
On reverse Faber-Krahn inequalities

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Abstract Payne-Weinberger showed that ‘among the class of membranes with given area A , free along the interior boundaries and fixed along the outer boundary of given length L_0 , the annulus $\Omega^\#$ has the highest fundamental frequency,’ where $\Omega^\#$ is a concentric annulus with the same area as Ω and the same outer boundary length as L_0 . We extend this result for the higher dimensional domains and p -Laplacian with $p \in (1, \infty)$, under the additional assumption that the outer boundary is a sphere. As an application, we prove that the nodal set of the second eigenfunctions of p -Laplacian (with mixed boundary conditions) on a ball and a concentric annulus cannot be a concentric sphere.

Keywords Reverse Faber-Krahn inequality · Interior parallels · Isoperimetric inequality · Nagy’s inequalities · First eigenvalue of p -Laplacian · Mixed boundary value problems · Elasticity problems · Non-radiality

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1. Introduction

Let us first recall the famous conjecture by Lord Rayleigh from his book ‘The theory of sound’ [23] published in 1877. He conjectured that ‘among all planar domains Ω of fixed area, the disk is the domain that minimises the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplacian.’ This conjecture was open for a very long time. The first proof of this conjecture was published in 1923 by Faber [8]. In 1925, Krahn [14] gave an independent proof for this conjecture, and later he extended the same to the higher dimension. Now this result is collectively known as the Faber-Krahn inequality, and it states that:

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega) \tag{1}$$

with the equality if and only if Ω is a ball (up to a measure zero set), where Ω^* is a ball of the same measure as Ω . For more on Faber-Krahn inequality and related results, we refer to [18] and [21].

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Among the class of domains having a fixed volume, $\lambda_1(\Omega)$ is not bounded above. For example, consider a class of arbitrary long rectangular domains with the same volume (see Remark 3.13 of [1]). However, if we fix both the surface area and the volume one can give upper bounds for $\lambda_1(\Omega)$, see [16] and [20]. In [19] Payne-Weinberger considered this problem for bounded planar membrane with fixed outer boundary having the interior boundaries along which it is free. They showed that ‘*among the class of membranes with given area A and given length L_0 of the outer boundary, the annulus $\Omega^\#$ has the highest fundamental frequency,*’ where $\Omega^\#$ is a concentric annulus with the same area as Ω and the same outer boundary length as L_0 . In other words, this result can be stated in the form of an inequality as below:

$$\nu_1(\Omega) \leq \nu_1(\Omega^\#), \quad (2)$$

where $\nu_1(\Omega)$, $\nu_1(\Omega^\#)$ are first eigenvalues of the following mixed eigenvalue problems:

$$\left. \begin{array}{l} -\Delta u = \nu u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0; \end{array} \right\} \quad \left. \begin{array}{l} -\Delta u = \nu u \quad \text{in } \Omega^\#, \\ u = 0 \quad \text{on } \Gamma_0^\#, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega^\# \setminus \Gamma_0^\#, \end{array} \right\}$$

where Γ_0 and $\Gamma_0^\#$ are the outer boundaries of Ω and $\Omega^\#$ respectively.

We call the inequality in (2), as the reverse Faber-Krahn (R-F-K) inequality. A similar inequality for the second Neumann eigenvalue (the first non-zero eigenvalue), namely $\mu_2(\Omega) \leq \mu_2(\Omega^*)$ is obtained in [26] by Szegő for planar domains, and in [27] for higher dimensions by Weinberger. The proof of Szegő mainly rely on the conformal mapping, and the proof of Weinberger based on construction of a test function using a radial function together with a suitable translation of the origin. In [19], for proving the inequality (2), authors used the interior parallels and an isoperimetric inequality, deduced from an inequality for the interior parallels due to B. Sz. Nagy [25]. Next, we briefly describe the interior parallels and Nagy’s inequality.

The parallel sets: Let Ω be a bounded domain in \mathbb{R}^2 and Γ_0 be its outer boundary. Let $L(\delta)$ be the measure of the inner parallel to Γ_0 -the set of all points that are inside Γ_0 and at a distance δ from Γ_0 . Nagy showed that the function $L(\delta)$ is defined for almost every δ and it satisfies

$$L(\delta) \leq |\Gamma_0| - 2\pi\delta. \quad (3)$$

Similarly, if $L(\delta)$ is the measure of the outer parallel curve to Γ_1 , an interior boundary; he showed that $L(\delta)$ satisfies

$$L(\delta) \leq |\Gamma_1| + 2\pi\delta. \quad (4)$$

Recall the Steiner formula $L(\delta) = |\Gamma_1| + 2\pi\delta$ for the convex planar domains, (see Chapter 4 of [24]). Indeed, Nagy’s second inequality is an extension of Steiner formula for the non-convex planar domains.

The *interior parallels* are the part of the parallels that lies in Ω . If $l(\delta)$ is the measure of the interior parallel at a distance δ , then $l(\delta) \leq L(\delta)$. The interior parallels and Nagy’s inequalities were used in [15], [16] and [20] for finding the upper bounds for $\lambda_1(\Omega)$ among the class of domains having a fixed volume and surface area. In [19], using Nagy’s inequality (3), authors derived the following isoperimetric inequality:

$$l(\delta)^2 \leq |\Gamma_0|^2 - 4\pi v(\delta), \quad (5)$$

where $v(\delta) = \int_0^\delta l(s) ds$. Further, using $v(\delta)$ and the first eigenfunction of the concentric domain $\Omega^\#$, they have constructed a test function on Ω whose level sets coincide with interior parallels and the Rayleigh quotient is smaller

than $\nu_1(\Omega^\#)$. In [12], Hersch gave another proof for (2) using a class of test functions known as ‘web functions’ (the test function that depends only on the distance from Γ_0), for more on web functions see [3] and [5] and the references therein.

Hersch also considered the mixed eigenvalue problem with the Dirichlet condition on an interior boundary and with the Neumann condition on the rest of the boundaries. By an ingenious parametrization $t(\delta) = \int_0^\delta \frac{ds}{t(s)}$, he related this problem with the fundamental frequency of a vibrating string of infinite length (with varying mass distribution) fixed at one end. Then he used Nagy’s inequality (4) to show that ‘among the class of membranes with given area A and given length L_1 of an inner boundary Γ_1 , the annulus $\Omega_\#$ has the maximal fundamental frequency,’ where $\Omega_\#$ is a concentric annulus with the same area as Ω and the same inner boundary length as L_1 . As before, this can be restated as a R-F-K inequality:

$$\tau_1(\Omega) \leq \tau_1(\Omega_\#), \quad (6)$$

where $\tau_1(\Omega)$, $\tau_1(\Omega_\#)$ are first eigenvalues of the following problems:

$$\left. \begin{array}{l} -\Delta u = \tau u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_1; \end{array} \right\} \quad \left. \begin{array}{l} -\Delta u = \tau u \quad \text{in } \Omega_\#, \\ u = 0 \quad \text{on } \Gamma_1^\#, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega_\# \setminus \Gamma_1^\#. \end{array} \right\}$$

The mixed eigenvalue problem with the Dirichlet condition on both Γ_1, Γ_0 , and with Neumann condition on the rest of the boundaries also studied in [12]. In this case, by an effective use of the ‘effectless cut’ due to Weinberger [28] together with (2) and (6), he showed that ‘among the class of membranes with given area A , given length L_0 of the inner boundary Γ_0 and given length L_1 of the outer boundary Γ_1 satisfying $L_1^2 - L_0^2 = 4\pi A$, the annulus has the highest fundamental frequency λ_1 .’ In particular, an annulus $\Omega = B_1 \setminus \overline{B_0} \subset \mathbb{R}^2$ satisfies this relation. Thus by Hersch’s result:

$$\lambda_1(B_1 \setminus \overline{B_0}) \leq \lambda_1(B_1^* \setminus \overline{B_0^*}). \quad (7)$$

In [22], Ramm and Shivakumar conjectured the inequality (7) for $N \geq 3$, with a numerical justification. In fact, they have conjectured a stronger result - ‘ $\lambda_1(B_1 \setminus \overline{B_0})$ strictly decreases when the inner ball moves towards the outer boundary.’ An analytic proof for this conjecture using an argument of M. Ashbaugh was published later in an arxiv paper (arxiv:math-ph/9911040) by the same authors. At the same time, Harrell et al. [10] and Kesavan [13] independently proved the strict monotonicity of $\lambda_1(B_1 \setminus \overline{B_0})$. All the proofs are based mainly on the Hadamard perturbation formula. In [4], the authors studied the monotonicity of the first Dirichlet eigenvalue of p -Laplacian, defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ for $p \in (1, \infty)$. They showed that ‘ $\lambda_1(B_1 \setminus \overline{B_0})$ decreases when the inner ball B_0 moves towards the boundary of the outer ball B_1 .’ Indeed, this result imply (7) for the p -Laplacian. The strict monotonicity is obtained in [1] and this result asserts that the equality happens in (7) only if the balls B_1 and B_0 are concentric.

