correlation functions involving scalars and tensors in a given inflationary model is the approach due to Maldacena [32]. In the Maldacena formalism, the three-point functions are evaluated using the standard rules of perturbative quantum field theory, based on the interaction Hamiltonian that depends cubically on the perturbations [32–38]. The resulting expressions for the three-point function of primary interest, viz. the scalar bi-spectrum, involves integrals over combinations of the background quantities such as the scale factor and the slow roll parameters as well as the modes describing the curvature perturbation (see, for instance, Refs. [39, 40]). Evaluating the bi-spectrum analytically for a generic inflationary model proves to be a non-trivial task.

But, as in the case of the power spectrum, the bi-spectrum can be calculated analytically under the slow roll approximation [32, 33, 35, 41–44].

As we have already mentioned, slow roll inflation driven by a single scalar field seems to be the most likely possibility to describe the early universe. Nevertheless, it is also interesting to consider other scenarios which lead to larger levels of non-Gaussianities. Such analyses can help us gain a better understanding of the constraints imposed by the Planck data on the parameters characterizing these class of models. Moreover, these exercises can actually allow us to assess the degree of fine-tuning implied by the CMB data from Planck and WMAP on the non-minimal alternatives (see, for example, Ref. [45–47]). Further, the recent claim of the detection of the imprints of the primordial tensor modes by BICEP2 and the indication of a relatively high tensor-to-scalar ratio [48, 49], if confirmed, implies that we cannot completely rule out non-trivial possibilities either. Studying non-standard scenarios is however not a simple task, since, in these situations, the calculation of non-Gaussianities can be highly non-trivial and one often has to rely on numerical calculations (for numerical analysis of specific models, see Refs. [50–56]; for a broader discussion on the procedures involved and applications to a few different classes of models, see Ref. [57]; in this context, also see Ref. [58]). However, occasionally, it is also possible to evaluate the bi-spectrum analytically in non-trivial situations such as scenarios involving departures from slow roll [59–61]. One such example that permits an analytic evaluation of the power spectrum and the complete bi-spectrum (at least, in the equilateral limit) even in the presence of fast roll, is the model originally due to Starobinsky [59]. This model has recently attracted quite a lot of attention, but different physical conclusions with regards to the shape of the bi-spectrum have been reached. The present paper is aimed at considering the question again in order to clarify the situation.

As we shall soon outline, the Starobinsky model involves a canonical scalar field and is described by a linear potential with a sudden change in the slope at a given point. The sharp change in the slope leads to a brief period of fast roll sandwiched between two epochs of slow roll. Typically, in such situations, the power spectrum is expected to turn scale invariant when slow roll has been restored, and it is indeed what happens in the case of the Starobinsky model. The scalar power spectrum has a step like feature with a burst of oscillations connecting the two levels of the step (see, for instance, Fig. 3 of Ref. [39]). The flat regions of the step reflect the two epochs of slow roll, while the oscillations in between arise as a result of the period of fast roll.

Naively, one would have expected that the scalar bi-spectrum too would exhibit a similar behavior, viz. that it would turn scale invariant when slow roll has been restored. However, strikingly, when considered without adequate care, it is found that the bi-spectrum grows linearly with the wavenumbers at small scales (see, Refs. [62, 63]; in this context, also see Figs. 7 and 11 in Ref. [57] and the recent work Ref. [64]). From a theoretical point of view, evidently, it is imperative to firmly establish the predictions of the model and settle upon the correct behavior in a physically relevant and realistic situation. It is also worth noting here that, given the extent of accuracy of the measurements of non-Gaussianities by Planck, upon comparing with the data, the two behavior mentioned above would probably lead to very different constraints on the Starobinsky model. With these motivations in mind, in this work, as we have already pointed out, we intend to revisit the issue.

Clearly, the fact that the scale invariance of the bi-spectrum is not restored on large scales must be unphysical and, in this paper, we shall show that this arises due to the discontinuity in the second derivative of the potential. In fact, the point that the growing

term is indeed unrealistic could have been easily guessed from the very beginning, since it exactly corresponds to a well-known and well-studied situation which was investigated long ago in the context of particle production by time-dependent, classical, gravitational fields [65]. In what follows, we shall quickly recall the main results and conclusions arrived at in the earlier work, as the phenomenon closely resembles the behavior of the bi-spectrum encountered in the Starobinsky model.

The earlier work [65] considers a scalar field, say, ψ , that is non-minimally coupled to gravity, and is evolving in a spatially, flat, Friedmann-Lemaître-Robertson-Walker (for convenience, simply FLRW, hereafter) metric. Such a scalar field is governed by the following equation of motion:

$$(\square - \xi R) \psi = 0, \quad (1.1)$$

where R denotes the scalar curvature, while ξ an arbitrary constant. In a time-dependent background such as the FLRW universe, it is common knowledge that, upon quantization, pairs of particles associated with the scalar field ψ will, in general, be produced, provided the coupling is *not* conformal, i.e. $\xi \neq 1/6$. Let $a(\eta)$ denote the scale factor of the FLRW universe, with η being the conformal time coordinate. Upon Fourier transforming the scalar field and redefining the Fourier modes, say, $\psi_{\mathbf{k}}$, as $\psi_{\mathbf{k}} \equiv \mu_{\mathbf{k}}/a(\eta)$, one finds that the differential equation satisfied by $\mu_{\mathbf{k}}$ can be written as

$$\mu_{\mathbf{k}}'' + k^2 \mu_{\mathbf{k}} = \mathcal{V}_{\mathbf{k}}(\eta) \mu_{\mathbf{k}}, \quad (1.2)$$

where an overprime represents differentiation with respect to the conformal time η , while k denotes the comoving wavenumber. The quantity $\mathcal{V}_{\mathbf{k}}(\eta)$ is given by

$$\mathcal{V}_{\mathbf{k}}(\eta) \equiv (1 - 6\xi) \frac{a''}{a} = \left(\frac{1}{6} - \xi\right) a^2 R. \quad (1.3)$$

The above differential equation for $\mu_{\mathbf{k}}$ can also be cast as an integro-differential equation as follows:

$$\mu_{\mathbf{k}}(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} + \frac{1}{k} \int_{-\infty}^{\eta} d\tau \mathcal{V}_{\mathbf{k}}(\tau) \sin[k(\eta - \tau)] \mu_{\mathbf{k}}(\tau). \quad (1.4)$$

At early stages of the expansion, the mode function can be expected to behave as $\mu_{\mathbf{k}} \rightarrow e^{-ik\eta}/\sqrt{2k}$, which essentially corresponds to choosing the field to be in the vacuum state initially. At late times, one has $\mu_{\mathbf{k}} \rightarrow (\mathcal{A}_{\mathbf{k}}/\sqrt{2k}) e^{-ik\eta} + (\mathcal{B}_{\mathbf{k}}/\sqrt{2k}) e^{ik\eta}$, where $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{B}_{\mathbf{k}}$ are the standard Bogoliubov coefficients that relate the modes at different times. Then, using Eq. (1.4), one can approximate the Bogoliubov coefficient $\mathcal{B}_{\mathbf{k}}$ at very late times to be

$$\mathcal{B}_{\mathbf{k}} \simeq \frac{i}{2k} \int_{-\infty}^{\infty} d\tau \mathcal{V}_{\mathbf{k}}(\tau) e^{-2ik\tau}. \quad (1.5)$$

This expression, in turn, permits one to evaluate the energy density of the created particles, which is arrived at by calculating the integral [65]

$$\rho = \frac{1}{2\pi^2 a^4} \int_0^{\infty} dk k^3 |\mathcal{B}_{\mathbf{k}}|^2. \quad (1.6)$$

Let us now consider the case wherein there is a sharp transition from a phase of de Sitter inflation to a radiation dominated era. Let the transition take place at the conformal time, say, η_* . In this scenario, the scalar curvature R is non-zero, but constant (being related

to the constant Hubble parameter during the de Sitter phase) for $\eta < \eta_*$, while R vanishes for $\eta > \eta_*$ (i.e. during the radiation dominated epoch). The integral (1.5) can be carried out explicitly in such a case and, one obtains that, $\mathcal{B}_k = 2(1 - 6\xi)\Gamma(-1, 2ik\eta_*)$, where $\Gamma(b, z)$ is the incomplete Euler function [66, 67]. For large values of k , one finds that $|\mathcal{B}_k|^2 \propto k^{-4}$, with the result that the corresponding energy density ρ diverges logarithmically. However, as discussed in the original work [65], this conclusion is unphysical, and it is just an artifact of the abruptness of the transition from the de Sitter phase to the epoch of radiation domination. Indeed, if we now ‘regularize’ the transition, for instance, by smoothening out the quantity $\mathcal{V}_k(\eta)$ to be, say, $\mathcal{V}_k(\eta) = 2(1 - 6\xi)/(\eta^2 + \eta_*^2)$, then the coefficient \mathcal{B}_k is found to be

$$\mathcal{B}_k = -\frac{i\pi}{k\eta_*} e^{2k\eta_*}. \quad (1.7)$$

In other words, one obtains an exponential cut-off in the spectrum, i.e. $|\mathcal{B}_k|^2$, of created particles (note that η_* is negative), which occurs as a result of smoothening out the sharp transition. If we now calculate the corresponding energy density, then we arrive at a finite result, viz. $\rho = (1 - 6\xi)^2/(32a^4\eta_*^4)$. This unambiguously illustrates the point that the original logarithmic divergence was indeed an artifact and, upon modeling the transition more realistically, one obtains a result that is perfectly finite and physical.

In the same manner, the indefinite growth of the bi-spectrum at small scales in the Starobinsky model ought to be just an artifact and should be considered to be unphysical. In this work, focusing on the equilateral limit, we shall analytically investigate the effects of smoothening out the discontinuity in the derivative of the potential on the scalar bi-spectrum. We shall also compare the analytical results with the numerical results from the code Bi-spectra and Non-Gaussianity Operator or, simply, BINGO, which we had recently put together to compute the scalar bi-spectrum in inflationary models involving the canonical scalar field [57]. As we shall illustrate, in the case of the bi-spectrum, smoothening out the discontinuity restores the scale invariance of the bi-spectrum at suitably large wavenumbers, depending on the extent of the smoothening. This allows us to conclude that the continued growth in the bi-spectrum at small scales, as was found earlier, can be attributed to the unrealistic assumption that the discontinuity in the derivative of the potential can be arbitrarily sharp.

The remainder of this paper is organized as follows. In the following two sections, we shall highlight a few essential aspects of the Starobinsky model and discuss the dominant contribution to the scalar bi-spectrum (in the equilateral limit) which arises due to the discontinuity in the first derivative of the potential in the model. In Sec. 4, we shall smoothen the discontinuity in a simple manner, which allows one to obtain the modes during the transition, and evaluate the corresponding contribution to the scalar bi-spectrum. We shall see that even the simplest of smoothening curtails the growth of the bi-spectrum on small scales. In Sec. 5, focusing on the limit of large wavenumbers, we shall discuss the effects of a more generic smoothening of the potential. We shall analytically illustrate that, if the potential is smoothened sufficiently, it ensures that the corresponding contributions to the bi-spectrum prove to be insignificant at suitably small scales. In Sec. 6, we shall compare the analytical expressions we obtain with the numerical results from BINGO. We shall conclude in Sec. 7 with a few general remarks.

