

Reoptimization of Path Vertex Cover Problem

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Abstract. Most optimization problems are notoriously hard. Considerable efforts must be spent in obtaining an optimal solution to certain instances that we encounter in the real world scenarios. Often it turns out that input instances get modified locally in some small ways due to changes in the application world. The natural question here is, given an optimal solution for an old instance I_O , can we construct an optimal solution for the new instance I_N , where I_N is the instance I_O with some local modifications. Reoptimization of NP-hard optimization problem precisely addresses this concern. It turns out that for some reoptimization versions of the NP-hard problems, we may only hope to obtain an approximate solution to a new instance. In this paper, we specifically address the reoptimization of path vertex cover problem. The objective in k -path vertex cover problem is to compute a minimum subset S of the vertices in a graph G such that after removal of S from G there is no path with k vertices in the graph. We show that when a constant number of vertices are inserted, reoptimizing unweighted k -path vertex cover problem admits a PTAS. For weighted 3-path vertex cover problem, we show that when a constant number of vertices are inserted, the reoptimization algorithm achieves an approximation factor of 1.5, hence an improvement from known 2-approximation algorithm for the optimization version. We provide reoptimization algorithm for weighted k -path vertex cover problem ($k \geq 4$) on bounded degree graphs, which is also an NP-hard problem. Given a ρ -approximation algorithm for k -path vertex cover problem on bounded degree graphs, we show that it can be reoptimized within an approximation factor of $(2 - \frac{1}{\rho})$ under constant number of vertex insertions.

Keywords: Reoptimization · Approximation algorithms · Path vertex cover

1.1 Introduction

Most combinatorial optimization problems are NP-hard. Efficient algorithms to find an optimal solution for such problems are not known. By efficient, we mean running in time polynomial in the input size. Hence, we resort to approximation algorithms which aim to efficiently provide a near-optimal solution. For minimization problems, a ρ -approximation algorithm ($\rho > 1$) efficiently outputs a

solution of cost at most ρ times the optimum, where ρ is called the approximation ratio. A family of $(1+\epsilon)$ approximation algorithms ($\forall \epsilon > 0$) with polynomial running times is called a polynomial time approximation scheme (PTAS).

In many practical applications, the problem instance can arise from small perturbations in the previous instance of an optimization problem. A naive approach is to work on the new problem instance from scratch using known ρ -approximation algorithm. But, with some prior knowledge of the solution for old instance, can we perform better? The computational paradigm of reoptimization addresses this question.

We consider the case where one has devoted a substantial amount of time to obtain an exact solution for the NP-hard optimization problem. Now, the goal is to reoptimize the solution whenever the modified instance is known. A reoptimization problem $Reopt(\pi)$ can be built over any optimization problem π . An input instance for $Reopt(\pi)$ is a triple $(I_N, I_O, OPT(I_O))$, where I_O is an old instance, I_N is a modified instance and $OPT(I_O)$ is an optimal solution for π on I_O .

Suppose I_N is a hard instance obtained via some perturbations in I_O and assume that we have an optimal solution of I_O . The natural question is, can we find an optimal solution of I_N ? In general this may not be the case and we specifically show in Lemma 3 that, if the optimization problem is *path vertex cover* and the perturbation is a single vertex insertion, then even possessing $OPT(I_O)$ does not help to find an optimal solution for I_N efficiently, unless $P = NP$. Hence, the objective of an efficient algorithm for $Reopt(\pi)$ is to either achieve a better approximation ratio or improve the running time of the known approximation algorithm. In this paper, the optimization problem we consider for reoptimization is the path vertex cover problem. This problem has its applications in traffic control and secure communication in wireless networks [7]. We briefly explain the optimization problem below:

A path of order k in a graph is a simple path containing k vertices. For a given graph $G = (V, E)$, $S \subseteq V$ is a feasible k -path vertex cover iff every path of order k in G contains at least one vertex from S . The problem of finding a feasible k -path vertex cover on a graph is known as k -path vertex cover problem (k -PVCP). This problem has two variants: weighted and unweighted. The goal in unweighted k -PVCP is to find a feasible subset of minimum cardinality whereas in weighted k -PVCP, the objective is to find minimum weighted subset of vertices that covers all the paths of order k or more.

1.2 Related Work and Contributions

For any fixed integer $k \geq 2$, the k -path vertex cover problem (k -PVCP) is known to be NP-complete for an arbitrary graph G and also it's NP-hard to approximate it within a factor of 1.3606, unless $P=NP$ [2]. However, unweighted and weighted k -path vertex cover problems on trees have polynomial time algorithms [2] [3]. The problem has been studied in [6] as k -path traversal problem which presents a $\log(k)$ -approximation algorithm for the unweighted version. For

$k = 2$, the k -PVCP corresponds to the conventional vertex cover problem. The 3-PVCP is a dual problem to the dissociation number of the graph. Dissociation number is the maximum cardinality of a subset of vertices that induce a subgraph with maximum degree at most 1. [8] provides a 2-approximation algorithm for weighted 3-PVCP and there is a 3-approximation algorithm for 4-PVCP [4].