In this article, we study the R-F-K inequalities (2) and (6) for the p -Laplacian with $p \in (1, \infty)$ and $N \geq 2$. We consider \mathcal{C}^1 -smooth multiply connected bounded domains in \mathbb{R}^N . For such a domain Ω , we denote the outer boundary by Γ_0 and the boundaries of the interior holes by $\Gamma_1, \gamma_1, \dots, \gamma_n$. We use the symbol $|\cdot|$ for both N dimensional volume measure and the $(N - 1)$ -dimensional surface measure. The Lebesgue measure of the unit ball in \mathbb{R}^N is denoted by ω_N . Further, we set

$$\Omega^* = \text{the ball centred at the origin with } |\Omega^*| = |\Omega|,$$

$\Omega^\# =$ the concentric annulus centred at the origin with the same volume as Ω and the outer surface measure as Γ_0 ,

$\Omega_\# =$ the concentric annulus centred at the origin with the same volume as Ω and the inner surface measure as Γ_1 .

In general, for a fixed domain Ω the sets $\Omega^\#$ and $\Omega_\#$ are different. However, they coincide if Ω satisfies the relation $|\Gamma_0|^{N'} - |\Gamma_1|^{N'} = C(N)|\Omega|$, where N' is the Hölder conjugate of N and $C(N) = N^{N'}\omega_N^{N'-1}$. We mainly consider two types of mixed eigenvalue problems:

- (i) the Dirichlet condition on the outer boundary Γ_0 ,
- (ii) the Dirichlet condition on an inner boundary Γ_1 ,

and the Neumann condition on the rest of the boundaries. More precisely, for $p \in (1, \infty)$ we consider the following eigenvalue problems:

$$\left. \begin{array}{l} -\Delta_p u = \nu |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Gamma_1 \cup (\cup_{i=1}^n \gamma_i); \end{array} \right\} \quad \text{(N-D)} \qquad \left. \begin{array}{l} -\Delta_p u = \tau |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Gamma_0 \cup (\cup_{i=1}^n \gamma_i). \end{array} \right\} \quad \text{(D-N)}$$

For $i = 0, 1$, let $W_{\Gamma_i} := \{u \in W^{1,p}(\Omega) : u|_{\Gamma_i} = 0\}$. We say a real number ν is an eigenvalue of **(N-D)**, if there exists $u \in W_{\Gamma_0} \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \nu \int_{\Omega} |u|^{p-2} uv \, dx, \quad \forall v \in W_{\Gamma_0}.$$

Similarly a real number τ is an eigenvalue of **(D-N)**, if there exists $u \in W_{\Gamma_1} \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \tau \int_{\Omega} |u|^{p-2} uv \, dx, \quad \forall v \in W_{\Gamma_0}.$$

The corresponding nonzero solutions are called the eigenfunctions associated to these eigenvalues. Now consider

$$J(u) := \int_{\Omega} |\nabla u|^p \, dx, \quad \text{and } \mathcal{S} := \left\{ v \in W^{1,p}(\Omega) : \int_{\Omega} |u|^p \, dx = 1 \right\}.$$

For $i = 0, 1$, one can easily see that the critical values of J on $\mathcal{S} \cap W_{\Gamma_i}$ are precisely the eigenvalues of **(N-D)** and **(D-N)** respectively. Thus the classical Ljusternik-Schnirrelman theory, ensures the existence of infinitely many eigenvalues for both **(N-D)** and **(D-N)**, see Proposition B.1. In particular, the first eigenvalues have the following variational characterisation:

$$\nu_1(\Omega) = \inf_{u \in \mathcal{S} \cap W_{\Gamma_0}} J(u), \quad \tau_1(\Omega) = \inf_{u \in \mathcal{S} \cap W_{\Gamma_1}} J(u).$$

Both $\nu_1(\Omega)$ and $\tau_1(\Omega)$ are simple and the first eigenfunctions are of constant sign. Moreover, the eigenfunctions corresponding to $\nu_1(\Omega^\#)$ and $\tau_1(\Omega_\#)$ are radial (see Appendix A.).

The R-F-K inequalities, except (7) for the annular regions, unfortunately did not get any attention for the domains in the higher dimensions and also for the operators different from the Laplacian. Unlike in the case of Faber-Krahn inequality, the uniqueness of the domains for which the equality holds in (2) (similarly in (6)) are not well understood, even for the planar domains. For $N = 2$, we extend the results of Payne-Weinberger and Hersch for the p -Laplacian. For $N \geq 3$, under an additional assumption on the boundary of Ω , we prove the R-F-K inequalities. To state this additional assumption, we make the following definition:

Definition: Let A, B be measurable sets in \mathbb{R}^N . We say that A is a μ -translate of B , if there exists $x \in \mathbb{R}^N$ such that $|(A+x) \triangle B| = 0$, where \triangle is the symmetric difference of sets.

Now we state the R-F-K inequality for ν_1 :

Theorem 1.1 *Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ be a C^1 -smooth multiply connected bounded domain. If the outer boundary Γ_0 of Ω be a sphere, then*

- (i) $\nu_1(\Omega) \leq \nu_1(\Omega^\#)$,
- (ii) if equality happens then Ω is a μ -translate of $\Omega^\#$.

This theorem generalises the result of Payne-Weinberger [19], and proves that ‘among the class of domains with given measure, free along the interior boundaries and fixed along a sphere of given radius as the outer boundary, the concentric annulus $\Omega^\#$ has the highest fundamental frequency’. For example, see below Figure 1.:

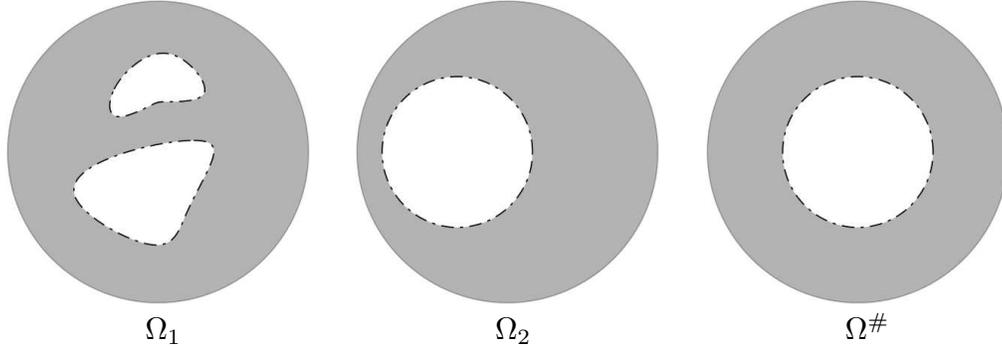


Fig. 1 The domains Ω_1, Ω_2 and $\Omega^\#$ have the same measure and the same outer boundary (up to a translation). Then $\nu_1(\Omega^\#)$ is strictly greater than $\nu_1(\Omega_i)$.

In general, the domain can have any finite number of interior boundaries. Comparing the values of $\nu_1(\Omega_i)$ with the number of interior boundaries is also an interesting problem. In a subsequent article, using the shape derivative technique, we also prove that the $\nu_1(B_1 \setminus B_0)$ strictly decreases when the inner ball B_0 moves towards the boundary of the outer ball B_1 .