Note that, we shall assume the background to be the spatially flat, FLRW line-element, which is described by the scale factor a and the Hubble parameter H . Also, we shall work with units such that $c = \hbar = 1$, and we shall set $M_{\text{Pl}}^2 = (8\pi G)^{-1}$. Moreover, t shall denote

the cosmic time coordinate, and we shall represent differentiation with respect to t by an overdot. As we have already mentioned, η represents the conformal time coordinate, while an overprime denotes differentiation with respect to η . Further, N shall denote the number of e-folds. Lastly, a plus sign, a zero or a minus sign in the sub-script or the super-script of any quantity shall denote its value or contribution before, during and after the field crosses the discontinuity in the derivative of the potential, respectively.

2 Essential aspects of the Starobinsky model

The Starobinsky model involves a canonical scalar field and it consists of a linear potential with a sudden change in its slope at a given point [59]. The potential that describes the model can be written as follows:

$$V(\phi) = \begin{cases} V_0 + A_+ (\phi - \phi_0) & \text{for } \phi > \phi_0, \\ V_0 + A_- (\phi - \phi_0) & \text{for } \phi < \phi_0. \end{cases} \quad (2.1)$$

Evidently, while the value of the scalar field where the slope, i.e. $V_\phi \equiv dV/d\phi$, changes abruptly is ϕ_0 , the slope of the potential above and below ϕ_0 are given by A_+ and A_- , respectively. Moreover, the quantity V_0 denotes the value of the potential at $\phi = \phi_0$. In this section, we shall highlight a few important points relating to the evolution of the background, in particular, the behavior of the slow roll parameters, in the Starobinsky model. We shall also discuss the behavior of the modes describing the curvature perturbation before and after the field crosses the point ϕ_0 .

2.1 Evolution of the background

An important assumption of the Starobinsky model is that the value of V_0 is sufficiently large that it dominates the energy of the scalar field as it rolls down the potential across ϕ_0 . As a result, the behavior of the scale factor proves to be essentially that of de Sitter. This, in turn, implies that the first slow parameter, viz. $\epsilon_1 = -\dot{H}/H^2$, remains much smaller than unity throughout the evolution, even as the field crosses the discontinuity in the potential. In fact, the first slow roll parameter before and after the transition, i.e. when the field crosses ϕ_0 , can be shown to be [39]

$$\epsilon_{1+} \simeq \frac{A_+^2}{18 M_{\text{Pl}}^2 H_0^4}, \quad (2.2)$$

$$\epsilon_{1-} \simeq \frac{A_-^2}{18 M_{\text{Pl}}^2 H_0^4} \left[1 - \frac{\Delta A}{A_-} e^{-3(N-N_0)} \right]^2, \quad (2.3)$$

respectively, where H_0 is a constant that is determined by the relation $H_0^2 \simeq V_0/(3 M_{\text{Pl}}^2)$, while N_0 denotes the e-fold at the transition, and $\Delta A \equiv A_- - A_+$.

Before the transition, the second slow roll parameter, viz. $\epsilon_2 = d \ln \epsilon_1 / dN$, is determined by the slow roll approximation and is found to be $\epsilon_{2+} \simeq 4 \epsilon_{1+}$. However, as the field crosses ϕ_0 , the change in the slope causes a short period of deviation from slow roll. After the transition, the second slow roll parameter ϵ_2 is found to be [39]

$$\epsilon_{2-} \simeq \frac{6 \Delta A}{A_-} \frac{e^{-3(N-N_0)}}{1 - (\Delta A/A_-) e^{-3(N-N_0)}} + 4 \epsilon_{1-}. \quad (2.4)$$

It is clear that ϵ_{2-} turns large immediately after the transition and, when slow roll is restored eventually, one finds that $\epsilon_{2-} \simeq 4 \epsilon_{1-}$, just as one would expect.

As we shall discuss in the following section, the dominant contribution to the scalar bi-spectrum arises due to the so-called fourth term in the Maldacena formalism (in this context, see, for instance, Refs. [39, 40, 57]). This contribution involves the time derivative of the second slow roll parameter ϵ_2 . Upon using the background equations, one can show that $\dot{\epsilon}_2$ can be written as [39, 62, 63]

$$\dot{\epsilon}_2 = -\frac{2V_{\phi\phi}}{H} + 12H\epsilon_1 - 3H\epsilon_2 - 4H\epsilon_1^2 + 5H\epsilon_1\epsilon_2 - \frac{H}{2}\epsilon_2^2, \quad (2.5)$$

where $V_{\phi\phi} \equiv d^2V/d\phi^2$, and we should stress here that this expression is an exact one. In the case of the Starobinsky model, due to the discontinuity in the slope V_ϕ of the potential, clearly, the first term in the expression for $\dot{\epsilon}_2$ above, which involves the second derivative of the potential, will lead to a Dirac delta function. The contribution to $\dot{\epsilon}_2$ due to this specific term can then be written as

$$\dot{\epsilon}_2 \simeq \frac{2\Delta A}{H_0} \delta^{(1)}(\phi - \phi_0) = \frac{6\Delta A}{A_+ a_0} \delta^{(1)}(\eta - \eta_0). \quad (2.6)$$

In fact, on large wavenumbers, as we shall soon discuss, it is this particular term that was found to lead to the dominant contribution to the scalar bi-spectrum [62, 63], if one works in the limit where the discontinuity in V_ϕ is infinitely sharp.

2.2 Evolution of the perturbations

Let us now turn to briefly discuss the behavior of the modes describing the scalar perturbations in the Starobinsky model.

Recall that, the Fourier modes of the curvature perturbations, say, $f_{\mathbf{k}}(\eta)$, are governed by the differential equation [11]

$$f_{\mathbf{k}}'' + 2\frac{z'}{z}f_{\mathbf{k}}' + k^2 f_{\mathbf{k}} = 0, \quad (2.7)$$

where $z = a M_{\text{Pl}} \sqrt{2\epsilon_1}$. In terms of the Mukhanov-Sasaki variable, $v_{\mathbf{k}} = z f_{\mathbf{k}}$, the above equation for $f_{\mathbf{k}}$ reduces to

$$v_{\mathbf{k}}'' + \left(k^2 - \frac{z''}{z}\right) v_{\mathbf{k}} = 0. \quad (2.8)$$

The ‘effective potential’ z''/z that appears in this differential equation can be written in terms of the slow roll parameters as follows:

$$\frac{z''}{z} = \mathcal{H}^2 \left(2 - \epsilon_1 + \frac{3\epsilon_2}{2} + \frac{\epsilon_2^2}{4} - \frac{\epsilon_1\epsilon_2}{2} + \frac{\epsilon_2\epsilon_3}{2} \right), \quad (2.9)$$

where $\mathcal{H} \equiv a'/a = aH$ is the conformal Hubble parameter, while ϵ_3 denotes the third slow roll parameter given by

$$\epsilon_3 = \frac{d \ln \epsilon_2}{dN} = \frac{\dot{\epsilon}_2}{H \epsilon_2}. \quad (2.10)$$

Also, it should be emphasized that the above expression for z''/z is exact, and no approximation has been made in arriving at it.

In the Starobinsky model, due to certain cancellations that occur under the approximations of interest, one finds that the quantity z''/z reduces to $2\mathcal{H}^2$ before *as well as* after the transition [39, 59, 62, 63]. This basically corresponds to the de Sitter limit, which then implies that the Mukhanov-Sasaki variable $v_{\mathbf{k}}$ during these regimes is essentially given by the conventional Bunch-Davies solutions [68]. However, it should be clear from the expressions (2.9), (2.10) and (2.6) that, at the transition, it is the last term involving the quantity ϵ_3 in z''/z above which will dominate. One finds that the corresponding effective potential is described by a Dirac delta function *at* the transition, and is given by [39]

$$\begin{aligned} \frac{z''}{z} &\simeq \frac{\mathcal{H}^2 \epsilon_2 \epsilon_3}{2} = \frac{\mathcal{H}^2 \dot{\epsilon}_2}{2H} = a_0^2 \Delta A \delta^{(1)}(\phi - \phi_0) \\ &= \frac{a_0^2 \Delta A}{|d\phi/d\eta|_{\eta_0}} \delta^{(1)}(\eta - \eta_0) = \frac{3a_0 H_0 \Delta A}{A_+} \delta^{(1)}(\eta - \eta_0), \end{aligned} \quad (2.11)$$

where η_0 and a_0 denote the conformal time and the scale factor at the transition. We should clarify that, while the strictly de Sitter term, viz. $z''/z \simeq 2\mathcal{H}^2$, remains, it is the above term which a priori dominates *at* the transition.

Due to slow roll, before the transition, the modes $v_{\mathbf{k}}$ can be described to a good approximation by following de Sitter solution:

$$v_{\mathbf{k}}^+(\eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \quad (2.12)$$

Though slow roll is indeed restored at late times, due to the intervening epoch of fast roll, post-transition, the modes $v_{\mathbf{k}}$ do not remain in the Bunch-Davies vacuum. Hence, after the transition, the solution to $v_{\mathbf{k}}$ takes the general form

$$v_{\mathbf{k}}^-(\eta) = \frac{\alpha_{\mathbf{k}}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} + \frac{\beta_{\mathbf{k}}}{\sqrt{2k}} \left(1 + \frac{i}{k\eta}\right) e^{ik\eta}, \quad (2.13)$$

where $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are the standard Bogoliubov coefficients. The expression (2.11) for z''/z then leads to the following matching conditions on the modes $v_{\mathbf{k}}$ and their derivatives $v'_{\mathbf{k}}$ at the transition:

$$v_{\mathbf{k}}^-(\eta_0) = v_{\mathbf{k}}^+(\eta_0). \quad (2.14)$$

and

$$v_{\mathbf{k}}^{-\prime}(\eta_0) - v_{\mathbf{k}}^{+\prime}(\eta_0) = \frac{3a_0 H_0 \Delta A}{A_+} v_{\mathbf{k}}^+(\eta_0). \quad (2.15)$$

These conditions then allow us to determine the Bogoliubov coefficients $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, which can be obtained to be

$$\alpha_{\mathbf{k}} = 1 + \frac{3i\Delta A}{2A_+} \frac{k_0}{k} \left(1 + \frac{k_0^2}{k^2}\right), \quad (2.16)$$

$$\beta_{\mathbf{k}} = -\frac{3i\Delta A}{2A_+} \frac{k_0}{k} \left(1 + \frac{ik_0}{k}\right)^2 e^{2ik/k_0}, \quad (2.17)$$

where $k_0 \equiv -1/\eta_0 = a_0 H_0$ corresponds to the mode that leaves the Hubble radius at the transition.

