

# ON EQUIVARIANT SERRE PROBLEM FOR PRINCIPAL BUNDLES

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**ABSTRACT.** Let  $E_G$  be a  $\Gamma$ -equivariant algebraic principal  $G$ -bundle over a normal complex affine variety  $X$  equipped with an action of  $\Gamma$ , where  $G$  and  $\Gamma$  are complex linear algebraic groups. Suppose  $X$  is contractible as a topological  $\Gamma$ -space with a dense orbit, and  $x_0 \in X$  is a  $\Gamma$ -fixed point. We show that if  $\Gamma$  is reductive, then  $E_G$  admits a  $\Gamma$ -equivariant isomorphism with the product principal  $G$ -bundle  $X \times_\rho E_G(x_0)$ , where  $\rho : \Gamma \rightarrow G$  is a homomorphism between algebraic groups. As a consequence, any torus equivariant principal  $G$ -bundle over an affine toric variety is equivariantly trivial. This leads to a classification of torus equivariant principal  $G$ -bundles over any complex toric variety, generalizing the main result of [2].

## 1. INTRODUCTION

Let  $\Gamma$  be a reductive complex algebraic group acting algebraically on a complex affine variety  $X$ . Let  $G$  be a complex linear algebraic group, and let  $E_G$  be a  $\Gamma$ -equivariant principal  $G$ -bundle on  $X$ . Given an algebraic group homomorphism  $\rho : \Gamma \rightarrow G$ , the product bundle  $X \times_\rho G$  equipped with diagonal action of  $\Gamma$  is the simplest example of a  $\Gamma$ -equivariant principal  $G$ -bundle. Here we are interested in the following question:

**Question 1.1.** *Is  $E_G$  equivariantly isomorphic to the above product (trivial) principal  $G$ -bundle  $X \times_\rho G$  for some homomorphism  $\rho$ ?*

When the structure group  $G = \mathrm{GL}(n, \mathbb{C})$  with  $\Gamma$  being the trivial group, and  $X$  is an affine space, Question 1.1 was first studied by Serre, and he proved that any vector bundle on an affine space is stably trivial [22]. Later Quillen [19] and Suslin [23] independently proved that the answer is affirmative for the above case of  $G = \mathrm{GL}(n, \mathbb{C})$ . This was generalized to the case when  $X$  is an affine toric variety by Gubeladze [6]. When  $\Gamma$  is trivial, but  $G$  is an arbitrary connected reductive algebraic group, and  $X$  is an affine space, Raghunathan, [20], showed that the answer is again affirmative.

When  $\Gamma$  is arbitrary, it is known that Question 1.1 has a negative answer in general even for vector bundles. This was first shown by Schwarz [21]; see also the work of Masuda and Petrie [17]. However, when  $\Gamma$  is a complex torus  $T$  with  $X$  being an affine toric variety under an action of  $T$ , and  $G = \mathrm{GL}(n, \mathbb{C})$ , Kaneyama, [8], and Klyachko, [9], gave affirmative answer to Question 1.1. Moreover, when  $\Gamma$  is abelian,

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$X$  is a linear representation space of  $\Gamma$ , and  $G = \mathrm{GL}(n, \mathbb{C})$ , Masuda, Moser and Petrie gave an affirmative answer in [16]. Later Masuda resolved affirmatively the case where  $\Gamma$  is abelian,  $X$  is an affine toric  $\Gamma$ -variety and  $G = \mathrm{GL}(n, \mathbb{C})$ ; thus this work of Masuda, [15], generalizes the results of Kaneyama and Klyachko.

In this paper we give an affirmative answer to Question 1.1 when the following conditions hold:

- $\Gamma$  is a reductive complex algebraic group,
- $X$  is a topologically contractible normal affine variety,
- $\Gamma$  acts on  $X$  with a dense orbit, and
- $G$  is an arbitrary linear algebraic group.

See Theorem 2.1.