We give two proofs for the above theorem. The first one mainly use some ideas from [19] and an isoperimetric inequality of the following form:

$$s(\delta)^{N'} \leq |\Gamma_0|^{N'} - C(N)v(\delta). \quad (8)$$

Even though Nagy’s inequality is not available for higher dimensions, our assumption on Γ_1 allows us to derive the above isoperimetric inequality. In the second proof, we borrow some ideas from [12] and give a proof without using the isoperimetric inequality. We feel that the proofs of Payne-Weinberger and Hersch need more appreciation and may be applicable for other related problems; this is the reason for presenting two proofs for Theorem 1.1.

Next we state the R-F-K inequality for τ_1 :

Theorem 1.2 *Let $p \in (1, \infty)$ and Ω be as in Theorem 1.1. If the inner boundary Γ_1 of Ω be a sphere, then*

- (i) $\tau_1(\Omega) \leq \tau_1(\Omega^\#)$,
- (ii) if equality happens then Ω is a μ -translate of $\Omega^\#$.

For proving the above result, we introduce a parametrization t , analogous to the one in [12] as

$$t(\delta) = \int_0^\delta \frac{dr}{s(r)^{p'-1}}.$$

This parametrization and the eigenfunction corresponding to $\tau_1(\Omega^\#)$ helps us to construct a test function on Ω , whose Rayleigh quotient is smaller than $\tau_1(\Omega^\#)$.

For $N = 2$, as we have Nagy's inequality for a general multiply connected bounded domain, we extend the results of Payne-Weinberger and Hersch for the p -Laplacian with $p \in (1, \infty)$.

Theorem 1.3 *Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^2$ be a C^1 -smooth multiply connected bounded domain. Then*

$$(i) \nu_1(\Omega) \leq \nu_1(\Omega^\#) \quad \text{and} \quad (ii) \tau_1(\Omega) \leq \tau_1(\Omega^\#).$$

The second eigenvalues: In [2], Anoop et al. proved that the eigenfunctions corresponding to the second Dirichlet eigenvalue of p -Laplacian are non-radial. As an application of above theorems, we prove the similar results for the second eigenvalues of p -Laplacian on a ball or an annulus with various boundary conditions. Let $\mu_2(\Omega)$ be the second (first non-zero) Neumann eigenvalue of p -Laplacian; let $\nu_2(\Omega)$ and $\tau_2(\Omega)$ be the second eigenvalues of (N-D) and (D-N) respectively. Now, we have the following results.

Theorem 1.4 *For $N \geq 2$ and $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be a ball or a concentric annulus in \mathbb{R}^N . Then the nodal set of any eigenfunction associated with $\mu_2(\Omega)$ cannot be a concentric sphere.*

Theorem 1.5 *For $N \geq 2$ and $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be a concentric annulus. Then the nodal set of any eigenfunction associated with $\nu_2(\Omega)$ and $\tau_2(\Omega)$ cannot be a concentric sphere.*

This article is organised as follows. In the first section, we briefly illustrate the interior parallels and derive Payne-Weinberger's type isoperimetric inequality. The proofs for Theorem 1.1, Theorem 1.2 and Theorem 1.3 are also given in this section. In the next section, we give the proofs of Theorem 1.4 and Theorem 1.5. In Section 4, we remark on a similar result for the elasticity problem and state a few open problems related to the mixed eigenvalue problems. In the appendix, we discuss some properties of the first eigenfunctions and give the variational characterisation of the second eigenvalues.

2. Proofs of main results

In this section we give proofs for all of our theorems. First, we briefly describe the interior parallels, and then obtain the inequalities analogous to Nagy's inequalities.

The parallel sets: For $N \geq 2$, let Ω be a C^1 -smooth multiply connected bounded domain in \mathbb{R}^N with interior holes, and let Γ_0 be its outer boundary and Γ_1 be an inner boundary. For $\delta \geq 0$, consider the following sets:

$\mathcal{A}_0(\delta)$ = the set of all points in Ω that are at a distance less than δ from Γ_0 ,

$\mathcal{A}_1(\delta)$ = the set of all points in Ω that are at a distance less than δ from Γ_1 ,

$\mathcal{L}_0(\delta) = \partial\mathcal{A}_0(\delta) \setminus \Gamma_0$; $\mathcal{L}_1(\delta) = \partial\mathcal{A}_1(\delta) \setminus \Gamma_1$.

The set $\mathcal{L}_0(\delta)$ is known as an *inner parallel surface* to Γ_0 and $\mathcal{L}_1(\delta)$ is an *outer parallel surface* to Γ_1 . The sets $\mathcal{L}_0(\delta) \cap \Omega$ and $\mathcal{L}_1(\delta) \cap \Omega$ are collectively called as the *interior parallels* in Ω . Since we will be dealing with only one type of parallel surfaces at a time, we use the same notations for the measures of interior parallels:

$$S(\delta) = |\mathcal{L}_i(\delta)|; \quad s(\delta) = |\mathcal{L}_i(\delta) \cap \Omega|, \quad \text{for } i = 0, 1.$$

Thus, for $\Omega \subset \mathbb{R}^2$ Nagy's inequality gives:

$$\begin{aligned} \text{Interior parallels to } \Gamma_0: s(\delta) &\leq |\Gamma_0| - 2\pi\delta \text{ a.e. on } \left[0, \frac{|\Gamma_0|}{2\pi}\right]; \\ \text{Interior parallels to } \Gamma_1: s(\delta) &\leq |\Gamma_1| + 2\pi\delta \text{ a.e. on } [0, \infty). \end{aligned}$$

The mixed eigenvalue problem with Dirichlet condition on Γ_0 :

For this case, we consider the interior parallels to Γ_0 . We make the assumption that Γ_0 is a sphere. Let $|\Gamma_0| = N\omega_N R_1^{N-1}$ and $\Omega^\# = B_{R_1}(0) \setminus \overline{B_{R_0}(0)}$ for some $R_1, R_0 > 0$. Now the following Nagy's type inequality is immediate:

$$S(\delta) = N\omega_N(R_1 - \delta)^{N-1}, \delta \in [0, R_1]. \quad (9)$$

Next we derive an isoperimetric inequality in higher dimension, analogous to that of Payne-Weinberger. First, define

$$\begin{aligned} \delta_\Omega &= \sup\{\delta : \mathcal{L}_0(\delta) \cap \Omega \neq \emptyset\}, \\ v(\delta) &:= \int_0^\delta s(t) dt; \quad V(\delta) := \int_0^\delta S(t) dt. \end{aligned}$$

Observe that $v(\delta) = |\mathcal{A}_0(\delta) \cap \Omega|$, $V(\delta) = |\mathcal{A}_0(\delta)|$, $v(\delta_\Omega) = |\Omega|$ and $\delta_{\Omega^\#} = R_1 - R_0$. Also the map v is differentiable a.e., and $v'(\delta) = s(\delta)$ a.e. on $[0, \delta_\Omega]$. Recall that, for $N \geq 2$, $N' = \frac{N}{N-1}$ and $C(N) = N^{N'} \omega_N^{N'-1}$.

Lemma 2.1 *If Γ_0 is a sphere then*

$$s(\delta)^{N'} \leq |\Gamma_0|^{N'} - C(N)v(\delta), \delta \in [0, \delta_\Omega]. \quad (10)$$

Proof As $s(\delta) \leq S(\delta)$, from (9) we have

$$s(\delta) \leq N\omega_N(R_1 - \delta)^{N-1}, \delta \in [0, \delta_\Omega].$$

By integrating the above inequality from 0 to δ yields

$$v(\delta) \leq \omega_N R_1^N - \omega_N(R_1 - \delta)^N.$$

Now multiply the above inequality by $C(N)$, then simple computations leads to the required inequality as below:

$$C(N)v(\delta) \leq N^{N'} \omega_N^{N'} R_1^N - N^{N'} \omega_N^{N'} (R_1 - \delta)^N = |\Gamma_0|^{N'} - L(\delta)^{N'} \leq |\Gamma_0|^{N'} - s(\delta)^{N'}. \quad (11)$$

□

Remark 2.2 If Ω is a μ -translate of $\Omega^\#$, then $s(\delta) = S(\delta)$ for $\delta \in [R_1, R_0]$ and hence the equality holds in the isoperimetric inequality. i.e.,

$$s(\delta)^{N'} = |\Gamma_0|^{N'} - C(N)v(\delta), \delta \in [0, \delta_\Omega].$$

The next lemma will show that, if Γ_0 is a sphere and the equality holds in (10), then Ω is necessarily a μ -translate of $\Omega^\#$.