One can arrive at the corresponding expressions for the modes $f_{\mathbf{k}}$ and the derivative $f'_{\mathbf{k}}$ before and after the transition from the above expressions for $v_{\mathbf{k}}$ and its time derivative $v'_{\mathbf{k}}$. Before the transition, the mode $f_{\mathbf{k}}$ and the derivative $f'_{\mathbf{k}}$ are given by

$$f_{\mathbf{k}}^+(\eta) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} (1 + i k \eta) e^{-i k \eta}, \quad (2.18)$$

and

$$f_{\mathbf{k}}^{+\prime}(\eta) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} \left[-\mathcal{H} \left(\epsilon_{1+} + \frac{\epsilon_{2+}}{2} \right) (1 + i k \eta) + k^2 \eta \right] e^{-i k \eta}. \quad (2.19)$$

Whereas, after the transition, one finds that

$$f_{\mathbf{k}}^-(\eta) = \frac{i H_0 \alpha_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} (1 + i k \eta) e^{-i k \eta} - \frac{i H_0 \beta_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} (1 - i k \eta) e^{i k \eta} \quad (2.20)$$

and

$$\begin{aligned} f_{\mathbf{k}}^{-\prime}(\eta) &= \frac{i H_0 \alpha_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} \left[-\mathcal{H} \left(\epsilon_{1-} + \frac{\epsilon_{2-}}{2} \right) (1 + i k \eta) + k^2 \eta \right] e^{-i k \eta} \\ &\quad - \frac{i H_0 \beta_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} \left[-\mathcal{H} \left(\epsilon_{1-} + \frac{\epsilon_{2-}}{2} \right) (1 - i k \eta) + k^2 \eta \right] e^{i k \eta}. \end{aligned} \quad (2.21)$$

Note that, unlike the case of the Mukhanov-Sasaki equation (2.8), the governing equation (2.7) for $f_{\mathbf{k}}$ involves only z'/z rather than z''/z . It should also be clear from the above arguments that z'/z will involve the Heaviside step function. This implies that the mode $f_{\mathbf{k}}$ and its derivative $f'_{\mathbf{k}}$ are both continuous at the transition. As we shall discuss in the next section, the most significant contribution to the dominant term in the scalar bi-spectrum in the Starobinsky model shall depend on the mode $f_{\mathbf{k}}$ and the derivative $f'_{\mathbf{k}}$ evaluated *at* the transition. Because of their simpler structure, it proves to be convenient to make use of the expressions (2.18) and (2.19) for the mode $f_{\mathbf{k}}$ and $f'_{\mathbf{k}}$ before the transition. At the transition, these reduce to

$$f_{\mathbf{k}}(\eta_0) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} \left(1 - \frac{i k}{k_0} \right) e^{i k/k_0}, \quad (2.22)$$

and

$$\begin{aligned} f_{\mathbf{k}}^{\prime}(\eta_0) &= -\frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} \left[3 \epsilon_{1+} k_0 \left(1 - \frac{i k}{k_0} \right) + \frac{k^2}{k_0} \right] e^{i k/k_0} \\ &\simeq -\frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} \frac{k^2}{k_0} e^{i k/k_0}, \end{aligned} \quad (2.23)$$

where we have made use of the fact that $\epsilon_{2+} = 4 \epsilon_{1+}$ to obtain the first expression, and have ignored the term involving ϵ_{1+} , as is done in the slow roll approximation, to arrive at the second.

3 The dominant contribution to the scalar bi-spectrum

For simplicity, we shall focus on the equilateral limit in this work. It is well known that, when deviations from slow roll occur, it is the fourth term in the Maldacena formalism that

leads to the dominant contribution to the bi-spectrum [40, 50–56]. In the equilateral limit of our interest, the fourth term, which we shall refer to as $G_4(k)$, is given by

$$G_4(k) = M_{\text{Pl}}^2 [f_{\mathbf{k}}^3(\eta_e) \mathcal{G}_4(k) + f_{\mathbf{k}}^{*3}(\eta_e) \mathcal{G}_4^*(k)], \quad (3.1)$$

where η_e denotes the end of inflation. The quantity $\mathcal{G}_4(k)$ is described by the integral

$$\mathcal{G}_4(k) = 3i \int_{\eta_i}^{\eta_e} d\eta a^3 \epsilon_1 \dot{\epsilon}_2 f_{\mathbf{k}}^{*2} f_{\mathbf{k}}', \quad (3.2)$$

where η_i denotes a very early time, say, when the initial conditions are imposed on the perturbations.

Recall that, in a generic situation, the complete expression for the quantity $\dot{\epsilon}_2$ is given by Eq. (2.5). As we have already discussed, in the Starobinsky model, the first term involving $V_{\phi\phi}$ in the exact expression for $\dot{\epsilon}_2$ leads to a delta function [cf. Eq. (2.6)]. It is then evident from the integral (3.2) that the corresponding contribution will be non-zero only *at* the transition. Actually, the contributions due to all the other terms, i.e. apart from the term involving $V_{\phi\phi}$ in Eq. (2.5), can be evaluated analytically (in this context, see Ref. [39]). However, we shall focus here only on the specific contribution due to the $V_{\phi\phi}$ term in $\dot{\epsilon}_2$, since it is this term that has been found to lead to the linear and indefinite growth on large wavenumbers in the bi-spectrum [62, 63]. We find that, with $\dot{\epsilon}_2$ given by Eq. (2.6), the quantity $\mathcal{G}_4(k)$ can be written as

$$\mathcal{G}_4^0(k) = \frac{i \Delta A A_+ k_0^2}{H_0^6 M_{\text{Pl}}^2} f_{\mathbf{k}}^{*2}(\eta_0) f_{\mathbf{k}}'(\eta_0). \quad (3.3)$$

Towards the end of inflation, i.e. as $\eta \rightarrow 0$, the mode $f_{\mathbf{k}}^-$ simplifies to

$$f_{\mathbf{k}}^-(\eta_e) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}(\eta_e)}} (\alpha_{\mathbf{k}} - \beta_{\mathbf{k}}), \quad (3.4)$$

where $\epsilon_{1-}(\eta_e)$ denotes the value of the first slow roll parameter at late times. Upon using the above two expressions for $\mathcal{G}_4^0(k)$ and $f_{\mathbf{k}}^-(\eta_e)$ in the expression (3.1) for $G_4(k)$, we find that we can write the contribution to the bi-spectrum due to the transition as follows:

$$\begin{aligned} k^6 G_4^0(k) &= -\frac{i \Delta A A_+}{64 M_{\text{Pl}}^6} \frac{k_0}{k} \frac{1}{\sqrt{\epsilon_{1+}^3 \epsilon_{1-}^3(\eta_e)}} \\ &\times \left\{ 3i \left[\left(\alpha_{\mathbf{k}}^2 \tilde{\beta}_{\mathbf{k}} + \alpha_{\mathbf{k}} \tilde{\beta}_{\mathbf{k}}^2 \right) \mathcal{C}(k) + \left(\alpha_{\mathbf{k}}^{*2} \tilde{\beta}_{\mathbf{k}}^* + \alpha_{\mathbf{k}}^* \tilde{\beta}_{\mathbf{k}}^{*2} \right) \mathcal{C}^*(k) \right] \sin\left(\frac{k}{k_0}\right) \right. \\ &- 3 \left[\left(\alpha_{\mathbf{k}}^2 \tilde{\beta}_{\mathbf{k}} - \alpha_{\mathbf{k}} \tilde{\beta}_{\mathbf{k}}^2 \right) \mathcal{C}(k) - \left(\alpha_{\mathbf{k}}^{*2} \tilde{\beta}_{\mathbf{k}}^* - \alpha_{\mathbf{k}}^* \tilde{\beta}_{\mathbf{k}}^{*2} \right) \mathcal{C}^*(k) \right] \cos\left(\frac{k}{k_0}\right) \\ &- i \left[\left(\alpha_{\mathbf{k}}^3 + \tilde{\beta}_{\mathbf{k}}^3 \right) \mathcal{C}(k) + \left(\alpha_{\mathbf{k}}^{*3} + \tilde{\beta}_{\mathbf{k}}^{*3} \right) \mathcal{C}^*(k) \right] \sin\left(\frac{3k}{k_0}\right) \\ &\left. + \left[\left(\alpha_{\mathbf{k}}^3 - \tilde{\beta}_{\mathbf{k}}^3 \right) \mathcal{C}(k) - \left(\alpha_{\mathbf{k}}^{*3} - \tilde{\beta}_{\mathbf{k}}^{*3} \right) \mathcal{C}^*(k) \right] \cos\left(\frac{3k}{k_0}\right) \right\}, \quad (3.5) \end{aligned}$$

where $\tilde{\beta}_{\mathbf{k}} = \beta_{\mathbf{k}} e^{-2ik/k_0}$ and the quantity $\mathcal{C}(k)$ is given by

$$\mathcal{C}(k) = \left(1 + \frac{ik}{k_0} \right)^2. \quad (3.6)$$

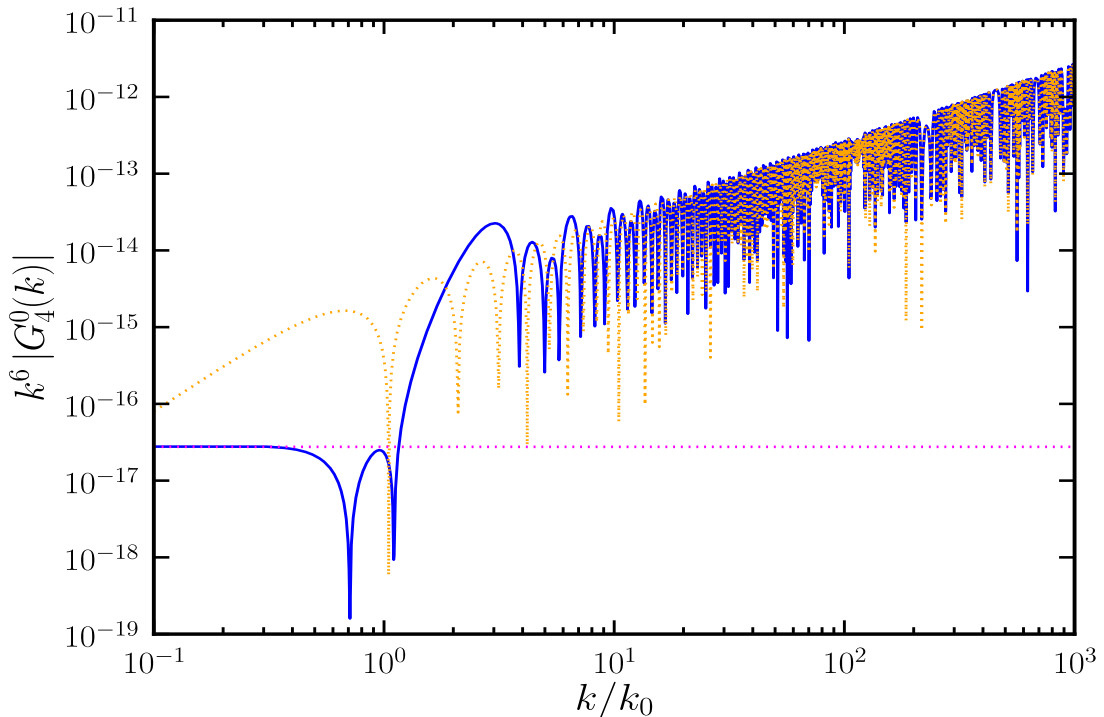


Figure 1. The behavior of the quantity $k^6 |G_4^0(k)|$ [cf. Eq. (3.5)] (in blue) as well as its behavior at small (in magenta) and large (in orange) wavenumbers in the Starobinsky model. The above plot corresponds to the following values of the parameters of the Starobinsky model: $V_0 = 2.36 \times 10^{-12} M_{\text{Pl}}^4$, $A_+ = 3.35 \times 10^{-14} M_{\text{Pl}}^3$, $A_- = 7.26 \times 10^{-15} M_{\text{Pl}}^3$ and $\phi_0 = 0.707 M_{\text{Pl}}$. The linear growth at large wavenumbers is evident.