For the reoptimization version, G. Ausiello et al. present an algorithm for reoptimizing unweighted vertex cover problem. Following the approach in [5], section 2 shows that reoptimization of unweighted k -PVCP admits a PTAS under the constant number of vertex insertions. In section 3.1, we extend the reoptimization paradigm for weighted vertex cover problem in [5] to weighted k -PVCP. As a use case for the subroutine in section 3.1, we show in section 3.2 that weighted 3-PVCP can be reoptimized with an approximation factor of 1.5 under constant number of vertex insertions. In section 3.3, we present an algorithm for reoptimization version of weighted k -PVCP ($k \geq 4$) on bounded degree graphs under constant number of vertex insertions. For a given ρ -approximation algorithm for weighted k -PVCP ($k \geq 4$), this algorithm achieves an approximation ratio of $(2 - \frac{1}{\rho})$ for such graphs. In Appendix A, we present the hardness results. In Appendix B, we present n and k -approximation algorithms that are used in our algorithm for weighted k -PVCP ($k \geq 5$).

1.3 Preliminaries

In this paper, the graphs we consider are simple undirected graphs. A graph G is a pair of sets (V, E) , where V is the set of vertices and E is the set of edges formed by unordered pairs of distinct vertices in V . For a vertex $v \in V$, we denote the set of neighbours of v in G by $N_G(v)$, where $N_G(v) = \{u \in V \mid (u, v) \in E\}$. For any $S \subseteq V$, we define $N_G(S)$ to be the neighbouring set of S in G , where $N_G(S) \subseteq (V - S)$ and $\forall u \in N_G(S) \exists v \in S$ such that $(u, v) \in E$. For any $S \subseteq V$, we use $G[S]$ to represent the subgraph induced on the vertex set S in G . Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. A degree of a vertex is the number of edges incident on it. We use $\Delta(G)$ to denote the maximum degree of the vertices in graph G . In the case of weighted graphs, with every vertex we associate a positive weight function $f : V \rightarrow \mathbb{R}^+$. For any $v \in V$, let $w(v)$ be the weight of the vertex and for any subset $S \subseteq V$, the weight of the subset $w(S)$ is $\sum_{v \in S} w(v)$. Size of a graph is defined as the number of vertices in it. A constant-size graph is a graph where number of vertices are constant and independent of input parameters of the algorithm. Two graphs are said to be disjoint if they do not share any common vertices.

Let $G = (V, E)$ and $G_A = (V_A, E_A)$ be two graphs where $V \cap V_A = \phi$. Given a set of attachment edges $E^a \subseteq (V \times V_A)$, insertion of G_A into G yields the undirected graph $G' = (V', E')$, where $V' = V \cup V_A$ and $E' = E \cup E_A \cup E^a$. Thus, a constant number of vertex insertions can be realized as a constant-size graph insertion. We define a vertex insertion in G as a special case of graph insertion where the inserted graph G_A is a single vertex $v \notin V[G]$. In general,

we denote $OPT(G)$ as the optimal solution and $ALG(G)$ as the solution output by an algorithm for the corresponding problem on G .

Let π denote the optimization problem and $Reopt(\pi)$ is the reoptimization version of it. The π we consider in this paper is the k -path vertex cover problem. The input instance of $Reopt(\pi)$ is a triple $(G_O, G_N, OPT(G_O))$, where G_O is the old graph, G_N is the new graph and $OPT(G_O)$ is an optimal solution for π on G_O . Let $A_\rho(\pi)$ be a known ρ -approximation algorithm for π . For the algorithms we give, the equality statements are considered as assignment from right to left.

2 Reoptimization of unweighted k -PVCP

Let π be unweighted k -PVCP. We consider the reoptimization version $Reopt(\pi)$ where a constant-size graph $G_A = (V_A, E_A)$ is inserted to the old graph $G_O = (V_O, E_O)$ to yield the new graph $G_N = (V_N, E_N)$. Let $|V_A| = c$. For a given ϵ , we design an algorithm $Unwtd-kpath$ for $Reopt(\pi)$ that outputs $ALG(G_N)$ as a solution.