Lemma 2.3 *If $\Omega^\#$ is not a μ -translate of Ω , then*

- (i) $R_1 - R_0 < \delta_\Omega$,
- (ii) there exists $\delta_0 \in [0, \delta_\Omega]$ such that $s(\delta)^{N'} < |\Gamma_0|^{N'} - C(N)v(\delta)$ for $\delta \geq \delta_0$.

Proof. (i) Since the outer boundary of Ω is a sphere without loss of generality, we may assume that the outer boundaries of Ω and $\Omega^\#$ are the same. If $R_1 - R_0 \geq \delta_\Omega$ then Ω must be a subset of $B_{R_1}(0) \setminus \overline{B_{R_1 - \delta_\Omega}(0)} \subseteq \Omega^\#$, since $\mathcal{L}_0(\delta) \cap \Omega = \emptyset$ for $\delta > \delta_\Omega$. If the strict inequality holds then the inclusion is strict, and if the equality holds, then Ω must be a μ -translate of $\Omega^\#$. In both cases, we get a contradiction and hence $R_1 - R_0 < \delta_\Omega$.

(ii) If $v = V$ on $[0, \delta_\Omega]$, then $|\Omega| = V(R_1 - R_0) < V(\delta_\Omega) = v(\delta_\Omega) = |\Omega|$. A contradiction and hence there exists

$\delta_0 \in [0, \delta_\Omega]$ such that $v(\delta_0) < V(\delta_0)$. Since $s(\delta) \leq S(\delta)$, we get $v(\delta) < V(\delta)$, for $\delta > \delta_0$. Now the same calculations as in (11), gives the required strict inequality. \square

From Remark 2.2, we have $s(\delta)^{N'} \leq |\Gamma_0|^{N'} - C(N)v(\delta)$, $\delta \in [0, \delta_\Omega]$. Motivated by this inequality, we define a function r as below:

$$C(N)\omega_N r(\delta)^N = |\Gamma_0|^{N'} - C(N)v(\delta), \delta \in [0, \delta_\Omega]. \quad (12)$$

Observe that, $r(0) = R_1$, $r(\delta_\Omega) = R_0$ and r is strictly decreasing on $[0, \delta_\Omega]$. Thus r maps $[0, \delta_\Omega]$ onto $[R_0, R_1]$.

Lemma 2.4 *Let r be defined as above. Then the map r is differentiable and $|r'(\delta)| \leq 1$.*

Proof By differentiating (12) we get $NC(N)\omega_N r(\delta)^{N-1} r'(\delta) = -C(N)s(\delta)$. Therefore,

$$|r'(\delta)|^{N'} = \frac{s(\delta)^{N'}}{N^{N'} \omega_N^{N'} r(\delta)^N} = \frac{s(\delta)^{N'}}{|\Gamma_0|^{N'} - C(N)v(\delta)}.$$

Now the conclusion follows easily from (10). \square

Next we give our first proof for Theorem 1.1, along the same lines in the proof of Payne-Weinberger (Section-II of [19]).

The first proof of Theorem 1.1.

(i) Let $u_0 \in W^{1,p}(\Omega^\#)$ be an eigenfunction corresponding to $\nu_1(\Omega^\#)$. Since u_0 is radial, there exists $\phi \in C^1(\mathbb{R})$ such that $\phi(R_1) = 0$ and $u_0(x) = \phi(|x|)$. Now consider the function $u(x) = \phi(r(\delta(x)))$, for $x \in \Omega$. Observe that $u \in W^{1,p}(\Omega)$ and $u|_{\Gamma_0} = 0$. Further,

$$|\nabla u(x)| = |\phi'(r(\delta(x)))| |r'(\delta)| |\nabla \delta| \leq |\phi'(r(\delta(x)))|, \quad (\text{since } |r'(\delta)| \leq 1, |\nabla \delta| = 1).$$

Therefore,

$$\begin{aligned} \int_{\Omega} |\nabla u(x)|^p dx &\leq \int_{\Omega} |\phi'(r(\delta(x)))|^p dx \\ &= \int_0^{\delta_\Omega} \left(\int_{\mathcal{L}(\delta)} |\phi'(r(\delta(x)))|^p d\sigma \right) d\delta = \int_0^{\delta_\Omega} |\phi'(r(\delta))|^p s(\delta) d\delta. \end{aligned}$$

From (12) we have $N\omega_N r^{N-1} dr = -s(\delta) d\delta$. Now by setting $r(\delta) = r$ we get

$$\int_{\Omega} |\nabla u(x)|^p dx \leq N\omega_N \int_{R_0}^{R_1} |\phi'(r)|^p r^{N-1} dr.$$

Similarly,

$$\int_{\Omega^\#} |\nabla u_0(x)|^p dx = N\omega_N \int_{R_0}^{R_1} |\phi'(r)|^p r^{N-1} dr.$$

Further,

$$\int_{\Omega} |u(x)|^p dx = N\omega_N \int_{R_0}^{R_1} |\phi(r)|^p r^{N-1} dr = \int_{\Omega^\#} |u_0(x)|^p dx.$$

Combining the above inequalities, we conclude that

$$\nu_1(\Omega) \leq \nu_1(\Omega^\#).$$

(ii) If Ω is not a μ -translate of $\Omega^\#$, then by Lemma 2.3 there exists $\delta \in (0, \delta_\Omega)$ such that $s(\delta)^{N'} < |\Gamma_0|^{N'} - C(N)v(\delta)$, $\delta \in (\delta_0, \delta_\Omega)$. Now from Lemma 2.4, we have $|r'(\delta)| < 1$ for $\delta \in (\delta_0, \delta_\Omega)$. Therefore,

$$\int_{\Omega} |\nabla u(x)|^p dx < N\omega_N \int_{R_0}^{R_1} |\phi'(r)|^p r^{N-1} dr = \int_{\Omega^\#} |\nabla u_0(x)| dx.$$

Thus $\nu_1(\Omega) < \nu_1(\Omega^\#)$. □

Next we give another proof of Theorem 1.1, without (explicitly) using the isoperimetric inequality (10). For this, first we prove the following lemma.

Lemma 2.5 *Let $h(\alpha) = s(v^{-1}(\alpha))$ and $H(\alpha) = S(V^{-1}(\alpha))$ for $\alpha \in [0, |\Omega|]$. Then*

(i) $h(\alpha) \leq H(\alpha)$, $\alpha \in [0, |\Omega|]$,

(ii) if Ω is not a μ -translate of $\Omega^\#$, then there exists $\alpha_0 \in [0, |\Omega|]$ such that $h(\alpha) < H(\alpha)$, $\alpha \geq \alpha_0$.

Proof (i) Since $v(\delta) \leq V(\delta)$ and S is monotonically decreasing, we have $v^{-1}(\alpha) \geq V^{-1}(\alpha)$ and hence $S(v^{-1}(\alpha)) \leq S(V^{-1}(\alpha))$. As $s(\delta) \leq S(\delta)$ we get the required conclusion.

(ii) If Ω is not a μ -translate of $\Omega^\#$, then as in the proof of (ii) of Lemma 2.3, there exists a $\delta_0 \in [0, \delta_\Omega]$ such that $v(\delta) < V(\delta)$ for all $\delta \geq \delta_0$. By setting $\alpha_0 = V(\delta_0)$, we obtain the strict inequality as S is strictly decreasing. □

The second proof of Theorem 1.1.