As $k/k_0 \rightarrow 0$, we find that $G_4^0(k)$ behaves as

$$\lim_{k/k_0 \rightarrow 0} k^6 G_4^0(k) = \frac{-27 \Delta A A_-^3 H_0^6}{8 A_+^5 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3(\eta_e)}}, \quad (3.7)$$

while, in the limit $k/k_0 \rightarrow \infty$, one obtains that

$$\lim_{k/k_0 \rightarrow \infty} k^6 G_4^0(k) = \frac{27 \Delta A H_0^6}{8 A_+^2 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3(\eta_e)}} \frac{k}{k_0} \sin\left(\frac{3k}{k_0}\right). \quad (3.8)$$

In Fig. 1, we have plotted the absolute values of the exact result (3.5) for the quantity k^6 times $G_4^0(k)$ as well as its asymptotic forms (3.7) and (3.8). Note the linear growth with k at large wavenumbers [57, 62, 63]. As we had discussed earlier, one physically expects the bi-spectrum to turn scale invariant for small scale modes that leave the Hubble radius at late times, when slow roll has been reestablished. However, one finds here that the bi-spectrum continues to grow indefinitely with the wavenumber. Evidently, this can be attributed to the fact that the potential contains an infinitely sharp transition, which can be considered to be unphysical, as discussed in the introductory section. As we shall illustrate in the following sections, the indefinite growth disappears as one smoothens the transition.

4 Effects of smoothening the discontinuity: A simple analytical treatment

In this section and the next, we shall analytically consider the effects of smoothening the discontinuity on the scalar bi-spectrum. We shall focus on the contribution due to the fourth term and we shall restrict ourselves to the specific term in ϵ_2 (viz. the one involving $V_{\phi\phi}$) that leads to the indefinite growth in the scalar bi-spectrum.

We shall first study the effects on the scalar bi-spectrum by smoothening the discontinuity in a specific fashion that permits a relatively complete analytical treatment of the problem. Essentially, we shall replace the delta function by one of its conventional representations. Let us write the delta function involved, viz. $\delta^{(1)}(\eta - \eta_0)$, in the following fashion:

$$\delta^{(1)}(\eta - \eta_0) = \begin{cases} 0 & \text{for } \eta < \eta_- \\ \frac{1}{\varepsilon} & \text{for } \eta_- < \eta < \eta_+, \\ 0 & \text{for } \eta > \eta_+, \end{cases} \quad (4.1)$$

where, for convenience, we have set

$$\eta_{\pm} = \eta_0 \pm \frac{\varepsilon}{2}, \quad (4.2)$$

with ε being a small quantity (not to be confused with the slow roll parameters) that determines the width and the height of the transition. Obviously, the limit $\varepsilon \rightarrow 0$ corresponds to the original sharp transition. In other words, instead of a function of infinite height and infinitesimal width, we shall alter the width and height suitably such that the area under the function is unity, as is required. In such a situation, in contrast to the infinitely sharp transition wherein there had existed just two domains, viz. the ones before and after the transition, there now exists a third domain corresponding to the period of the transition. It is then clear that, in the two original domains, i.e. when $\eta < \eta_-$ and $\eta > \eta_+$, we have $z''/z \simeq 2\mathcal{H}^2$, just as we had before. Hence, the earlier solutions for $v_{\mathbf{k}}$, viz. (2.12) and (2.13) continue to remain valid during these domains. However, during the transition, i.e. when $\eta_- < \eta < \eta_+$, we have

$$\frac{z''}{z} \simeq 2\mathcal{H}^2 + \frac{3a_0 H_0 \Delta A}{A_+ \varepsilon}. \quad (4.3)$$

In order to be able to solve for the modes analytically corresponding to the z''/z above and also to be able to evaluate the integral describing the quantity \mathcal{G}_4 [see Eq. (3.2)], we shall assume a few further points. Recall that, the delta function encountered in z''/z arises essentially due to its dependence on ϵ_2 [cf. Eq. (2.11)]. Therefore, by altering the delta function, we have essentially modified the behavior of ϵ_2' during the transition to be

$$\epsilon_2^{0'} = \frac{6\Delta A}{A_+ \varepsilon}. \quad (4.4)$$

In such a case, clearly, during the transition, we would have

$$\epsilon_2^0(\eta) = \gamma(\eta - \eta_-) + 4\epsilon_{1+}, \quad (4.5)$$

where, for convenience, we have set $\gamma = 6\Delta A/(A_+ \varepsilon)$. Actually, such a modification would also result in a change in the behavior of the first slow roll parameter ϵ_1 and, needless to add, the scale factor as well. But, we shall assume that the scale factor continues to behave

as that of de Sitter, and that the first slow roll parameter remains small and largely constant during the transition. As we shall see, these assumptions allow us to arrive at a complete analytical form for the resulting bi-spectrum, with the expected limit as $\varepsilon \rightarrow 0$.

Under the above assumptions, during the transition, the quantity z''/z is given by

$$\frac{z''}{z} \simeq \frac{2}{\eta^2} + \frac{3 a_0 H_0 \Delta A}{A_+ \varepsilon}, \quad (4.6)$$

and the corresponding solution to the Mukhanov-Sasaki equation can be written as

$$v_{\mathbf{k}}^0(\eta) = \frac{\bar{\alpha}_{\mathbf{k}}}{\sqrt{2q}} \left(1 - \frac{i}{q\eta}\right) e^{-iq\eta} + \frac{\bar{\beta}_{\mathbf{k}}}{\sqrt{2q}} \left(1 + \frac{i}{q\eta}\right) e^{iq\eta}, \quad (4.7)$$

where $\bar{\alpha}_{\mathbf{k}}$ and $\bar{\beta}_{\mathbf{k}}$ denote the Bogoliubov coefficients during the transition, while

$$q^2 = k^2 - \frac{3 a_0 H_0 \Delta A}{A_+ \varepsilon}. \quad (4.8)$$

It is important to stress that, since $\Delta A < 0$ for the parameter values of our interest (in this context, see the caption of Fig. 1) and, as a_0 and H_0 are positive quantities, q^2 is a positive definite quantity. The corresponding mode $f_{\mathbf{k}}^0$ and its derivative $f_{\mathbf{k}}^{0'}$ are given by

$$f_{\mathbf{k}}^0(\eta) = \frac{i H_0 \bar{\alpha}_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{q^3} \epsilon_1^0} (1 + iq\eta) e^{-iq\eta} - \frac{i H_0 \bar{\beta}_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{q^3} \epsilon_1^0} (1 - iq\eta) e^{iq\eta}, \quad (4.9)$$

and

$$\begin{aligned} f_{\mathbf{k}}^{0'}(\eta) &= \frac{i H_0 \bar{\alpha}_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{q^3} \epsilon_1^0} \left[-\mathcal{H} \left(\epsilon_1^0 + \frac{\epsilon_2^0}{2} \right) (1 + iq\eta) + q^2 \eta \right] e^{-iq\eta} \\ &\quad - \frac{i H_0 \bar{\beta}_{\mathbf{k}}}{2 M_{\text{Pl}} \sqrt{q^3} \epsilon_1^0} \left[-\mathcal{H} \left(\epsilon_1^0 + \frac{\epsilon_2^0}{2} \right) (1 - iq\eta) + q^2 \eta \right] e^{iq\eta}, \end{aligned} \quad (4.10)$$

where ϵ_1^0 and ϵ_2^0 represent the first two slow roll parameters during the transition.

The expressions for the Bogoliubov coefficients during the transition, viz. $\bar{\alpha}_{\mathbf{k}}$ and $\bar{\beta}_{\mathbf{k}}$, are obtained by matching the modes $v_{\mathbf{k}}$ and their derivatives $v_{\mathbf{k}}'$ on either side at η_- . It should also be clear that the Bogoliubov coefficients after the transition, i.e. $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, will no more be given by the original expressions [viz. Eqs. (2.16) and (2.17)], but will be modified. They are arrived at by matching the modes at η_+ . We find that the Bogoliubov coefficients $\bar{\alpha}_{\mathbf{k}}$ and $\bar{\beta}_{\mathbf{k}}$ are given by

$$\bar{\alpha}_{\mathbf{k}} = \frac{1}{2\eta_-} \frac{1}{(kq)^{3/2}} (k+q) (kq\eta_- + ik - iq) e^{-i(k-q)\eta_-}, \quad (4.11)$$

$$\bar{\beta}_{\mathbf{k}} = -\frac{1}{2\eta_-} \frac{1}{(kq)^{3/2}} (k-q) (kq\eta_- - ik - iq) e^{-i(k+q)\eta_-}. \quad (4.12)$$

The Bogoliubov coefficients, say, $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ in the domain $\eta > \eta_+$ can be calculated to be

$$\begin{aligned} \alpha_{\mathbf{k}} &= \frac{1}{2\eta_+} \frac{1}{(kq)^{3/2}} \left[(k+q) (kq\eta_+ - ik + iq) \bar{\alpha}_{\mathbf{k}} e^{i(k-q)\eta_+} \right. \\ &\quad \left. + (k-q) (kq\eta_+ + ik + iq) \bar{\beta}_{\mathbf{k}} e^{i(k+q)\eta_+} \right], \end{aligned} \quad (4.13)$$

$$\begin{aligned} \beta_{\mathbf{k}} &= \frac{1}{2\eta_+} \frac{1}{(kq)^{3/2}} \left[(k-q) (kq\eta_+ - ik - iq) \bar{\alpha}_{\mathbf{k}} e^{-i(k+q)\eta_+} \right. \\ &\quad \left. + (k+q) (kq\eta_+ + ik - iq) \bar{\beta}_{\mathbf{k}} e^{-i(k-q)\eta_+} \right]. \end{aligned} \quad (4.14)$$

One can easily show that, as $\varepsilon \rightarrow 0$, these expressions simplify to the original expressions, viz. (2.16) and (2.17), for $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$.