Algorithm 1 $Unwtd-kpath(G_O, G_N, OPT(G_O), \epsilon)$

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1:  $V_A = V(G_N) - V(G_O)$ 
2:  $c = |V_A|$ 
3:  $m = \lceil c/\epsilon \rceil$ 
4:  $S_1 = V(G_N)$ 
5: for each subset  $X$  of  $V(G_N)$  where  $|X| \leq m$  do
6:   if ( $X$  covers all  $k$ -paths in  $G_N$  and  $|X| < |S_1|$ )
7:      $S_1 = X$ 
8:  $S_2 = OPT(G_O) \cup V_A$ 
9:  $ALG(G_N) = \min(|S_1|, |S_2|)$ 
10: return  $ALG(G_N)$ 

```

Theorem 1. *Unwtd-kpath for $Reopt(\pi)$ under constant-size graph insertion admits a PTAS.*

Proof. Since $OPT(G_N) \cap V(G_O)$ and $OPT(G_O) \cup V_A$ is a feasible k -path vertex cover on G_O and G_N respectively, we get

$$|OPT(G_O)| \leq |OPT(G_N)| \leq |OPT(G_O)| + c \quad \dots (1)$$

If $OPT(G_N)$ has size at most m , it would have been found in step 7 of $Unwtd-kpath$. We know,

$$|ALG(G_N)| \leq |OPT(G_O)| + c = |S_2|$$

and S_2 is picked when $|OPT(G_N)| \geq m \geq \frac{c}{\epsilon}$. Thus, approximation factor for $ALG(G_N)$ using inequality (1) and above observation is,

$$\frac{|ALG(G_N)|}{|OPT(G_N)|} \leq \frac{|OPT(G_O)| + c}{|OPT(G_N)|} \leq \frac{|OPT(G_N)| + c}{|OPT(G_N)|} \leq 1 + \epsilon$$

Further, we analyze the runtime. Enumerating all possible k -paths in a graph of n vertices takes $O(n^k)$ time. Thus for a given set X , we can decide in polynomial time whether all paths of order k are covered by the set. There are $O(n^m)$ subsets of size at most m , where $n = |V_N|$. The runtime of the algorithm is $O(n^m \cdot n^k) = O(n^{\frac{m}{\epsilon}} \cdot n^k)$, and hence a valid PTAS. Note that the runtime can be improved by using color coding algorithm for finding a k -path [1], which runs in $O(2^k n^{O(1)})$ time.

3.1 Subroutine for Reoptimization of Weighted k -PVCP

Let π_k be weighted k -PVCP. $A_\rho(\pi_k)$ be a known ρ -approximation algorithm for π_k . In reoptimization version of the problem $Reopt(\pi_k)$, a new graph G_N is obtained by inserting a graph G_A to G_O .

Definition: A family $\mathcal{F} = \{F_1, F_2, \dots, F_\psi\}$, where $\psi = |\mathcal{F}|$ and of subsets of V_N is called a **good** family if it satisfies the following two properties:

- **Property 1:** $\exists F_i \in \mathcal{F}$ such that $F_i \subseteq OPT(G_N)$ and,
- **Property 2:** $\forall F_i \in \mathcal{F}$, F_i covers all the k -paths which contains at-least one vertex from $V(G_A)$ in graph G_N .

We give below a generic algorithm that works on the good family \mathcal{F} . This family of sets will be constructed in different ways for different problems. The details are provided in the respective sections.

An algorithm for $Reopt(\pi_k)$ constructs the good family \mathcal{F} and feeds it to the subroutine *Construct-Sol*. The algorithm *Construct-Sol* iteratively prepares a solution S_i for each set $F_i \in \mathcal{F}$. The inputs to the algorithm *Construct-Sol* are: modified graph G_N , inserted graph G_A , old optimal solution $OPT(G_O)$, a good family \mathcal{F} and $A_\rho(\pi_k)$.

Algorithm 2 *Construct-Sol*($G_N, G_A, OPT(G_O), \mathcal{F}, A_\rho(\pi_k)$)

- 1: **for** $i = 1$ **to** $|\mathcal{F}|$ **do**
 - 2: $S_i^1 = OPT(G_O) \cup F_i$
 - 3: $G' = G_N[(V_N - V(G_A)) - F_i]$
 - 4: Run $A_\rho(\pi_k)$ on G' and denote the output set as S_i^2
 - 5: $S_i^2 = S_i^2 \cup F_i$
 - 6: $S_i = minWeight(S_i^1, S_i^2)$
 - 7: $ALG(G_N) = minWeight(S_1, S_2, \dots, S_{|\mathcal{F}|})$
 - 8: **return** $ALG(G_N)$
-

Lemma 1. *If $OPT(G)$ is an optimal solution for weighted k -PVCP for G , then for any $S \subseteq OPT(G)$, $w(OPT(G[V - S])) \leq w(OPT(G)) - w(S)$.*

Proof. If F is a feasible k -path cover for $G[V]$, then for any $V^* \subseteq V$, $F \cap V^*$ is a feasible k -path cover for $G[V^*]$.

$OPT(G) - S$ is a feasible solution for $G[V - S]$ because $(V - S) \cap OPT(G) = OPT(G) - S$. Since $S \subseteq OPT(G)$, $w(OPT(G) - S) = w(OPT(G)) - w(S)$. Hence, $w(OPT(G[V - S])) \leq w(OPT(G)) - w(S)$.

Theorem 2. *The algorithm Construct-Sol outputs a solution $ALG(G_N)$ with an approximation factor of $(2 - \frac{1}{\rho})$, running in $O(|V(G_N)|^2 \cdot \psi \cdot T(A_\rho(\pi_k), G_N))$ steps, where ρ is the approximation factor of a known $A_\rho(\pi_k)$.*

Proof. A graph G_A is inserted to G_O to yield the new graph G_N . By property 1 of the good family \mathcal{F} , the optimal solution for G_N must include at least one set in $\mathcal{F} = \{F_1, \dots, F_\psi\}$, where $\psi = |\mathcal{F}|$. Thus, at least one $S_i (1 \leq i \leq \psi)$ is prepared by the subroutine.

Let $OPT(G_N)_i$ be the optimal solution which includes F_i and not $(V(G_A) - F_i)$. We prepare ψ number of solutions for the graph G_N .

S_i^1 is a feasible k -path cover for G_N , where feasibility follows from property 2 of the family. We can write the following inequalities:

$$w(OPT(G_O)) \leq w(OPT(G_N)_i)$$

$$w(S_i^1) = w(OPT(G_O) \cup F_i) \leq w(OPT(G_O)) + w(F_i)$$

From above two inequalities,

$$w(S_i^1) \leq w(OPT(G_N)_i) + w(F_i) \quad \dots (1)$$

Another solution S_i^2 is prepared. From Lemma 1 and construction of S_i^2 , we can write the following inequality:

$$w(S_i^2) \leq \rho(w(OPT(G_N)_i) - w(F_i)) + w(F_i) \quad \dots (2)$$

Since $\rho > 1$, adding $(\rho - 1) \times (1)$ and (2), we get

$$(\rho - 1)w(S_i^1) + w(S_i^2) \leq (2\rho - 1)(w(OPT(G_N)_i))$$

Minimum weighted subset between S_i^1 and S_i^2 is chosen to be S_i . Then,

$$(\rho - 1)w(S_i) + w(S_i) \leq (2\rho - 1)(w(OPT(G_N)_i))$$

$$\implies \forall i \in [1, \psi], w(S_i) \leq (2 - \frac{1}{\rho})(w(OPT(G_N)_i)) \quad \dots (3)$$

We have prepared a set of ψ number of solutions that is, $\{S_1, S_2, \dots, S_\psi\}$. By definition of $OPT(G_N)_i$ and property 2 of good family \mathcal{F} , we get that, if $F_i \subseteq OPT(G_N)$, then $w(OPT(G_N)_i) = w(OPT(G_N))$. By the property 1 of \mathcal{F} , there exists an F_i such that $F_i \subseteq OPT(G_N)_i$. Hence, the following inequality for such an i holds true:

$$w(S_i) \leq (2 - \frac{1}{\rho})(w(OPT(G_N)_i)) = (2 - \frac{1}{\rho})(w(OPT(G_N)))$$

We know, $\forall i \in [1, \psi]$, $w(ALG(G_N)) \leq w(S_i)$. So,

$$w(ALG(G_N)) \leq (2 - \frac{1}{\rho})(w(OPT(G_N)))$$

Thus, algorithm *Construct-Sol* outputs a solution with an approximation factor of $(2 - \frac{1}{\rho})$. Note that step 3 of the algorithm takes $O(|V_N|^2)$ time. Moreover, if the running time of $A_\rho(\pi_k)$ on input graph G_N is $T(A_\rho(\pi_k), G_N)$, then the running time of algorithm *Construct-Sol* is $O(|V(G_N)|^2 \cdot \psi \cdot T(A_\rho(\pi_k), G_N))$.

3.2 Reoptimization of weighted 3-PVCP

Let π_3 be weighted 3-PVCP. A constant-size graph $G_A = (V_A, E_A)$ is inserted to G_O to yield the new graph $G_N = (V_N, E_N)$. Let $T(A_2(\pi_3), G_N)$ denote the runtime of 2-approximation algorithm [8] for π_3 . Let $|V_N| = n$.

Algorithm 3 *Wtd-3pathGI*($G_N, G_A, OPT(G_O), A_2(\pi_3)$)