(i) Let $u_0 \in W^{1,p}(\Omega^\#)$ be the eigenfunction corresponding to $\nu_1(\Omega^\#)$ with $u_0 > 0$. Since u_0 is radial, there exists a $\psi \in C^1(\mathbb{R})$ such that $\psi(0) = 0$ and

$$u_0(x) = \psi(R_1 - |x|) = (\psi \circ V^{-1})(V(\delta(x))), \quad x \in \Omega^\#.$$

By setting $\phi = \psi \circ V^{-1}$, we have $u_0(x) = \phi(V(\delta(x)))$. Clearly, $\phi \in C^1(\mathbb{R})$ with $\phi(0) = 0$. Now, define a function u on Ω as $u(x) = \phi(v(\delta(x)))$, $x \in \Omega$. Observe that, $u \in W^{1,p}(\Omega)$, $u|_{\Gamma_0} = 0$, and

$$|\nabla u(x)| = |\phi'(v(\delta(x)))| s(\delta(x)), \text{ a.e. on } \Omega.$$

Therefore,

$$\int_{\Omega} |\nabla u(x)|^p dx = \int_0^{\delta_\Omega} \left(\int_{\mathcal{L}_\delta} |\nabla u(x)|^p d\sigma \right) d\delta = \int_0^{\delta_\Omega} |\phi'(v(\delta))|^p s(\delta)^{p+1} d\delta.$$

Using the change of variable $\alpha = v(\delta)$, we deduce

$$\int_{\Omega} |\nabla u(x)|^p dx = \int_0^{|\Omega|} |\phi'(\alpha)|^p h(\alpha)^p d\alpha.$$

A similar calculation yields,

$$\int_{\Omega^\#} |\nabla u_0(x)|^p dx = \int_0^{|\Omega|} |\phi'(\alpha)|^p H(\alpha)^p d\alpha.$$

Further, $\int_{\Omega^\#} |u_0(x)|^p dx = \int_0^{|\Omega|} \phi(\alpha)^p d\alpha = \int_{\Omega} |u(x)|^p dx$. By Lemma 2.5, $h(\alpha) \leq H(\alpha)$, so we get

$$\int_0^{|\Omega|} |\phi'(\alpha)|^p h(\alpha)^p d\alpha \leq \int_0^{|\Omega|} |\phi'(\alpha)|^p H(\alpha)^p d\alpha.$$

Now, by the variational characterisation of $\nu_1(\Omega)$, we conclude $\nu_1(\Omega) \leq \nu_1(\Omega^\#)$.

(ii) If Ω is not a μ -translate of $\Omega^\#$, then (ii) of Lemma 2.5 and the above calculations gives $\nu_1(\Omega) < \nu_1(\Omega^\#)$. □

Remark 2.6 Notice that, in our case (Γ_0 is a sphere), the equality in the isoperimetric inequality implies Ω is a μ -translate of $\Omega^\#$ and hence $\nu_1(\Omega) = \nu_1(\Omega^\#)$.

2.1 The mixed eigenvalue problem with Dirichlet condition on Γ_1 :

In this case, we consider the interior parallel surfaces from the inner boundary Γ_1 . Let $\delta_\Omega := \sup\{\delta : \mathcal{L}_1(\delta) \cap \Omega \neq \emptyset\}$. We assume that Γ_1 is a sphere. Let $|\Gamma_1| = N\omega_N R_0^{N-1}$ and $\Omega_\# = B_{R_1}(0) \setminus \overline{B_{R_0}(0)}$ for some $R_1, R_0 > 0$. Now, we clearly have the following Nagy's type inequality:

$$S(\delta) = N\omega_N (R_0 + \delta)^{N-1}, \quad \delta \in [0, \delta_\Omega]. \quad (13)$$

For $p \in (1, \infty)$, we define a parametrization t as

$$t(\delta) = \int_0^\delta \frac{dr}{s(r)^{p'-1}}; \quad T(\delta) = \int_0^\delta \frac{dr}{S(r)^{p'-1}}, \quad \delta \in [0, \delta_\Omega],$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p . Let $T_\# := T(R_1 - R_0)$; $t_\Omega := t(\delta_\Omega)$. Notice that t, T and S are strictly increasing and $s(\delta) \leq S(\delta)$. Now consider the maps $g(\alpha) = s(t^{-1}(\alpha))$ and $G(\alpha) = S(T^{-1}(\alpha))$ on $[0, |\Omega|]$.

Lemma 2.7 *Let $p \in (1, \infty)$ and Ω be as in Theorem 1.2. If Γ_1 is a sphere, then*

- (i) $\delta_\Omega \geq R_1 - R_0$ and $T_\# \leq t_\Omega$,
- (ii) $g(\alpha) \leq G(\alpha)$, $\alpha \in (0, T_\#)$,
- (iii) $\int_0^{t_\Omega} g(\alpha)^{p'} d\alpha = \int_0^{T_\#} G(\alpha)^{p'} d\alpha = |\Omega|$.

Proof (i) If $\delta_\Omega < R_1 - R_0$, then Ω must be a μ -translate of a subset of the annulus $B_{R_0+\delta_\Omega}(0) \setminus \overline{B_{R_0}(0)}$. A contradiction and hence $\delta_\Omega \geq R_1 - R_0$. Now as $T(\delta) \leq t(\delta)$, we get $T_\# \leq t_\Omega$.

(ii) For $\alpha \in [0, T_\#]$, clearly $t^{-1}(\alpha) \leq T^{-1}(\alpha)$ and hence

$$g(\alpha) = s(t^{-1}(\alpha)) \leq S(t^{-1}(\alpha)) \leq S(T^{-1}(\alpha)) = G(\alpha).$$

(iii) By changing the variable $\alpha = t(\delta)$, we get

$$\int_0^{t_\Omega} g(\alpha) d\alpha = \int_0^{t_\Omega} s(t^{-1}(\alpha))^{p'} d\alpha = \int_0^{\delta_\Omega} s(\delta)^{p'} \frac{1}{s(\delta)^{p'-1}} d\delta = \int_0^{\delta_\Omega} s(\delta) d\delta = |\Omega|.$$

Similarly $\int_0^{T_\#} G(\alpha)^{p'} d\alpha = |\Omega_\#| = |\Omega|$. □

Lemma 2.8 *Let Ω be as in Theorem 1.2. If Ω is not a μ translate of $\Omega_\#$, then*

- (i) $\delta_\Omega > R_1 - R_0$,
- (ii) $T_\# < t_\Omega$,
- (iii) there exists $\alpha_0 \subset [0, T_\#]$ such that $g(\alpha) < G(\alpha)$, $\alpha \in (\alpha_0, T_\#]$.

Proof (i) By the previous lemma, $\delta_\Omega \geq R_1 - R_0$. If $\delta_\Omega = R_1 - R_0$, then Ω must be a μ -translate of the annulus $\Omega_\#$. Hence $\delta_\Omega > R_1 - R_0$.

(ii) Follows from (i) as T and t are strictly increasing.

(iii) If $t = T$ on $[0, T_\#]$, then $s = S$ a.e. on $[0, T_\#]$ and this implies that $|\Omega| = \int_0^{R_1-R_0} S(r) dr \leq \int_0^{R_1-R_0} s(r) dr < \int_0^{\delta_\Omega} s(r) dr = |\Omega|$. A contradiction and hence there exists a $\delta_0 \in [0, \delta_\Omega]$ such that $t(\delta_0) > T(\delta_0)$. Since $s(\delta) \leq S(\delta)$, we get $t(\delta) > T(\delta)$, for $\delta > \delta_0$. By setting $\alpha_0 = v(\delta_0)$, we obtain the required result. □

Proof of Theorem 1.2.

(i) Let u_1 be a non-negative eigenfunction corresponding to $\tau_1(\Omega_\#)$. Since u_1 is radial in $\Omega_\#$, there exists $\psi \in C^1(\mathbb{R})$ such that $\psi(0) = 0$ and

$$u_1(x) = \psi(|x| - R_0) = \psi \circ T^{-1}(T(\delta(x))) = \phi(T(\delta(x))), \quad x \in \Omega_\#,$$

where $\phi = \psi \circ T^{-1}$. Notice that u_1 has its maximum on the outer boundary Γ_0 , and hence

$$\phi(T_\#) \geq \phi(\alpha), \quad \alpha \in [0, T_\#]. \quad (14)$$

Now, define a function u on Ω as

$$u(x) = \begin{cases} \phi(t(\delta(x))), & t(\delta(x)) \in [0, T_\#], \\ \phi(T_\#), & t(\delta(x)) \in (T_\#, t_\Omega]. \end{cases}$$

Clearly, $u \in W^{1,p}(\Omega)$ and $u|_{\Gamma_1} = 0$. Further,

$$\nabla u(x) = \begin{cases} \frac{\phi'(t(\delta(x)))}{s(\delta(x))^{p'-1}} \nabla \delta(x), & t(\delta(x)) \in [0, T_\#], \\ 0, & t(\delta(x)) \in (T_\#, t_\Omega]. \end{cases}$$

Therefore,

$$\begin{aligned} \int_{\Omega} |\nabla u(x)|^p dx &= \int_0^{t^{-1}(T_\#)} \left(\int_{\mathcal{L}(\delta)} |\nabla u(t(\delta(x)))|^p d\sigma \right) d\delta \\ &= \int_0^{t^{-1}(T_\#)} |\phi'(t(\delta))|^p \frac{d\delta}{s(\delta)^{p'-1}}. \end{aligned}$$

By the change of variable $\alpha = t(\delta)$, we get $\int_{\Omega} |\nabla u(x)|^p dx = \int_0^{T_\#} |\phi'(\alpha)|^p d\alpha$.