Recall that, our aim is to evaluate contribution to the bi-spectrum *during* the transition, when it has been smoothened. It is now a matter of substituting the mode (4.9) and the corresponding derivative (4.10) in the expression (3.2) and evaluating the integral involved from η_- to η_+ . We find that, we can write $\mathcal{G}_4^0(k)$ as

$$\mathcal{G}_4^0(k) = 3i \left[\bar{\alpha}_{\mathbf{k}}^{*3} I_4^0(k) + \bar{\beta}_{\mathbf{k}}^{*3} I_4^{0*}(k) + \bar{\alpha}_{\mathbf{k}}^{*2} \bar{\beta}_{\mathbf{k}}^* J_4^0(k) + \bar{\alpha}_{\mathbf{k}}^* \bar{\beta}_{\mathbf{k}}^{*2} J_4^{0*}(k) \right], \quad (4.15)$$

where the quantities $I_4^0(k)$ and $J_4^0(k)$ are described by the integrals

$$I_4^0(k) = \frac{i H_0 \gamma}{16 M_{\text{Pl}}^3 q^3 \sqrt{q^3} \varepsilon_1^0} \int_{\eta_-}^{\eta_+} \frac{d\eta}{\eta^3} e^{3iq\eta} (1 - iq\eta)^2 \times \left\{ [\gamma (\eta - \eta_-) + 4\varepsilon_{1+}] (1 - iq\eta) + 2q^2 \eta^2 \right\}, \quad (4.16)$$

$$J_4^0(k) = \frac{-i H_0 \gamma}{16 M_{\text{Pl}}^3 q^3 \sqrt{q^3} \varepsilon_1^0} \int_{\eta_-}^{\eta_+} \frac{d\eta}{\eta^3} e^{iq\eta} (1 - iq\eta) \times \left\{ 3 [\gamma (\eta - \eta_-) + 4\varepsilon_{1+}] (1 + q^2 \eta^2) + 2q^2 \eta^2 (1 - iq\eta) + 4q^2 \eta^2 (1 + iq\eta) \right\}. \quad (4.17)$$

We should mention that, in arriving at these integrals, we have ignored the term involving ε_1^0 in the expression for $f_{\mathbf{k}}^{0'}$ [the one within the square brackets in Eq. (4.10)], and we have made use of the expressions (4.4) and (4.5) for $\varepsilon_2^{0'}$ and ε_2^0 , respectively. If we also ignore the term involving ε_2^0 [within the square brackets in Eq. (4.10)], we find that the above integrals can be trivially integrated to arrive at the following results¹:

$$I_4^0(k) = \frac{i H_0 \gamma}{16 M_{\text{Pl}}^3 q^3 \sqrt{q^3} \varepsilon_1^0} e^{3iq\eta_0} \left\{ - \left(\frac{4q^3 \eta_0}{3} + \frac{28iq^2}{9} \right) \sin \left(\frac{3q\varepsilon}{2} \right) + \frac{2iq^3 \varepsilon}{3} \cos \left(\frac{3q\varepsilon}{2} \right) + 2q^2 e^{-3iq\eta_0} [\text{Ei}(3iq\eta_+) - \text{Ei}(3iq\eta_-)] \right\}, \quad (4.18)$$

$$J_4^0(k) = \frac{-i H_0 \gamma}{16 M_{\text{Pl}}^3 q^3 \sqrt{q^3} \varepsilon_1^0} e^{iq\eta_0} \left\{ - (4iq^2 - 4q^3 \eta_0) \sin \left(\frac{q\varepsilon}{2} \right) - 2iq^3 \varepsilon \cos \left(\frac{q\varepsilon}{2} \right) + 6q^2 e^{-iq\eta_0} [\text{Ei}(iq\eta_+) - \text{Ei}(iq\eta_-)] \right\}, \quad (4.19)$$

where $\text{Ei}(x)$ denotes the exponential integral function [66, 67].

The resulting contribution to the bi-spectrum can be obtained by substituting the above results for $I_4^0(k)$ and $J_4^0(k)$ in the expression (4.15) for $\mathcal{G}_4^0(k)$ and, in turn, substituting the resulting $\mathcal{G}_4^0(k)$ in the expression (3.1) and making use of the behavior (3.4) of the modes $f_{\mathbf{k}}^-$ at late times [with $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ now being given by Eqs. (4.13) and (4.14)]. The complete

¹We should clarify a point here. We find that the final results and conclusions we have presented below remain largely unaffected, even if we retain the term involving ε_2^0 .

expression for $G_4^0(k)$ is quite long and unwieldy and, hence, we will not write it down here. However, its form in the limit of $k/k_0 \rightarrow \infty$, which is the behavior of our principal focus, can be arrived at easily. One obtains that

$$\lim_{k/k_0 \rightarrow \infty} k^6 G_4^0(k) = \frac{27 \Delta A H_0^6}{8 A_+^2 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3 (\eta_e)}} \left[\frac{2}{3 \epsilon k} \sin \left(\frac{3 k \epsilon}{2} \right) \right] \frac{k}{k_0} \sin \left(\frac{3 k}{k_0} \right). \quad (4.20)$$

It ought to be highlighted that, in the large k limit, the only additional factor [as compared to the original result (3.8)] that arises due to the smoothening of the transition is the one that appears within the square brackets in the above expression. Note that, the quantity ϵ has dimensions of time, and the width as well as the sharpness of the step are determined by the ratio $\epsilon/|\eta_0|$ or, equivalently, ϵk_0 . If we write $\epsilon = \kappa/k_0$, where κ is a dimensionless quantity, then we arrive at

$$\lim_{k/k_0 \rightarrow \infty} k^6 G_4^0(k) = \frac{27 \Delta A H_0^6}{8 A_+^2 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3 (\eta_e)}} \left[\frac{2 k_0}{3 \kappa k} \sin \left(\frac{3 \kappa k}{2 k_0} \right) \right] \frac{k}{k_0} \sin \left(\frac{3 k}{k_0} \right), \quad (4.21)$$

an expression that can be said to be the first important result of this paper.

Three points need to be emphasized regarding the result we have arrived at above. Firstly, it should be evident that the additional factor [when compared to the original expression (3.8)] reduces to unity in the limit ϵ tends to zero, exactly as is required. This suggests that the assumptions and methods we have adopted to smoothen the step seem reasonable. Secondly, the above expression does not grow indefinitely, but turns finite at large k . It saturates at a scale invariant amplitude that is inversely proportional to the value of κ and, moreover, the quantity turns scale invariant at $k/k_0 \simeq \kappa^{-1}$. In Fig. 2, we have plotted the complete expression for the absolute value of the quantity $k^6 G_4^0(k)$ for different values of κ . A close look clearly indicates that the figure completely corroborates the two points we have made above. Thirdly and lastly, it is interesting to compare the above result with the contribution to the bi-spectrum that arises when all the other terms in $\dot{\epsilon}_2$, i.e. apart from the term involving $V_{\phi\phi}$, are taken into account. In such a situation, one obtains that (in this context, see Eq. (106) of Ref. [39])

$$\lim_{k/k_0 \rightarrow \infty} k^6 \bar{G}_4(k) = \frac{27 \Delta A A_- H_0^6}{8 A_+^3 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3 (\eta_e)}} \cos \left(\frac{3 k}{k_0} \right). \quad (4.22)$$

If one neglects the trigonometric functions that are of order unity in the above two expressions, one finds that

$$\lim_{k/k_0 \rightarrow \infty} \frac{k^6 \bar{G}_4(k)}{k^6 G_4^0(k)} \simeq \frac{3 \kappa}{2} \frac{A_-}{A_+} \simeq \frac{\Delta k}{k_0} \frac{A_-}{A_+}, \quad (4.23)$$

where we have set $\Delta k \simeq \epsilon k_0^2 = \kappa k_0$, at first order in ϵ . This suggests that the contribution originating from the unphysical, growing term can be negligible provided $\Delta k/k_0 \gg A_+/A_-$. In other words, if the transition is sufficiently smooth, then the growing term cannot rise to be too large. For the values of the parameters we have worked with in Figs. 1 and 2, viz. $A_+ = 3.35 \times 10^{-14} M_{\text{Pl}}^3$ and $A_- = 7.26 \times 10^{-15} M_{\text{Pl}}^3$, the condition we have arrived at above leads to $\Delta k/k_0 \simeq \kappa \gg 4.6$. It remains to be seen if this represents a reasonable value in the context of a realistic model. But, in any case, we have explicitly shown here that the bi-spectrum turns scale invariant on suitably small scales, when the step is smoothened.

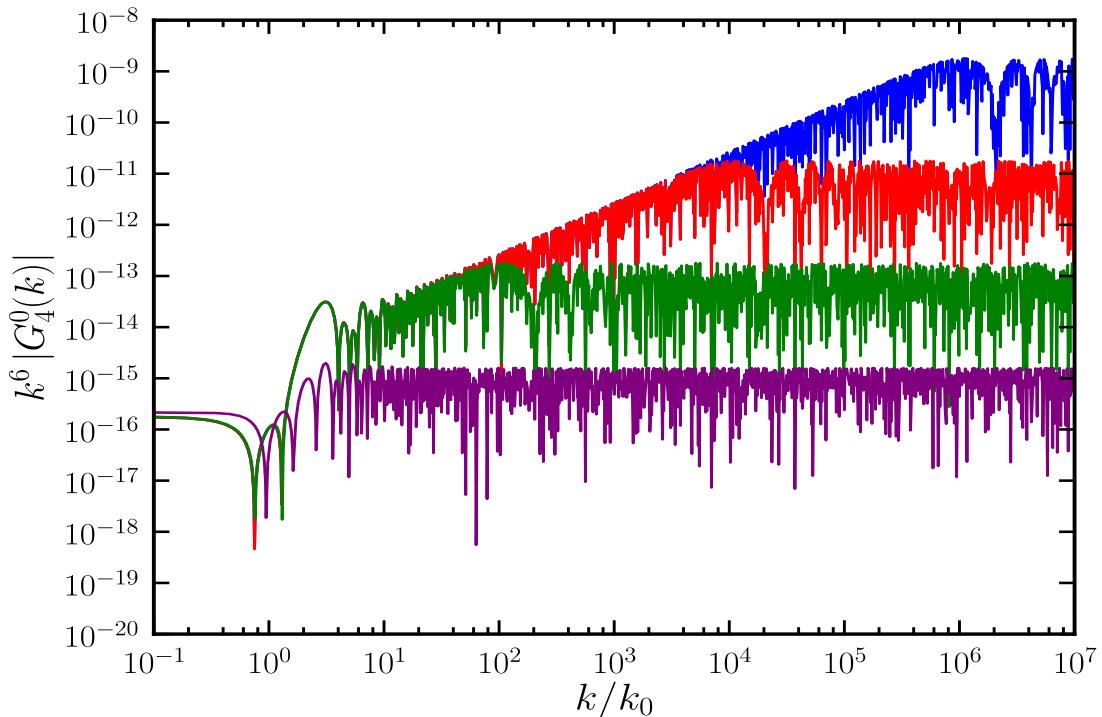


Figure 2. The behavior of the quantity $k^6 |G_4^0(k)|$ in the Starobinsky model, when the delta function is represented by Eq. (4.1). We have worked with the same values for the set of parameters that describe the original Starobinsky model (viz. V_0 , A_+ and A_-) as in the last figure. The different plots correspond to the following values of the dimensionless parameter $\kappa = \varepsilon k_0$: 10^{-6} (in blue), 10^{-4} (in red), 10^{-2} (in green) and 1 (in purple). It is evident that the smoothed step curtails the growth at $k/k_0 \simeq \kappa^{-1}$. Also, it is clear that the ratio of the scale invariant amplitudes at large k/k_0 is inversely proportional to the values of κ , thereby confirming the limiting behavior (4.21) that we had arrived at analytically.

5 Smoothing the transition: A more general treatment

The calculation of the last section was based on a specific representation of the regularized Dirac delta function, as given by Eq. (4.1). One may wonder as to what happens if we alter the fashion in which the transition is smoothed. In particular, it will be interesting to examine whether the plateau at small scales, as seen in Fig. 2, generically appears whenever the transition is no longer infinitely sharp. In the previous section, we had chosen to work with the simple representation (4.1) of the regularized delta function, since it had permitted us to analytically determine the modes during the transition. But, the modes prove to be difficult to obtain for a generic representation of the delta function. However, as we are only interested in the small scale limit of the bi-spectrum, we find that, fortunately, it turns out to be possible to arrive at its behavior analytically in this regime in a simple manner, as we shall now describe.