```

1:  $\mathcal{F} = \phi$ 
2: for all  $X \subseteq (V_A = V(G_A))$  and  $X$  is a 3-path cover for  $G_A$  do
3:    $V_I =$  Set of isolated  $v$  in  $G_A[V_A - X]$  and  $N_{G_N}(v) \cap V(G_O) \neq \phi$ 
4:    $E_I =$  Set of isolated  $(u, v)$  in  $G_A[V_A - X]$  and  $N_{G_N}(\{u, v\}) \cap V(G_O) \neq \phi$ 
5:   for all  $(u, v) \in E_I$  do
6:      $X = X \cup N_{G_O}(\{u, v\})$ 
7:    $Y = N_{G_N}(V_A) - X$ 
8:   for all  $Y' \subseteq Y$ ,  $|Y'| \leq |V_I|$  do
9:     if  $(Y - Y')$  is a 3-path cover for  $G_N[V_I \cup Y]$ 
10:       $X' = X \cup (Y - Y')$ 
11:       $X' = X' \cup N_{G_O}(Y')$ 
12:       $\mathcal{F} = \mathcal{F} \cup \{X'\}$ 
13: return Construct-Sol( $G_N, G_A, OPT(G_O), \mathcal{F}, A_2(\pi_3)$ )

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Theorem 3. *Algorithm Wtd-3path is a 1.5 approximation for Reopt(π_3) under constant-size graph insertion.*

Proof. The algorithm works in 3 phases to construct the good family \mathcal{F} . The algorithm prepares a subset X for each feasible 3-path cover X for G_A because the optimal solution for G_N must contain one subset among the feasible X 's. In the first phase, if an edge in $G_A[V_A - X]$ has a neighbour in G_O , it must be included in X .