Further, we obtain

$$\begin{aligned} \int_{\Omega_\#} |\nabla u_1(x)|^p dx &= \int_0^{T_\#} |\phi'(\alpha)|^p d\alpha, \quad \int_{\Omega_\#} |u_1(x)|^p dx = \int_0^{T_\#} |\phi(\alpha)|^p G(\alpha)^{p'} d\alpha, \\ \int_{\Omega} |u(x)|^p dx &= \int_0^{T_\#} |\phi(\alpha)|^p g(\alpha)^{p'} d\alpha + |\phi(T_\#)|^p \int_{T_\#}^{t_\Omega} g(\alpha)^{p'} d\alpha. \end{aligned}$$

Using the facts $g(\alpha) \leq G(\alpha)$ and $\phi(t) \leq \phi(T_\#)$ (by Lemma 2.7 and (14)), we estimate the following:

$$\begin{aligned} \int_{\Omega_\#} |u_1(x)|^p dx - \int_{\Omega} |u(x)|^p dx &= \int_0^{T_\#} |\phi(\alpha)|^p \left(G(\alpha)^{p'} - g(\alpha)^{p'} \right) d\alpha - \int_{T_\#}^{t_\Omega} |\phi(T_\#)|^p g(\alpha)^{p'} d\alpha \\ &\leq |\phi(T_\#)|^p \left(\int_0^{T_\#} \left(G(\alpha)^{p'} - g(\alpha)^{p'} \right) d\alpha - \int_{T_\#}^{t_\Omega} g(\alpha)^{p'} d\alpha \right) = 0. \end{aligned}$$

Therefore, we obtain $\int_{\Omega_\#} |u_1(x)|^p dx \leq \int_{\Omega} |u(x)|^p dx$. Now the above inequalities and the variational characterisation of τ_1 gives

$$\tau_1(\Omega) \leq \tau_1(\Omega_\#).$$

(ii) If Ω is not a μ -translate of $\Omega_\#$, then from Lemma 2.8, there exists $\alpha_0 \in [0, T_\#]$ such that $g(\alpha) < G(\alpha)$, $\alpha \in (\alpha_0, T_\#]$. Therefore, $\int_{\Omega_\#} |u_1(x)|^p dx < \int_{\Omega} |u(x)|^p dx$ and hence $\tau_1(\Omega) < \tau_1(\Omega_\#)$. \square

Remark 2.9 Consider the annular domains of the form $B \setminus \overline{B_0} \subset \mathbb{R}^N$ with $N \geq 2$. For $p \in (1, \infty)$, the first eigenvalues $\nu_1(B \setminus \overline{B_0})$ and $\tau_1(B \setminus \overline{B_0})$ are maximum only if the balls are concentric.

2.2 The mixed eigenvalue problems in dimension 2:

For both the annular regions $\Omega^\#$ and $\Omega_{\#}$, we denote the lengths of the interior parallels by $\bar{S}(\delta)$.

Proof of Theorem 1.3.

(i) In this case, we consider the interior parallels from Γ_0 . Thus we have $\bar{S}(\delta) = |\Gamma_0| - 2\pi\delta$. Further, we define

$$\bar{V}(\delta) = \int_0^\delta \bar{S}(t) dt, \quad \delta \in [0, R_1 - R_0]; \quad \bar{H}(\alpha) = \bar{S}(\bar{V}^{-1}(\alpha)), \quad \alpha \in [0, |\Omega|],$$

As before, we obtain $\bar{S}(\delta)^2 = |\Gamma_0|^2 - 4\pi\bar{V}(\delta)$, $\delta \in [0, R_1 - R_0]$ and this yields

$$\bar{H}(\alpha)^2 = |\Gamma_0|^2 - 4\pi\alpha, \quad \alpha \in [0, |\Omega|].$$

From Nagy's inequality, we have $s(\delta) \leq \bar{S}(\delta)$ and this gives the following isoperimetric inequality

$$s(\delta)^2 \leq |\Gamma_0|^2 - 4\pi v(\delta), \quad \delta \in [0, \delta_\Omega].$$

Therefore, $h(\alpha)^2 \leq \bar{H}(\alpha)^2$, $\alpha \in [0, |\Omega|]$. Now the proof follows using a similar set of arguments as in the second proof of part (i) of Theorem 1.1.

(ii) For this case, we consider the interior parallels from Γ_1 . Thus we have $\bar{S}(\delta) = |\Gamma_1| + 2\pi\delta$, $\delta \in (0, \delta_\Omega)$ and Nagy's inequality gives $s(\delta) \leq \bar{S}(\delta)$. Now define a parametrization as

$$\bar{T}(\delta) := \int_0^\delta \frac{dt}{\bar{S}(t)^{p'-1}}, \quad \delta \in [0, \delta_\Omega].$$

Let $\bar{G}(\alpha) = \bar{S}(\bar{T}^{-1}(\alpha))$, $\alpha \in [0, T_\#]$. Now from (ii) of Lemma 2.7, we get $g(\alpha) \leq \bar{G}(\alpha)$, $\alpha \in [0, T_\#]$. Now the rest of the proof is same as the proof of part (i) of Theorem 1.2. □

3. Applications

In this section we prove Theorem 1.4 and Theorem 1.5. First, we give a variational characterisation of the second eigenvalue. For this, let

$$\mathcal{M} := \mathcal{S} \cap \left\{ u \in W^{1,p}(\Omega) : \int_\Omega |u|^{p-2}u = 0 \right\},$$

$$\mathcal{F}_2 := \{h(S^1) : h \text{ is an odd continuous map from } S^1 \text{ into } \mathcal{S} \cap W_{\Gamma_0}\},$$

$$\mathcal{G}_2 := \{h(S^1) : h \text{ is an odd continuous map from } S^1 \text{ into } \mathcal{S} \cap W_{\Gamma_1}\}.$$

Then

$$\mu_2(\Omega) = \inf_{u \in \mathcal{M}} J(u), \quad \nu_2(\Omega) = \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} J(u), \quad \tau_2(\Omega) = \inf_{\mathcal{A} \in \mathcal{G}_2} \sup_{u \in \mathcal{A}} J(u).$$

Proof of Theorem 1.4. By the translation invariance of p -Laplacian, we can take $\Omega = B_{R_1}(0) \setminus \overline{B_{R_0}(0)}$ for some $0 \leq R_0 < R_1 < \infty$. Let u be the eigenfunction associated with $\mu_2(\Omega)$, then u must change its sign in Ω . Suppose the nodal set $\{x \in \Omega : u(x) = 0\}$ of u is a sphere of radius $r \in (R_0, R_1)$ centred at origin. Then we have

$$\tau_1(B_r(0) \setminus \overline{B_{R_0}(0)}) = \mu_2(\Omega); \quad \nu_1(B_{R_1}(0) \setminus \overline{B_r(0)}) = \mu_2(\Omega).$$

Now for $s \in (0, r - R_0)$, by Theorem 1.1 and Theorem 1.2 we have

$$\tau_1(B_r(se_1) \setminus \overline{B_{R_0}(0)}) \leq \mu_2(\Omega) \text{ and } \nu_1(B_{R_1}(0) \setminus \overline{B_r(se_1)}) < \mu_2(\Omega).$$

Let ϕ_1 and ϕ_2 be positive eigenfunctions corresponding to $\tau_1(B_r(se_1) \setminus \overline{B_{R_0}(0)})$ and $\nu_1(B_{R_1}(0) \setminus \overline{B_r(se_1)})$ respectively, with the normalisation $\int |\phi_1|^{p-1} = 1 = \int |\phi_2|^{p-1}$. Let $\tilde{\phi}$ denotes the extension of ϕ to Ω by zero. Then $\phi = \tilde{\phi}_1 - \tilde{\phi}_2 \in W^{1,p}(\Omega)$ with $\int_{\Omega} |\phi|^{p-2} \phi = 0$, and

$$\begin{aligned} J(\phi) &= \int_{\Omega} |\nabla \phi|^p = \int |\nabla \phi_1|^p + \int |\nabla \phi_2|^p \\ &= \tau_1(B_r(se_1) \setminus \overline{B_{R_0}(0)}) \int |\phi_1|^p + \nu_1(B_{R_1}(0) \setminus \overline{B_r(se_1)}) \int |\phi_2|^p \\ &< \mu_2(\Omega) \left(\int |\phi_1|^p + \int |\phi_2|^p \right) = \mu_2(\Omega) \int_{\Omega} |\phi|^p. \end{aligned}$$

This contradicts the variational characterisation of $\mu_2(\Omega)$.