To begin with, note that, for very large wavenumbers, i.e. in the extreme small scale limit, the modes are completely unaffected by the background. As a result, in this limit, the

modes f_k^0 and their time derivative $f_k^{0'}$ during the transition will be given by

$$f_{\mathbf{k}}^0(\eta) = \frac{-H_0 k \eta}{2 M_{\text{Pl}} \sqrt{k^3} \epsilon_1^0} e^{-i k \eta} \quad (5.1)$$

and

$$f_{\mathbf{k}}^{0'}(\eta) = \frac{i H_0 k^2 \eta}{2 M_{\text{Pl}} \sqrt{k^3} \epsilon_1^0} e^{-i k \eta}, \quad (5.2)$$

which are basically the behavior of the modes *before* the transition. Actually, for the extremely small scale modes, the above form will be valid *even after* the transition. In other words, the form of the modes remain unchanged throughout the evolution. Moreover, it should be clear that this behavior is independent of the details of the transition. Due to this reason, the Bogoliubov transformations turn trivial, with $\bar{\alpha}_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}$ reducing to unity, while $\bar{\beta}_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ vanish. It is straightforward to check that the above-mentioned behavior are indeed satisfied by the modes and the Bogoliubov coefficients for the specific form of the regularized delta function representation considered in the previous section. For instance, as $k \rightarrow \infty$, one finds that $\bar{\alpha}_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}$ [as given by Eqs. (4.11) and (4.13)] reduce to unity, whereas $\bar{\beta}_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ [as given by Eqs. (4.12) and (4.14)] vanish, just as expected. Further, since $q \rightarrow k$ for large k , the mode (4.9) and its derivative (4.10) indeed reduce to the above forms for f_k^0 and $f_k^{0'}$.

Therefore, for the extreme small scale modes, we find that the quantity $\mathcal{G}_4^0(k)$, i.e. the integral characterizing the contribution to the bi-spectrum during the transition, reduces to the following simple form:

$$\mathcal{G}_4^0(k) = \frac{3 H_0 k}{8 M_{\text{Pl}}^3 \sqrt{k^3} \epsilon_1^0} \int_{\eta_-}^{\eta_+} d\eta \eta \epsilon_2^{0'} e^{3 i k \eta}, \quad (5.3)$$

which is essentially the main result of this article. It is important to appreciate the point that this expression for applies to *any* smooth transition in the small scale limit. Note that, $\mathcal{G}_4^0(k)$ is basically the Fourier transform of the combination $\eta \epsilon_2^{0'}$. Moreover, since $\alpha_{\mathbf{k}}$ is unity, while $\beta_{\mathbf{k}}$ vanishes for small scales, the mode $f_{\mathbf{k}}^-(\eta_e)$ at late times [cf. Eq. (3.4)] simplifies to

$$f_{\mathbf{k}}^-(\eta_e) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3} \epsilon_{1-}(\eta_e)}. \quad (5.4)$$

As a consequence, if we can carry out the integral (5.3) describing $\mathcal{G}_4^0(k)$, then, upon using the above expression for $f_{\mathbf{k}}^-(\eta_e)$, we can easily determine the small scale behavior of the quantity $k^6 \mathcal{G}_4^0(k)$. Recall that, according to the representation (4.1) of the delta function that we had considered in the previous section, $\epsilon_2^{0'}$ is a constant during the transition [cf. Eq. (4.4)]. In such a situation, the above integral for $\mathcal{G}_4^0(k)$ turns out to be trivial to evaluate and, if we make use of the late time modes (5.4), we find that we indeed recover the large wavenumber behavior (4.21) that we had mentioned earlier.

5.1 The case with the exponential cut-off

The main advantage of the approach described above to arrive at the small scale behavior of the bi-spectrum should be obvious. We can now make use of the procedure to analytically test the behavior of the bi-spectrum at large wavenumbers on the details of the transition. With

this motivation, let us replace the quantity $\epsilon_2^{0'}$ by the following alternative representation of the original delta function:

$$\epsilon_2^{0'}(\eta) = \frac{6 \Delta A}{A_+} \frac{1}{2 \varepsilon} \exp\left(-\frac{|\eta - \eta_0|}{\varepsilon}\right). \quad (5.5)$$

If we substitute this expression in Eq. (5.3) describing $\mathcal{G}_4^0(k)$, it takes the form

$$\mathcal{G}_4^0(k) = \frac{3 H_0 k}{8 M_{\text{Pl}}^3 \sqrt{k^3 \epsilon_1^0}} \frac{6 \Delta A}{A_+} \frac{1}{2 \varepsilon} \int_{\eta_-}^{\eta_+} d\eta \eta \exp\left(-\frac{|\eta - \eta_0|}{\varepsilon} + 3 i k \eta\right), \quad (5.6)$$

where

$$\eta_{\pm} = \eta_0 \pm r \frac{\varepsilon}{2}, \quad (5.7)$$

as before, represent the boundaries of the transition. However, notice that this definition differs from that of Eq. (4.2). Specifically, we have introduced the dimensionless quantity r to control the duration of the transition, with $r = 1$ leading to the case of the step transition that we had considered in the last section. The justification for the introduction of the additional parameter r being that, with the parameterization (5.5), the duration of the transition is not necessarily related to the height of $\epsilon_2^{0'}$, as it was in the case before. In order for Eq. (5.5) to represent the Dirac delta function faithfully, one should actually choose $\eta_{\pm} = \pm\infty$, viz. the limits wherein $\epsilon_2^{0'}$ vanishes. If this condition is not satisfied, there will arise spurious contributions proportional to $\epsilon_2^{0'}(\eta_{\pm})$. In what follows, we shall also evaluate these contributions (in order to highlight a specific point), but they can always be made negligible by suitably tuning the parameter r . The integral (5.6) can be performed in a straightforward manner, and we obtain that

$$\begin{aligned} \mathcal{G}_4^0(k) = & \frac{3 H_0 k}{8 M_{\text{Pl}}^3 \sqrt{k^3 \epsilon_1^0}} \left\{ \frac{6 \Delta A}{A_+} \left[\frac{-1/k_0}{1 + 9 k^2 \varepsilon^2} + \frac{6 i k \varepsilon^2}{(1 + 9 k^2 \varepsilon^2)^2} \right] e^{-3 i k/k_0} \right. \\ & \left. + \epsilon_2^{0'}(\eta_{\pm}) \left[\frac{\eta_+ e^{3 i k \eta_+}}{3 i k - 1/\varepsilon} - \frac{\eta_- e^{3 i k \eta_-}}{3 i k + 1/\varepsilon} + \frac{e^{3 i k \eta_-}}{(3 i k + 1/\varepsilon)^2} - \frac{e^{3 i k \eta_+}}{(3 i k - 1/\varepsilon)^2} \right] \right\} \quad (5.8) \end{aligned}$$

and, to arrive at this result, we have assumed that the transition is symmetric with $\epsilon_2^{0'}(\eta_+) = \epsilon_2^{0'}(\eta_-)$. The above expression is evidently made of two terms, with the second one, as already mentioned, originating from the non-zero contributions to $\epsilon_2^{0'}$ at the boundaries of the transition.

Having arrived at the above expression, we can now evaluate the corresponding contribution to the bi-spectrum. A somewhat lengthy, but straightforward calculation, leads

to

$$\begin{aligned}
k^6 G_4^0(k) &= \frac{27 \Delta A H_0^6}{8 A_+^2 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3}(\eta_e)} \frac{k}{k_0} \\
&\times \left[\frac{1}{1 + 9 \kappa^2 k^2/k_0^2} \sin\left(\frac{3k}{k_0}\right) + \frac{6 \kappa^2 k/k_0}{(1 + 9 \kappa^2 k^2/k_0^2)^2} \cos\left(\frac{3k}{k_0}\right) \right] \\
&- \frac{9 H_0^6 \kappa k/k_0^2}{16 A_+ M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3}(\eta_e)} \epsilon_2^{0'}(\eta_{\pm}) \left\{ \frac{2}{1 + 9 \kappa^2 k^2/k_0^2} \cos\left(\frac{3r \kappa k}{2k_0}\right) \sin\left(\frac{3k}{k_0}\right) \right. \\
&- \frac{6 \kappa k/k_0}{1 + 9 \kappa^2 k^2/k_0^2} \sin\left(\frac{3r \kappa k}{2k_0}\right) \sin\left(\frac{3k}{k_0}\right) \\
&+ \left. \left[\frac{r \kappa}{1 + 9 \kappa^2 k^2/k_0^2} + \frac{2 \kappa (1 - 9 \kappa^2 k^2/k_0^2)}{(1 + 9 \kappa^2 k^2/k_0^2)^2} \right] \sin\left(\frac{3r \kappa k}{2k_0}\right) \cos\left(\frac{3k}{k_0}\right) \right. \\
&+ \left. \left[\frac{3r \kappa^2 k/k_0}{1 + 9 \kappa^2 k^2/k_0^2} + \frac{12 \kappa^2 k/k_0}{(1 + 9 \kappa^2 k^2/k_0^2)^2} \right] \cos\left(\frac{3r \kappa k}{2k_0}\right) \cos\left(\frac{3k}{k_0}\right) \right\}, \quad (5.9)
\end{aligned}$$

where we should remind that, as earlier, ε has been written as κ/k_0 . It is easy to check from the above expression that, in the limit corresponding to that of the exact Dirac delta function (i.e. as $\varepsilon \rightarrow 0$) one recovers the original result that $k^6 G_4^0(k)$ grows linearly as k/k_0 . Let us now assume that we indeed have a faithful representation of the Dirac function so that $\epsilon_2^{0'}(\eta_{\pm}) = 0$. In such a case, we find the leading contribution at large k to be

$$\lim_{k/k_0 \rightarrow \infty} k^6 G_4^0(k) = \frac{3 \Delta A H_0^6}{8 A_+^2 M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3}(\eta_e)} \frac{k_0}{\kappa^2 k} \sin\left(\frac{3k}{k_0}\right). \quad (5.10)$$

Note that, importantly, we no longer obtain a plateau in the small scale limit, but a term which decreases as k^{-1} . This clearly illustrates the point that the small scale behavior of the bi-spectrum depends on the manner in which the original Dirac delta function and, thereby the transition, is smoothed.