Let $Y = N_{G_N}(V_A) - X$. In the second phase, $G_N[V_I \cup Y]$ is made free from all 3-paths by removing a feasible subset $Y - Y'$. In the third phase, the neighbours of the vertices in Y' are included in X' because such a neighbour will form 3-path with the vertices in Y' and V_I .

Since we consider all feasible subsets X and feasible $Y - Y'$ for the corresponding X , the constructed family \mathcal{F} satisfies both the properties of good family. Thus from Theorem 2 and $A_2(\pi_3)$, we get the desired approximation. Further, we analyze the running time. Let $c = |V(G_A)|$. The maximum cardinality of Y is n . Then, the steps 3 to 12 in the algorithm run in $O(n^3 n^{c+1})$ because $|Y'| \leq c$. Thus $|\mathcal{F}| \in O(c^2 \cdot 2^c \cdot n^{c+4})$. Hence the algorithm *Wtd-3path* runs in $O(n^{c+6} \cdot 2^c \cdot T(A_2(\pi_3), G))$.

3.3 Reoptimization of weighted k -PVCP ($k \geq 4$) for bounded degree graphs

A graph free from 2-paths contains only isolated vertices. A graph that does not have any 3-path contains isolated vertices and isolated edges. But, in the case of graphs that are free from k -paths ($k \geq 4$), star graph is a possible component. As the number of subsets needed to be considered for preparation of \mathcal{F} would be exponential in the vertex degree, we restrict the reoptimization of weighted k -PVCP ($k \geq 4$) to bounded degree graphs. Lemma 3 in Appendix A shows that the problem on bounded degree graphs is NP-complete.

The local modification which we consider for reoptimization is constant-size graph insertion. Let $G_O = (V_O, E_O)$ be the old graph. Given G_O , constant-size graph $G_A = (V_A, E_A)$ and attachment edges E^a , the new graph $G_N = (V_N, E_N)$ is obtained. Let $|V_N| = n$ and $|V_A| = c$. Let the maximum degree of the graph G_N be Δ . We use $P_k(G, V')$ to denote the collection of k -paths in graph G containing at least one vertex from $V' \cap V(G)$. For a set of vertices V' , a graph is said to be V' -connected graph if every connected component in the graph contains at least one vertex from V' . Let $A_\rho(\pi_k)$ be a ρ -approximation algorithm for weighted k -PVCP (π_k) and $T(A_\rho(\pi_k), G_N)$ be the running time of $A_\rho(\pi_k)$ on G_N .

Definition: We define a variation of *BFS* on a graph G , where traversal starts by enqueueing a set of vertices V' instead of a single root vertex. Initially, all the vertices in V' are at the same level and unvisited. Now the unvisited nodes are explored in breadth first manner. In this variation, we obtain the *BFS* forest for the input (G, V') , where the vertices of V' are at level 1 and the subsequent levels signify the order in which the vertices are explored.

Consider the *BFS* forest obtained from V_A in G_N . We use L_i to denote the set of vertices at level i ($i \geq 0$) of the *BFS* forest. Let $S_j = \bigcup_{i=0}^j L_i$, where $L_0 = \phi$ and $L_1 = V_A$. Then $L_i = N_{G_N}(S_{i-1})$ for $i \geq 2$. Note that this *BFS* forest has $|V_A|$ number of disjoint *BFS* trees, where the trees have distinct root vertices from V_A .

Lemma 2. *In a BFS forest obtained after performing BFS traversal from a set of vertices $V_A \subseteq V$ in a graph $G = (V, E)$ having no k -paths, the number of vertices at each level is at most $|V_A| \Delta (\Delta - 1)^{\lceil \frac{k-5}{2} \rceil}$.*

Proof. Consider the case when *BFS* is performed from a single vertex set $V_A = \{v_1\}$ to obtain a *BFS* tree. For any level i , $|L_i| \leq \Delta(\Delta - 1)^{i-2}$. Thus the statement holds true for $i \leq \lceil \frac{k-1}{2} \rceil$. For the case when $i > \lceil \frac{k-1}{2} \rceil$, let $j = i - \lceil \frac{k-3}{2} \rceil$. We claim that there exists a vertex v in L_j such that v is a common ancestor for all the vertices in L_i . Assume to contrary that the claim is false. If $|L_i| = 1$ the claim is trivially true. Otherwise we have two distinct vertices v_x and $v_y \in L_i$ such that they have the lowest common ancestor in $L_{j'}$, where $1 \leq j' \leq j - 1 = i - \lceil \frac{k-1}{2} \rceil$. This imposes a path $\langle v_i, \dots, v, \dots, v_j \rangle$ of order $\lceil \frac{k-1}{2} \rceil + 1 + \lceil \frac{k-1}{2} \rceil \geq k$. But it contradicts the fact that G has no paths of order k or more. Hence $|L_i| \leq \Delta(\Delta - 1)^{i-j-1} = \Delta(\Delta - 1)^{\lceil \frac{k-5}{2} \rceil}$.