Proof of Theorem 1.5. We give the result only for $\nu_2(\Omega)$, a similar proof holds for $\tau_2(\Omega)$. By the translation invariance of p -Laplacian, we can take $\Omega = B_{R_1}(0) \setminus \overline{B_{R_0}(0)}$ for some $0 < R_0 < R_1 < \infty$. Let u be an eigenfunction associated with $\nu_2(\Omega)$. Suppose the nodal set $\{x \in \Omega : u(x) = 0\}$ of u is a sphere of radius $r \in (R_0, R_1)$ centred at the origin. Then we have

$$\lambda_1(B_r(0) \setminus \overline{B_{R_0}(0)}) = \nu_2(\Omega); \quad \tau_1(B_{R_1}(0) \setminus \overline{B_r(0)}) = \nu_2(\Omega).$$

Thus, for $s \in (0, r - R_0)$, by Theorem 1.1 of [1] and Theorem 1.2 we have

$$\lambda_1(B_r(se_1) \setminus \overline{B_{R_0}(0)}) < \nu_2(\Omega) \quad \text{and} \quad \tau_1(B_{R_1}(0) \setminus \overline{B_r(se_1)}) < \nu_2(\Omega).$$

Let ϕ_1 and ϕ_2 be the positive eigenfunctions corresponding to the eigenvalues $\lambda_1(B_r(se_1) \setminus \overline{B_{R_0}(0)})$ and $\tau_1(B_{R_1}(0) \setminus \overline{B_r(se_1)})$ respectively, with the normalisation $\int |\phi_1|^p = 1 = \int |\phi_2|^p$. Now consider the set $\mathcal{A} := \{a\tilde{\phi}_1 + b\tilde{\phi}_2 : |a|^p + |b|^p = 1\}$. Then $\mathcal{A} \in \mathcal{F}_2$ and for $\phi \in \mathcal{A}$ we have

$$\begin{aligned} J(\phi) &= \int_{\Omega} |\nabla \phi|^p = \int |\nabla \phi_1|^p + \int |\nabla \phi_2|^p \\ &= \lambda_1(B_r(se_1) \setminus \overline{B_{R_0}(0)}) |a|^p + \tau_1(B_{R_1}(0) \setminus \overline{B_r(se_1)}) |b|^p \\ &< \nu_2(\Omega) (|a|^p + |b|^p) = \nu_2(\Omega). \end{aligned}$$

A contradiction to the variational characterisation of $\nu_2(\Omega)$. □

4. Some remarks and open problems

Remark 4.1 Let Ω be a doubly connected planar domain whose inner hole is a convex set and the outer boundary is a parallel curve to the inner boundary. Now one can use Steiner's formula for the convex domain and get the equality in Nagy's inequality. i.e., $S(\delta) = |\Gamma_1| + 2\pi\delta$. For such a domain Ω , without being a μ -translate of $\Omega_{\#}$, indeed we have $T_{\#} = t_{\Omega}$ and $\overline{G}(\alpha) = g(\alpha)$. However, it is not clear whether $\tau_1(\Omega) = \tau_1(\Omega_{\#})$ or not.

Remark 4.2 (The elasticity problem) For $p \in (1, \infty)$ and for a multiply connected domain Ω in \mathbb{R}^N with $N \geq 2$, one can consider the following elasticity problem:

$$\left. \begin{aligned} -\Delta_p u &= \Lambda |u|^{p-2} u \text{ in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} + k |u|^{p-2} u &= 0 \text{ on } \Gamma_0, \\ \frac{\partial u}{\partial \eta} &= 0 \text{ on } \Gamma_1 \cup (\cup_{i=1}^n \gamma_i), \end{aligned} \right\}$$

where $k > 0$ is the elasticity constant. The first eigenvalue $\Lambda_1(\Omega)$ has the following variational characterisation:

$$\Lambda_1(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p + k \int_{\Gamma_1} |u|^p}{\int_{\Omega} |u|^p} : u \in W^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Then using the similar set of arguments as in the proof of Theorem 1.1 one can show that if Γ_0 is a sphere, then $\Lambda_1(\Omega) \leq \Lambda_1(\Omega^\#)$ and the equality holds if and only if Ω is a μ -translate of $\Omega^\#$.

Next, we state a few open problems that are related to the mixed eigenvalue problems that we considered in this article. For more open problems related to the extremum of eigenvalues of various operators, we refer to the book [11].

Open problems:

- (i) The uniqueness for $N = 2$: Our results ensures the uniqueness of the domain (up to μ -translates) that gives the equality in the reverse F-K inequality. However, for the general domains ($N = 2$), Payne-Weinberger's or Hersch's results are not making any claim on the uniqueness of the domain. In the best of our knowledge this question is open for $N = 2$ and for every p . See also our Remark 4.1.
- (ii) The mixed eigenvalue problems for the general multiply connected domains: We proved our results under the assumptions that the boundary on which the Dirichlet condition is specified are spheres. We feels that the isoperimetric inequalities may obtained for other domains. For an excellent review on isoperimetric inequalities and related results, we refer to [17,21].
- (iii) The lower bounds for the first mixed eigenvalues $\nu_1(\Omega)$: Is it possible to find a constant C (depending on the domain) such that $C\nu_1(\Omega^\#) \leq \nu_1(\Omega)$ and hence an upper bound for the isoperimetric deficit $\frac{\nu_1(\Omega^\#) - \nu_1(\Omega)}{\nu_1(\Omega^\#)}$. The similar problems for Dirichlet eigenvalue are studied in [3] and [6] for various operators.
- (iv) The Dirichlet eigenvalue problem for the general multiply connected domains: Except for the annular region, the R-F-K is open for the general doubly connected domains in the higher dimension, even for the case one of the boundaries is a sphere.
- (v) One can also study the mixed eigenvalue problems with the Dirichlet condition specified on more than one interior boundaries. An upper bound for the case when $N = 2$, $p = 2$ is given by Hersch, see Section 3 of [12].

Appendices

A. The existence of the first eigenvalue and some of its properties

For $N \geq 2$, let $\Omega \subseteq \mathbb{R}^N$ be a multiply connected smooth domain such that $\partial\Omega = \Gamma_0 \sqcup \Gamma_1$ with $|\Gamma_0| > 0$, the $(N - 1)$ -dimensional measure. For $1 < p < \infty$, consider the following mixed eigenvalue problem:

$$\left. \begin{aligned} -\Delta_p u &= \alpha |u|^{p-2} u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_1. \end{aligned} \right\} \quad (15)$$

We say a real number α is an eigenvalue of (15) if there exists a $u \in W_{\Gamma_0} \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \alpha \int_{\Omega} |u|^{p-2} uv, \quad \text{for all } v \in W_{\Gamma_0},$$

where $W_{\Gamma_0} = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_0} = 0\}$. Recall that

$$J(u) = \int_{\Omega} |\nabla u|^p dx, \quad \mathcal{S} = \left\{ u \in W^{1,p}(\Omega) : \int_{\Omega} |u|^p dx = 1 \right\}.$$

It is easy to verify that the eigenvalues of (15) are precisely the critical values of J on $\mathcal{S} \cap W_{\Gamma_0}$.