At this stage, a couple of clarifying remarks are in order regarding the result we have obtained above. If there remains a contribution at the boundary of the transition, that is to say, if $\epsilon_2^{0'}(\eta_{\pm})$ are not exactly zero, we find that the corresponding contribution to the bi-spectrum at large k is given by

$$\begin{aligned}
\lim_{k/k_0 \rightarrow \infty} k^6 G_4^0(k) &= \frac{3 H_0^6/k_0}{8 A_+ M_{\text{Pl}}^3 \sqrt{2 \epsilon_{1-}^3}(\eta_e)} \epsilon_2^{0'}(\eta_{\pm}) \left[\sin\left(\frac{3r \kappa k}{2k_0}\right) \sin\left(\frac{3k}{k_0}\right) \right. \\
&- \left. \frac{\kappa r}{2} \cos\left(\frac{3r \kappa k}{2k_0}\right) \cos\left(\frac{3k}{k_0}\right) \right]. \quad (5.11)
\end{aligned}$$

Since this term depends on the wavenumber only through the trigonometric functions, it will lead to a plateau (as in the case of Fig. 2) at very small scales, when it begins to dominate the original term (5.10) which falls as k^{-1} . This will occur at a wavenumber that depends on the overall amplitude which, in turn, depends on r and $\epsilon_2^{0'}(\eta_{\pm})$. Let us assume that κr is small so that the second term in Eq. (5.11) above is negligible. In such a case, upon equating

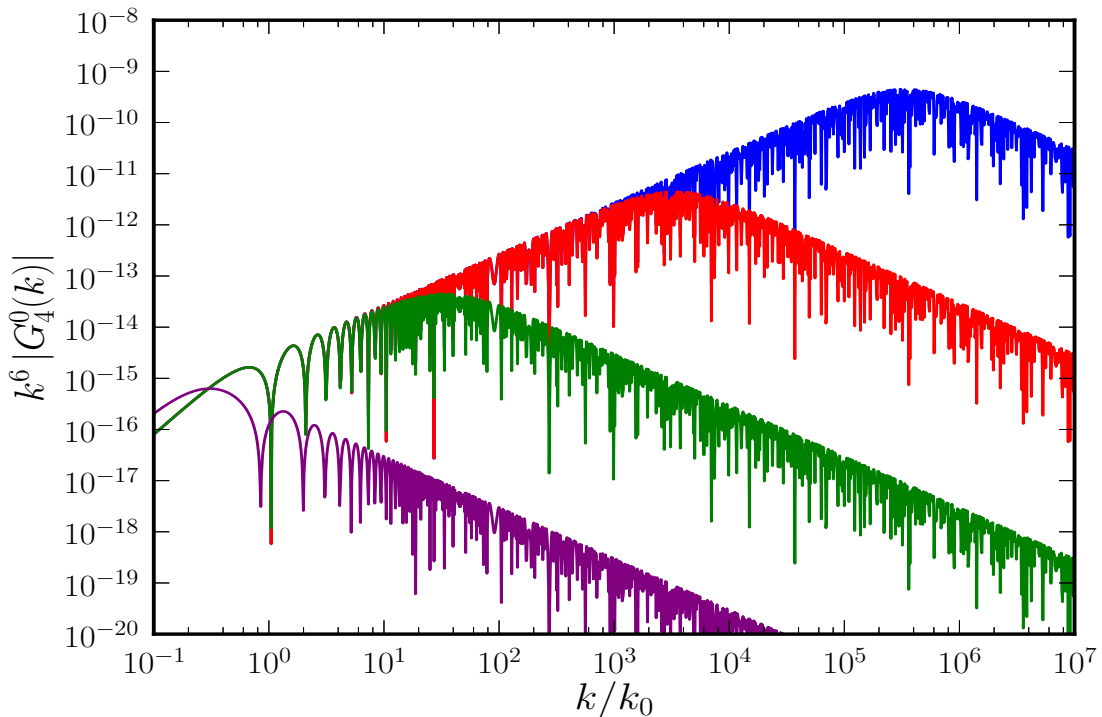


Figure 3. The behavior of the quantity $k^6 |G_4^0(k)|$ in the Starobinsky model, when the transition has been smoothed according to the exponential representation (5.5) of the original delta function. We have worked with the same set of values for the various κ parameters and the same colors to represent the results corresponding to the different values of κ , as in the previous figure. The k^{-1} fall-off at large wavenumbers is evident.

the amplitudes of the first terms in the above two equations, we find that the plateau will occur at the wavenumber of $k/k_0 \simeq \exp(r/2)/(3\kappa)$. Upon plotting the complete analytical result (5.9) for sufficiently large wavenumbers, we do observe the plateau, and we also find that the plateau indeed begins at wavenumbers corresponding to the above estimate. But, it ought to be clear that, physically, the plateau should not be present since $\epsilon_2^{0'}(\eta_{\pm}) = 0$ is necessary in order to have a faithful representation of the Dirac delta function. In summary, with the original Dirac delta function smoothed and represented in terms of an exponential function, we arrive at a behavior wherein the contribution to the bi-spectrum due to the transition falls off as k^{-1} at large wavenumbers. In Fig. 3, we have plotted the corresponding results, which explicitly illustrate this behavior.

5.2 Working with a Gaussian representation

The exponential representation (5.5) of the Dirac delta function has a cusp at η_0 , and it will be worthwhile to extend the above analysis for an even smoother representation. Towards this end, let us replace the delta function appearing in the original $\epsilon_2^{0'}(\eta)$ [cf. Eq. (2.6)] with the following Gaussian representation:

$$\epsilon_2^{0'}(\eta) = \frac{6 \Delta A}{\sqrt{\pi} A_+ \varepsilon} \exp \left[-\frac{(\eta - \eta_0)^2}{\varepsilon^2} \right]. \quad (5.12)$$

In such a case, upon carrying out the integral (5.3) from $\eta_- = \eta_0 - r\varepsilon/2$ to $\eta_+ = \eta_0 + r\varepsilon/2$, and ignoring the contributions due to the end points of the transition (for reasons discussed in the previous subsection), we obtain that

$$\begin{aligned} \mathcal{G}_4^0(k) &= \frac{3H_0 k}{8M_{\text{Pl}}^3 \sqrt{k^3 \epsilon_1^0}} \frac{3\Delta A}{A_+} \left(\frac{3ik\varepsilon^2}{2} - \frac{1}{k_0} \right) e^{-9k^2\varepsilon^2/4} e^{-3ik/k_0} \\ &\quad \times \left[\operatorname{erf} \left(\frac{r}{2} - \frac{3ik\varepsilon}{2} \right) - \operatorname{erf} \left(-\frac{r}{2} - \frac{3ik\varepsilon}{2} \right) \right], \end{aligned} \quad (5.13)$$

where $\operatorname{erf}(z)$ denotes the error function [66, 67]. One can then obtain the corresponding contribution to the bi-spectrum to be

$$\begin{aligned} k^6 G_4^0(k) &= \frac{27\Delta A H_0^6}{32A_+^2 M_{\text{Pl}}^3 \sqrt{2\epsilon_{1-}^3(\eta_e)}} e^{-9k^2\varepsilon^2/4} \\ &\quad \times \left\{ \left[\operatorname{erf} \left(\frac{r}{2} - \frac{3ik\varepsilon}{2} \right) - \operatorname{erf} \left(-\frac{r}{2} - \frac{3ik\varepsilon}{2} \right) \right] \left(\frac{3k^2\varepsilon^2}{2} + \frac{ik}{k_0} \right) e^{-3ik/k_0} \right. \\ &\quad \left. + \left[\operatorname{erf} \left(\frac{r}{2} + \frac{3ik\varepsilon}{2} \right) - \operatorname{erf} \left(-\frac{r}{2} + \frac{3ik\varepsilon}{2} \right) \right] \left(\frac{3k^2\varepsilon^2}{2} - \frac{ik}{k_0} \right) e^{3ik/k_0} \right\}. \end{aligned} \quad (5.14)$$

If we now assume r to be sufficiently large, then we find that this expression simplifies to

$$\begin{aligned} k^6 G_4^0(k) &= \frac{27\Delta A H_0^6}{8A_+^2 M_{\text{Pl}}^3 \sqrt{2\epsilon_{1-}^3(\eta_e)}} e^{-9k^2\varepsilon^2/4} \\ &\quad \times \left[\frac{k}{k_0} \sin \left(\frac{3k}{k_0} \right) + \frac{3k^2\varepsilon^2}{2} \cos \left(\frac{3k}{k_0} \right) \right], \end{aligned} \quad (5.15)$$

which reduces to the original result [viz. Eq. (3.8)] involving the sharp step in the limit $\varepsilon \rightarrow 0$. The reason for assuming r to be sufficiently large is the same as the reason we had attributed in the case of the exponential representation discussed in the previous sub-section. If $\epsilon_2^0(\eta_{\pm})$ do not vanish, then the Gaussian (5.12) ceases to be a faithful representation of the original delta function, and it can then lead to incorrect contributions. In fact, it can be easily established analytically that the immediate sub-leading term (for a finite r) contains an additional Gaussian growth and, when taken into account along with the overall Gaussian suppression encountered above, it leads to a spurious plateau at large wavenumbers, just as in the exponential case. It should now be evident from the examples we have considered that the smoother $\epsilon_2'(\eta)$ is during the transition, the sharper is the cut-off in the scalar bi-spectrum at large wavenumbers.

More generally, it should be clear from the above discussion that, the contribution to the bi-spectrum due to the transition contains a cut-off at large wavenumbers for an arbitrary smooth transition. The specific case that we had considered in the last section wherein the form of smoothening had led to a plateau therefore appears to be a very particular situation. The exact nature of the cut-off, of course, depends on the precise form of the transition (and is therefore not necessarily proportional to k^{-1} or suppressed exponentially), but our analysis unambiguously shows that a cut-off is generically present. As we shall illustrate in the next section, these conclusions are also corroborated by numerical calculations.

6 Comparison with the numerical results from BINGO

Recently, we have developed an efficient and accurate numerical code, called BINGO, to evaluate the scalar bi-spectrum in inflationary models involving the canonical scalar field [57]. In this section, we shall make use of BINGO to numerically investigate the effects of smoothening the sharp transition in the Starobinsky model. In place of the actual potential (2.1), we shall work with the following potentials that have been smoothened in two different fashion:

$$V_1(\phi) = V_0 + \frac{1}{2}(A_+ + A_-)(\phi - \phi_0) + \frac{1}{2}(A_+ - A_-)(\phi - \phi_0) \tanh\left(\frac{\phi - \phi_0}{\Delta\phi}\right), \quad (6.1)$$

$$V_2(\phi) = V_0 + \frac{1}{2}(A_+ + A_-)(\phi - \phi_0) + \frac{1}{2}(A_+ - A_-)\Delta\phi \ln\left[\cosh\left(\frac{\phi - \phi_0}{\Delta\phi}\right)\right], \quad (6.2)$$

both of which, as is required, reduce to the shape of the original potential in the limit $\Delta\phi \rightarrow 0$. Also, it should be clear that, in the above potentials, instead of the original, infinitely sharp transition, the field will make the transition over the width $\Delta\phi$ in field space.

While the details of the numerical procedures to compute the scalar bi-spectrum can be found in our earlier work [57], we believe that a couple of brief and generic remarks are in order at this stage of our discussion. Given a potential and the value of the parameters that describe it, the background evolution is completely determined by the initial conditions on the scalar field. If one further assumes that the perturbations are in the Bunch-Davies vacuum at sufficiently early times, the quantities that characterize the perturbations—such as the power spectrum and the bi-spectrum—are uniquely determined as well. In order to compare with the analytical results we have obtained in the previous section, using BINGO, we numerically compute the contribution to the fourth term of the bi-spectrum, viz. $G_4(k)$, assuming that the quantity $\dot{\epsilon}_2$ [cf. Eq. (2.6)] is determined *only* by the term involving $V_{\phi\phi}$ corresponding to the smoothened potentials (6.1) and (6.2). We work with the same values of the original parameters V_0 , A_+ and A_- that we had considered in the previous three figures, but vary $\Delta\phi$ over a suitable range. In Fig. 4, we have plotted the resulting contribution to the bi-spectrum for a few different values of $\Delta\phi$. It should be clear from the figure that the smoother the transition the more stunted is the growth at large wavenumbers.

At this stage, it is important that we highlight the results we have obtained and also discuss earlier efforts in similar situations. As we had outlined in the introductory section, our main aim had been to illustrate that an indefinite growth in the bi-spectrum is unphysical and that it is related to the infinitely sharp transition that one encounters in the original Starobinsky model. Moreover, we had intended to show that, for any finite and smooth transition, the scale invariance of the bi-spectrum will be restored at suitably large wavenumbers. Evidently, we have been able to establish these two points both analytically and numerically. While it seems natural to expect that the indefinite growth will be suppressed if the transition is smoothened [63], it had not been established earlier. We have been able to explicitly show that this is indeed the case. However, it should be clear from our discussion in the last two sections that, whereas the contribution due to a smooth transition is generically suppressed at large wavenumbers, the details of the suppression depends on the way in which the discontinuity is smoothened.