Now, when *BFS* is performed for the case when $|V_A| > 1$, the *BFS* forest obtained has $|V_A|$ number of disjoint *BFS* trees where each tree satisfies the above argument. Hence the number vertices in each level in the *BFS* forest is at most $|V_A|\Delta(\Delta - 1)^{\lceil \frac{k-5}{2} \rceil}$

Algorithm 4 *Construct-F*($X, V, L, level, V_A, G_N, \mathcal{F}, k$)

```

1:  $\mathcal{F} = \mathcal{F} \cup \{X \cup L\}$ 
2: if  $level \geq k - 1$ 
3:   return  $\mathcal{F}$ 
4:  $b = (|V_A|\Delta(\Delta - 1)^{\lceil \frac{k-5}{2} \rceil})$ 
5: for each non-empty subset  $V'$  of  $L$  and  $|V'| \leq b$  do
6:   if  $G_N[V \cup V']$  is a  $k$ -path free  $V_A$ -connected graph
7:      $X'' = X \cup (L - V')$ 
8:      $V'' = V \cup V'$ 
9:      $L'' = N_{G_N}(V'') - X''$ 
10:     $\mathcal{F} = \text{Construct-}(X'', V'', L'', level + 1, V_A, G_N, \mathcal{F}, k)$ 
11: return  $\mathcal{F}$ 
    
```

Algorithm 5 *Wtd-kpath*($G_N, G_A, OPT(G_O), A_\rho(\pi_k), k$)

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1: Initialization:  $\mathcal{F} = \phi$ ,  $level = 1$ ,  $X = \phi$  and  $V = \phi$ .
2:  $\mathcal{F} = \text{Construct-F}(X, V, V(G_A), level, V(G_A), G_N, \mathcal{F}, k)$ 
3:  $ALG(G_N) = \text{Construct-Sol}(G_N, G_A, OPT(G_O), \mathcal{F}, A_\rho(\pi_k))$ 
4: return  $ALG(G_N)$ 
    
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Theorem 4. *Algorithm Wtd-kpath is a $(2 - \frac{1}{\rho})$ approximation for Reopt(π_k) under graph insertion and runs in $O(n^{O(1)} \cdot 2^{k(\Delta+1)b} \cdot T(A_\rho(\pi_k), G_N))$, where $b = |V_A|\Delta(\Delta - 1)^{\lceil \frac{k-5}{2} \rceil}$ and $((\Delta + 1)b) \in O(\log n)$.*

Proof. We first prove that for every call to the function *Construct-F*($X, V, L, level, V_A, G_N, \mathcal{F}, k$), the following invariants on V , X and L are maintained:

- $V \subseteq S_{level}$ and $G_N[V]$ is a k -path free V_A -connected graph.
- X is the set of neighbours of V in $G_N[S_{level}]$
- L is the set of neighbours of V in graph G_N that are also in $L_{level+1}$, i.e $L = N_{G_N}(V) - S_{level}$.

The above invariants trivially hold true during the first call to the function *Construct-F*. Assuming the invariants to be true during a call to *Construct-F*, we show that the subsequent recursive calls maintain the invariants. Note that the parameter '*level*' is incremented to $level + 1$ during the recursive call. $G_N[V \cup V'] = G_N[V'']$ is a k -path free V_A -connected subgraph in G_N . Also, $V'' \subseteq S_{level+1}$ because $V \subseteq S_{level}$ and $L \subseteq L_{level+1}$. The invariance property of X and L implies $X'' = X \cup (L - V')$ is the set of neighbours of V'' in $G_N[S_{level+1}]$. From previous observation about X'' and V'' , we get that $L'' = N_{G_N}(V'') - X''$ is the set of neighbours of V'' in G_N which are also in $L_{level+2}$. Thus, the invariants are maintained.

Note that X covers all the paths in $P_k(G_N[S_{level}], V_A)$. $X \cup L$ is a k -path cover for G_N because the paths in $P_k(G_N, V_A) - P_k(G_N[S_{level}], V_A)$ contain at least one vertex from L . Thus, $\{X \cup L\}$ is included in \mathcal{F} to satisfy property 2 of good family.

By Lemma 2, it is sufficient to consider non empty subsets V' of size at most $(|V_A| \Delta (\Delta - 1)^{\lceil \frac{k-5}{2} \rceil})$ from subsequent level L to construct V'' . For each recursive call, the case when V' or L is empty is handled in the step 1. A V_A -connected graph that has no k -paths will have a maximum level of $k - 1$ in the *BFS* forest. The algorithm explores all feasible subsets V' for each $level \leq k - 1$. Thus the property 1 of good family holds true for \mathcal{F} , because the family includes all possibilities for $\{X \cup L\}$ that covers $P_k(G_N, V_A)$. Thus, the constructed family \mathcal{F} is indeed a good family.