Proposition A.1 *Let $p \in (1, \infty)$ and Ω as above. Let $\alpha_1 = \inf_{u \in \mathcal{S} \cap W_{\Gamma_0}} J(u)$. Then*

- (i) α_1 is an eigenvalue of (15),
- (ii) α_1 is simple and principal.

Proof. (i) Let $(u_n)_{n \in \mathbb{N}}$ be a minimising sequence, i.e., $J(u_n) = \int_{\Omega} |\nabla u_n|^p \rightarrow \alpha_1$. Then (u_n) is a bounded sequence in W_{Γ_0} and hence by the reflexivity, $u_n \rightharpoonup u$ in W_{Γ_0} for some $u \in W_{\Gamma_0}$. From the compact embedding $W_{\Gamma_0} \hookrightarrow L^p(\Omega)$, $u_n \rightarrow u$ in $L^p(\Omega)$, so $\int_{\Omega} |u|^p = 1$ hence $u \in \mathcal{S}$. Now, by the weakly lower semi continuity of J we have

$$\alpha_1 \leq J(u) \leq \liminf_{n \rightarrow \infty} J(u_n) = \alpha_1.$$

Hence α_1 is attained for some $u \in W_{\Gamma_0}$. Now it is easy to verify that α_1 is a critical value of J and hence an eigenvalue of (15). If $\alpha_1 = 0$ then $\nabla u = 0$, this will imply that $u \equiv 0$, as $u \equiv 0$ on the boundary, a contradiction. Hence α_1 is the first non-zero eigenvalue.

(ii) To prove the simplicity of α_1 we use the Picone's identity. Let us recall the Picone's identity: *Let $u \geq 0$, $v > 0$ a.e., such that $|\nabla u|$, $|\nabla v|$ exists. Define*

$$L(u, v) := |\nabla u|^p + (p-1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} |\nabla v|^{p-2} \nabla v,$$

and

$$R(u, v) := |\nabla u|^p - |\nabla v|^{p-2} \nabla \left(\frac{u^p}{v^{p-1}} \right) \cdot \nabla v.$$

Then we have the following:

$$L(u, v) = R(u, v) \text{ a.e.}$$

Furthermore, $L(u, v) \geq 0$ and $L(u, v) = 0$ if and only if u, v are constant multiples on each connected component of Ω .

Let $u, v > 0$ be two eigenfunctions corresponding to α_1 . By Picone's identity we get

$$0 \leq \int_{\Omega} L(u, v) dx = \int_{\Omega} R(u, v) dx = \int_{\Omega} (|\nabla u|^p - \alpha_1 |u|^p) dx = 0.$$

Therefore $u = kv$ for some $k > 0$, hence α_1 is simple.

For principality of α_1 , let $u \in \mathcal{S} \cap W_{\Gamma_0}$ be any eigenfunction corresponding to α_1 . We can verify that $|u| \in \mathcal{S} \cap W_{\Gamma_0}$ and $|u|$ is a minimiser of J over $\mathcal{S} \cap W_{\Gamma_0}$. Thus $|u|$ are also an eigenfunction corresponding to α_1 . From the strong maximum principle either $|u| \equiv 0$ or $|u| > 0$ in Ω . Since $u \neq 0$ we have $|u| > 0$ in Ω , hence α_1 is a principal eigenvalue of (15). \square

Proposition A.2 *Let p, Ω, α_1 as in Proposition A.1. Let u be an eigenfunction associated with α_1 . If Ω and Γ_0 are symmetric with respect to a hyperplane H , then u is also symmetric with respect to the hyperplane H .*

Proof. Let σ_H be the reflection with respect to the hyperplane H . Given Ω and Γ_0 are symmetric with respect to a hyperplane H , so we have $x \in \Omega$ (or Γ_0) if and only if $\sigma_H(x) \in \Omega$ (or Γ_0). Consider an eigenfunction u corresponding to α_1 and define $v(x) := u(\sigma_H(x))$ for $x \in \Omega$. Then $v \in W_{\Gamma_0}$ with $u = v$ on H and $J(v) = J(u) = \alpha_1$. By the simplicity of α_1 , we have $v = ku$ a.e. in Ω for some $k > 0$. Since $\|u\|_p = \|v\|_p$, we have $k = 1$. Therefore $v = u$ a.e. in Ω , i.e., $u(x) = u(x^*)$ a.e. $x \in \Omega$. Hence u is symmetric with respect to H . \square

Remark A.3 In particular if Ω and Γ_0 are radially symmetric, then the eigenfunction u associated with α_1 is radial in Ω . Thus the eigenfunctions associated with $\nu_1(\Omega^\#)$ and $\tau_1(\Omega_\#)$ are radial in $\Omega^\#$ and $\Omega_\#$ respectively. Notice that, the standard arguments using the schwartz symmetrization and Pólya-Szegő inequality does not work, if the domain Ω is not a ball.

B. The existence of infinitely many eigenvalues:

Using standard variational methods, as in [9] for Dirichlet eigenvalues using Krasnoselskii genus, we can obtain a set of critical values of J on $\mathcal{S} \cap W_{\Gamma_0}$. For a symmetric closed subset $\mathcal{A} \subset \mathcal{S}$, Krasnoselskii genus of \mathcal{A} is defined as

$$\gamma(\mathcal{A}) := \inf\{n \in \mathbb{N} : \exists \text{ an odd continuous map from } \mathcal{A} \text{ into } \mathbb{R}^n \setminus \{0\}\}$$

with the convention that $\inf\{\emptyset\} = \infty$. For $n \in \mathbb{N}$, let

$$\mathcal{E}_n := \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} = \overline{\mathcal{A}}, \mathcal{A} = -\mathcal{A} \text{ and } \gamma(\mathcal{A}) \geq n\},$$

$$\lambda_n := \inf_{\mathcal{A} \in \mathcal{E}_n} \sup_{u \in \mathcal{A}} J(u).$$

For each $n \in \mathbb{N}$, let

$$\mathcal{F}_n := \{h(S^{n-1}) : h \text{ is an odd continuous map from } S^{n-1} \text{ to } \mathcal{S} \cap W_{\Gamma_0}\}.$$

Another set of critical values of J can be obtained as follows:

$$\lambda_n^* := \inf_{\mathcal{A} \in \mathcal{F}_n} \sup_{u \in \mathcal{A}} J(u).$$

Since $\gamma(S^{n-1}) = n$, and γ is invariant under odd homeomorphisms, we have $\mathcal{F}_n \subseteq \mathcal{E}_n$ and hence $\lambda_n \leq \lambda_n^*$, $\forall n \in \mathbb{N}$. Then, with similar arguments as in Proposition 5.3 of [9] and Theorem 5 of [7], we can prove the following:

Proposition B.1 *Let $p \in (1, \infty)$ and Ω as above. Then for $n \in \mathbb{N}$, λ_n, λ_n^* are eigenvalues of (15) such that $\lambda_n, \lambda_n^* \nearrow \infty$, as $n \rightarrow \infty$.*

C. The variational characterisation of second eigenvalue:

We have that $\alpha_i = \lambda_i = \lambda_i^*$, $i = 1, 2$. This follows for $i = 1$, since the set $\{u, -u\}$ belongs to both \mathcal{E}_1 and \mathcal{F}_1 for $u \in \mathcal{S} \cap W_{\Gamma_0}$. For $i = 2$, this follows from the facts that the set $\{au_2^+ + bu_2^- : |a|^p \|u_2^+\|^p + |b|^p \|u_2^-\|^p = 1\}$ belongs to both \mathcal{E}_2 and \mathcal{F}_2 , where u_2 is the second eigenfunction, and α_1 is isolated (see Section 2 of [2] for a similar characterisation of second Dirichlet eigenvalue). Now we have the following proposition.

Proposition C.1 *Let $p \in (1, \infty)$ and Ω be as in Theorem 1.1. Let ν_2, τ_2 be the second eigenvalues of (N-D) and (D-N) respectively. Then ν_2 and τ_2 have the following variational characterisation:*

$$\nu_2 = \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} J(u), \quad \tau_2 = \inf_{\mathcal{A} \in \mathcal{G}_2} \sup_{u \in \mathcal{A}} J(u),$$

where $\mathcal{G}_2 := \{h(S^1) : h \text{ is an odd continuous map from } S^1 \text{ to } \mathcal{S} \cap W_{\Gamma_1}\}$.

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