In fact, we should also emphasize here that there has also been prior efforts in studying the bi-spectrum in similar scenarios. These previous efforts had made use of the so-called generalized slow roll approximation, which is a generic analytical method to study cases involving short periods of fast roll (see, for instance, Refs. [60, 61]). Using the approach, it

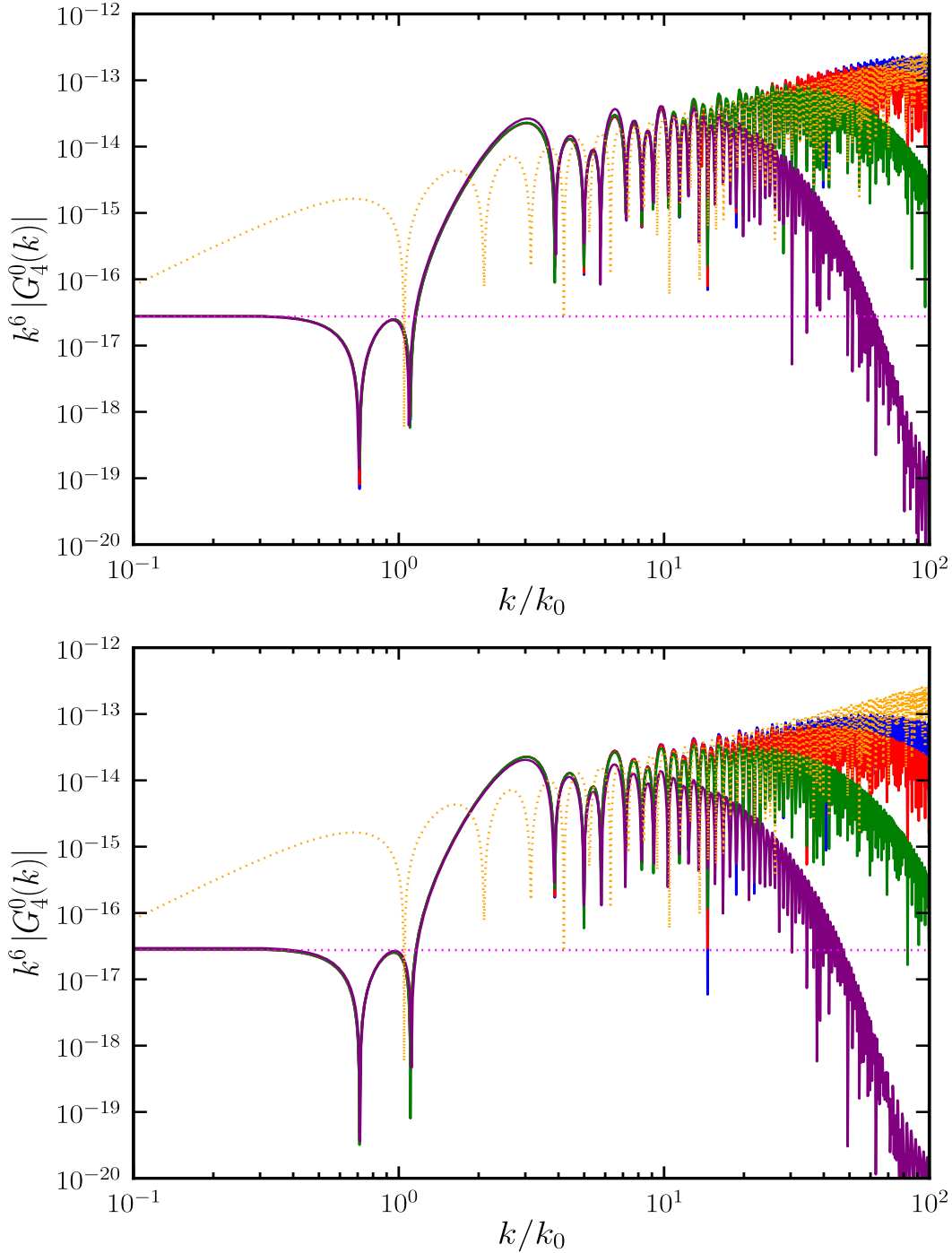


Figure 4. The behavior of the quantity $k^6 |G_4^0(k)|$ with a smoothed Starobinsky model that is described by the potentials (6.1) (on top) and (6.2) (below). These results have been obtained using BINGO, which is a recently developed numerical code to evaluate the scalar bi-spectrum [57]. We have worked with the same values for the parameters V_0 , A_+ and A_- as in the earlier figures, but have varied $\Delta\phi$. The plots correspond to $\Delta\phi/\phi_0$ of 1/7500 (in blue), 1/5000 (in red), 1/2500 (in green) and 1/1000 (in purple). As in Fig. 1, we have also plotted the asymptotic behavior of the analytical result from the original, unsmoothed, Starobinsky model in the limit of small (in magenta) and large (in orange) wavenumbers. It should be clear from the two figures that, as the step is smoothed or, equivalently, the transition is widened, the growth due to term involving $V_{\phi\phi}$ is considerably reduced and its contribution ceases to be important at suitably large k .

has been shown that sufficiently sharp steps in the potential can lead to a strong burst of oscillations before the bi-spectrum turns scale invariant at small scales (in this context, see Refs. [56, 69]). There exist some similarities and certain differences between our work and the earlier efforts based on the generalized slow roll approximation. Evidently, the results from the generalized slow roll approach can be expected to be broadly applicable to the linear potential of our interest here and, in this sense, our effort can be considered to be similar to the prior efforts, but without the constant term V_0 . However, the presence of the constant V_0 in the potential (which we have assumed to be the dominant term) turns out to be important in our approach as it ensures that the background evolution is rather close to that of de Sitter. Moreover, due to this reason (and also because of the linear nature of the potential), the de Sitter modes prove to be a very good approximation to the scalar perturbations². Further, while the bi-spectrum in the earlier efforts was evaluated up to the first order in the generalized slow roll approximation, such a limitation does not apply to our approach. We believe that these three points make the approximations we have adopted work well, as is confirmed by the numerical analysis. But, our approach is specifically designed for the case of the linear potential, dominated by a constant term. In contrast, the generalized slow roll approach can be applied to a wider class of potentials. We feel it will be interesting to examine if the linear growth in the original Starobinsky model can be reproduced, say, at a certain order, in the generalized slow roll approach. Nevertheless, we should stress the fact that the restoration of scale invariance in the scenarios studied using the generalized slow roll approximation [56, 69] corroborate the main conclusions that we have arrived at here.

7 Discussion

It is well known that periods of departure from slow roll will lead to certain features in the inflationary scalar power spectrum corresponding to modes that leave the Hubble radius during the epochs of fast roll. However, in the case of the power spectrum, scale invariance is always restored when slow roll has been reestablished (see, for example, Refs. [70–72]). Such a behavior occurs independent of the sharpness or the extent of the deviation from slow roll. In complete contrast to the behavior of the power spectrum, in the case of the Starobinsky model, it has been found that the abrupt transition that occurs leads to a term which grows linearly at large wavenumbers in the scalar bi-spectrum. Importantly, this occurs despite the fact that slow roll is restored a little while after the field crosses the discontinuity in the Starobinsky model.

Clearly, the continuing growth of the scalar bi-spectrum in the Starobinsky model is an artifact of the infinitely sharp transition, and one would imagine that the bi-spectrum will turn scale invariant at sufficiently small scales if the transition is made smoother. In this work, we have shown, both analytically and numerically, that this expected behavior indeed occurs. Analytically, we have been able to show that a sufficient smoothening of the transition leads to a truncation of the growth and an eventual sharp fall-off at a suitably

²It is well known that a de Sitter background and the de Sitter solutions for the scalar modes are generally a good approximation in most slow roll scenarios. However, in the Starobinsky model, one finds that, since V_0 is the dominant term and the fact that the potential is linear make them particularly good approximations. A large V_0 will, evidently, ensure that the background behaves essentially as that of de Sitter. Before the transition, since ϵ_1 is small and constant, $\epsilon_2 \simeq 4\epsilon_1$ is small and constant as well, while ϵ_3 vanishes. Interestingly, it can be shown that, after the transition, as ϵ_1 continues to remain small (because of the dominant V_0 term) and the potential is linear, certain cancellations occur in the expression for the quantity z''/z as a result of which the scalar modes are described very well by the de Sitter solutions [39].

large wavenumber which is related to the width of the transition. Numerically, we find that, as the sharpness of the transition is decreased, the width of the feature that arises as a result reduces, with the contribution due to the transition ceasing to be important at large wavenumbers. In such situations, the various remaining contributions to the bi-spectrum that we have calculated in some detail earlier (see Ref. [39]) become important.

We should point out here that the restoration of scale invariance of the bi-spectrum at large wavenumbers can occur in two different ways. Evidently, the complete bi-spectrum is the sum of the contribution due to the $V_{\phi\phi}$ term in ϵ'_2 (which had been the focus of our attention here) and the contributions due to all the other terms (that have been calculated earlier in Ref. [39]). The amplitude of the remaining contributions goes to a constant value at small scales [39]. Hence, scale invariance of the complete bi-spectrum is restored if the contribution due to the $V_{\phi\phi}$ term either itself turns a constant or vanishes at large wavenumbers. Our analysis suggests that both cases can arise depending on the manner in which the transition is actually smoothed, i.e. it depends on the microphysics of the transition. We should emphasize that a generic smoothing of the transition does not necessarily lead to an exponential cut-off in the contribution due to the $V_{\phi\phi}$ term, as one might naively be tempted to deduce, possibly guided by the example of particle production presented in the introductory section. But, such a behavior is nevertheless consistent with the scale invariance of the total bi-spectrum at small scales. This is because of the reason that there exist different ways to arrive at a scale invariant behavior at large k , as we have explained above.

Before concluding, it is worthwhile that we touch upon two related points. Firstly, it would be interesting to examine if there can exist conditions under which the power spectrum itself may exhibit the behavior as the scalar bi-spectrum does in the case of the Starobinsky model, i.e. grow indefinitely at large wavenumbers. The second point that is worth considering concerns the implication of such behavior for the higher order correlation functions such as the tri-spectrum. Let us now turn to discuss these two points.

The fact that the discontinuity in the second derivative of the potential leads to the growth in the bi-spectrum suggests that one can expect such a discontinuity in the first derivative of the potential to influence the power spectrum. However, a discontinuity in the first derivative of the potential would imply that the first slow roll parameter itself would grow large, thereby even ending inflation. In such a case, two possibilities can arise. Either, inflation is completely terminated, never to be restored. Or, the departure from the accelerated expansion may be of an extremely short duration and the shape of the potential permits inflation to restart. The former situation does not help, whereas the latter scenario, dubbed punctuated inflation [73, 74], is indeed a genuine possibility. However, inflation reestablished in such situations proves to be of the slow roll type, which also restores scale invariance of the power spectrum.

One can expect that the tri-spectrum will involve one higher order slow parameter beyond the third. If so, then the tri-spectrum can, in fact, be expected to diverge in the case of the Starobinsky model, since the fourth slow roll parameter ϵ_4 would. Actually, not only the tri-spectrum, this conclusion may apply to all the higher order correlations functions as well. For any transition of a finite width, one can expect the tri-spectrum and the higher order correlation functions to exhibit a rather sharp rise for modes that leave the Hubble scale during the transition. These issues are worth studying closer.

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