Let $RT(l)$ be the running time of the function *Construct-F*, where l is the parameter '*level*'. Let $C = \sum_{i=1}^{i=b} \binom{\Delta b}{i}$. Observe that $|L| \leq (\Delta \cdot b)$ due to the construction of L'' in the previous recursion. As we are choosing sets of size at most b from L , we get the recursion $RT(l) = O(n^{O(1)} \cdot C^k \cdot RT(l+1))$ for $1 \leq l \leq (k-1)$ and $RT(k) = O(n^{O(1)})$. Thus step 6 in *Wtd-kpath* runs in $O(n^{O(1)} \cdot C^k)$ time. In each function call, $|\mathcal{F}|$ is incremented by one element. Thus, $|\mathcal{F}| \leq 2^{kb}$ because $|V''| \leq b$ for each level. Note that $C \leq 2^{b\Delta}$. Hence using Theorem 2, the algorithm *Weighted-kpath* runs in $O(n^{O(1)} \cdot C^k \cdot 2^{kb} \cdot T(A_\rho(\pi_k), G_N)) = O(n^{O(1)} \cdot 2^{k(\Delta+1)b} \cdot T(A_\rho(\pi_k), G_N))$ and achieves the desired approximation.

Using 3-approximation algorithm for weighted 4-*PVCP* [4] and Theorem 4, we get the following corollary:

Corollary 1. *Algorithm Wtd-kpath is a $\frac{5}{3}$ approximation for $Reopt(\pi_4)$ under constant-size graph insertion, where $(\Delta) \in O(1)$ and $A_\rho(\pi_4)$ is $A_3(\pi_4)$.*

In Appendix B, we present approximation algorithms for weighted k -path vertex cover problem. The n -approximation algorithm runs in $O(2^k n^{O(1)})$ time, where n is the size of input graph. The k -approximation algorithm is a primal dual based algorithm. Using Theorems 5 and 6 of Appendix, we get the following corollaries:

Corollary 2. *Algorithm Wtd-kpath is a $(2 - \frac{1}{n})$ approximation for $Reopt(\pi_k)$ under constant-size graph insertion, where $(\Delta) \in O(1)$ and $A_\rho(\pi_k)$ is $A_n(\pi_k)$.*

Corollary 3. *Algorithm Wtd-kpath is a $(2 - \frac{1}{k})$ approximation for $Reopt(\pi_k)$ under constant-size graph insertion, where $(\Delta) \in O(1)$ and $A_\rho(\pi_k)$ is $A_k(\pi_k)$.*

Note that the algorithm only explores the vertices till level $k - 1$ that is, the vertices in the set S_{k-1} . Thus $|\mathcal{F}|$ is at most $2^{|S_{k-1}|}$. Therefore the algorithm will also run efficiently for the scenarios where the graph G_A is attached to a 'sparse' part of G_O , that is for $|S_{k-1}| \in O(\log n)$.

Corollary 4. *Algorithm Wtd-kpath is a $(2 - \frac{1}{\rho})$ approximation for $Reopt(\pi_k)$ under graph insertion, where $|S_{k-1}| \in O(\log n)$.*

4 Concluding Remarks

In this paper, we have given a PTAS for reoptimization of unweighted k -PVCP under constant number of vertex insertions. When constant-size graph is inserted to the old graph, we have presented 1.5-approximation algorithm for reoptimization of weighted 3-PVCP. Restricting our inputs to bounded degree graphs, we have presented $\frac{5}{3}$ -approximation for reoptimization of weighted 4-PVCP under constant-size graph insertion. For the reasons we mentioned in Section 3.3, our technique for reoptimization of weighted k -PVCP ($k \geq 4$) cannot be extended to arbitrary graphs. Hence, reoptimization of weighted k -PVCP ($k \geq 4$) for arbitrary graphs under constant number of vertex insertions is an intriguing open problem.

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Appendix

A Reductions

Lemma 3. *Minimum unweighted k -path vertex cover problem on graphs with maximum degree $\Delta(G) \geq 3$ is NP-complete.*

Proof. k -path vertex cover is in NP as enumerating over all paths of order k would verify a k -path vertex cover instance, where run time of verification is $O(n^k)$. We will show it is NP-hard by reducing vertex cover problem for cubic graphs to it, which is known to be NP-complete [9]. Applying the same reduction given in Theorem 1 of [2], for the input instance of a cubic graph G we get the reduced graph instance G' . Since the proof of reduction given in Theorem 2 of [2] is independent of the $\Delta(G)$ and $\Delta(G') = \Delta(G) + 1 = 4$, hence the reduction implies NP-hardness for k -path vertex cover on bounded degree graphs too.

Corollary 5. *Minimum weighted k -path vertex cover for bounded degree graphs is NP-hard.*

Lemma 4. *Unless $P = NP$, reoptimization under vertex insertion of k -path vertex cover problem for bounded degree graphs (π) does not admit a polynomial time optimal algorithm.*

Proof. Let Oracle A output optimal solution for $Reopt(\pi)$ under vertex insertion for bounded degree graphs. Given oracle access to A , we design a polynomial time algorithm B to obtain an optimal solution for problem π on an arbitrary non-empty graph $G = (V, E)$. Here $|V| = n$.

Algorithm 6 $B(\pi, G)$

- 1: Let $\langle v_1, \dots, v_n \rangle$ be an arbitrary sequence of $V(G)$
 - 2: $G_1 = (V_1, E_1) = (\{v_1\}, \phi)$
 - 3: $ALG(G_1) = \phi$
 - 4: **for** i **from** 2 **to** n **do**
 - 5: $G_i = \{V_i, E_i\}$
 - 6: $E_i^a = \{(u, v_i) \mid u \in V(G_{i-1}) \text{ and } (u, v_i) \in E(G)\}$
 - 7: $G_i = (V_{i-1} \cup \{v_i\}, E_{i-1} \cup E_i^a)$
 - 8: $ALG(G_i) = A(G_{i-1}, G_i, ALG(G_{i-1}))$
 - 9: **return** $ALG(G_n)$
-

We will claim by the principle of mathematical induction that $ALG(G_i)$ is the optimal solution for G_i , where $1 \leq i \leq n$. For the base case of $G_1 = (\{v_1\}, \phi)$, ϕ is the optimum for G_1 . For the inductive step, assume $ALG(G_{i-1})$ is optimum for G_{i-1} . Since $G_i = (V_{i-1} \cup \{v_i\}, E_{i-1} \cup E_i^a)$ for $v_i \notin V[G_{i-1}]$ and $E_i^a \subseteq (V[G_{i-1}] \times \{v_i\})$, the change is a valid vertex insertion. Therefore,

algorithm $A(G_{i-1}, G_i, ALG(G_{i-1}))$ indeed outputs the optimal solution for G_i as $ALG(G_i)$, thus proving the induction hypothesis. Next, we claim that $G_n = G$. Clearly $V(G_n) = V(G)$, thus we only need to show $E(G_n) = E(G)$. $\forall e \in E(G_n)$, E_i^a is the only set which contributes to $E(G_n)$, enforcing $e \in E(G)$. For any $(v_i, v_j) \in E(G)$ such that v_i comes before v_j in the vertex sequence, in the j^{th} iteration of loop $(v_i, v_j) \in E_j^a$, thus $(v_i, v_j) \in E(G_n)$. Thus the algorithm B outputs $OPT(G)$. All steps in the algorithm B are polynomial in input size. Hence, we obtain a polynomial time Turing reduction from π to $Reopt(\pi)$ under vertex insertion. Thus, the proof of *Lemma* follows since π is known to be an NP-complete problem (refer Lemma 3 in Appendix A).

B Approximation algorithms for weighted k -PVCP

In this section, we give the approximation algorithms for weighted k -PVCP (π_k) on input graph $G = (V, E)$. Here, $|V| = n$ and subroutine $getKPath(G)$ outputs a path of order k in G if it is present and ϕ otherwise.

Algorithm 7 *Approx-Alg*(π_k, G)

```

1:  $V' = V(G)$ 
2:  $ALG(G) = \phi$ 
3:  $P = getKPath(G[V'])$ 
4: while ( $P \neq \phi$ ) do
5:    $v_m =$  minimum weight vertex in  $P$ 
6:    $ALG(G) = ALG(G) \cup \{v_m\}$ 
7:    $V' = V' - \{v_m\}$ 
8:    $P = getKPath(G[V'])$ 
9: return  $ALG(G)$ 

```

Theorem 5. *Algorithm *Approx-Alg* achieves n -approximation for π_k on graph $G = (V, E)$ and runs in $O(2^k n^{O(1)})$ time, where n is $|V|$.*

Proof. In the i^{th} iteration of while loop, we find a k -path, P_i , by calling subroutine $getKPath$. Let the vertex with minimum weight in P_i be v_m^i . Since, $OPT(G)$ must include at least one vertex from P_i , implying $w(v_m^i) \leq w(OPT(G))$, where $OPT(G)$ is an optimal solution for π_k . Algorithm *Approx-Alg* always terminates because $|ALG(G)|$ is at most $n - k + 1$. The while loop continues to run till $ALG(G)$ is a feasible k -path vertex cover.

$$w(ALG(G)) = \sum_{n=1}^{|ALG(G)|} w(v_m^i) \leq \sum_{n=1}^{|ALG(G)|} w(OPT(G)) \leq (n - k + 1)w(OPT(G))$$

Thus, Algorithm *Approx-Alg* achieves n -approximation for the problem π_k on G . The function $getKPath(G)$ uses color coding algorithm for finding a *path*

on k vertices [1] and runs in $O(2^k n^{O(1)})$. Hence, algorithm *Approx-Alg* runs in time $O(2^k n^{O(1)})$.

Theorem 6. *There is a k -approximation algorithm for weighted k -PVCP (π_k) for graph $G = (V, E)$.*

Proof. Let P_k be the collection of all k -paths in a graph G . Note that $|P_k|$ is at most n^k . For each vertex $v \in V$, we denote subset v_P to be the collection of k -paths containing v . π_k is a special case of set cover problem where P_k is the universe of elements and $V_P = \{v_P | v \in V\}$ is family of subsets. Since each path contains exactly k vertices, the frequency of the element is k in the family of subsets. Thus, we have k -approximation algorithm from Theorem 15.3 in [10].