The proof of Theorem 2.1 combines the classically known topological triviality of related bundles with an equivariant Oka principle established by Heinzner and Kutzschebauch, [7], to infer analytic triviality; after that, applying the classical fact, mainly due to Borel, that analytic representations of a reductive algebraic group are actually algebraic, the algebraic triviality of bundles in question is inferred. As an application we prove that any  $T$ -equivariant principal  $G$ -bundle over an affine toric variety is  $T$ -equivariantly trivial. This was proved earlier in [2] under the extra assumption that  $X$  is a nonsingular affine toric variety. Using Theorem 2.1 and the method of [2], we give a Kaneyama-type classification of  $T$ -equivariant principal  $G$ -bundles (up to isomorphism) over an arbitrary toric variety.

After this article was accepted for publication, Gerald W. Schwarz informed us of a more recent and stronger version of equivariant Oka principle due to Kutzschebauch-Lárusson-Schwarz [13] that would apply to generalized holomorphic principal bundles over a reduced Stein space. See also their related works [10, 11, 12].

## 2. MAIN RESULTS

All the objects are defined over the field  $\mathbb{C}$  of complex numbers unless mentioned specifically. Let  $\Gamma$  be a complex reductive algebraic group with a maximal compact subgroup  $K$ . This means that  $\Gamma = K^{\mathbb{C}}$ , the complexification of  $K$ . Let  $G$  be a complex affine algebraic group and  $X$  is an affine variety. Assume that  $\Gamma$  acts on  $X$  with a dense orbit  $O$  of it in  $X$ . In particular,  $\Gamma$  may be an algebraic torus and  $X$  a toric variety.

Let  $E_G$  be a  $\Gamma$ -equivariant algebraic principal  $G$ -bundle over  $X$ . This means that the action of  $\Gamma$  on  $X$  lifts to an action on  $E_G$  which commutes with the action of  $G$  on  $E_G$  defining the principal  $G$ -bundle structure of  $E_G$ , in other words,

$$\gamma(eg) = (\gamma e)g \quad \text{for all } \gamma \in \Gamma, e \in E_G, g \in G.$$

Given an algebraic group homomorphism  $\rho : \Gamma \rightarrow G$ , the product bundle

$$X \times G \rightarrow X$$

equipped with the diagonal action of  $\Gamma$  naturally becomes a  $\Gamma$ -equivariant principal  $G$ -bundle; this  $\Gamma$ -equivariant principal  $G$ -bundle is denoted by  $X \times_{\rho} G$ . We call  $X \times_{\rho} G$  a  $\Gamma$ -equivariant product bundle.

Let  $X(\mathbb{R})$  denote the underlying topological space for  $X$  with real norm topology. Let  $\Gamma(\mathbb{R})$  and  $G(\mathbb{R})$  denote the Lie groups underlying  $\Gamma$  and  $G$  respectively. Note that the action of  $\Gamma$  on  $X$  induces an action of  $\Gamma(\mathbb{R})$  on  $X(\mathbb{R})$ . The affine variety  $X$  is said to be equivariantly contractible as a topological  $\Gamma$ -space if there exists a point  $x_0 \in X(\mathbb{R})$  and a  $\Gamma(\mathbb{R})$ -equivariant continuous map

$$F : X(\mathbb{R}) \times [0, 1] \longrightarrow X(\mathbb{R}),$$

such that  $F(x, 1) = x$  and  $F(x, 0) = x_0$  for all  $x \in X(\mathbb{R})$ . Note that  $x_0$  is fixed by the action of  $\Gamma(\mathbb{R})$ . Since  $\Gamma$  is reductive, the good quotient  $X/\Gamma$  exists. As  $X$  has a dense orbit of  $\Gamma$ ,  $X/\Gamma$  is a point. Since fixed points inject into the good quotient,  $\Gamma$  has a unique fixed point in  $X$ . It follows that that  $x_0$  is the unique  $\Gamma(\mathbb{R})$ -fixed point in  $X(\mathbb{R})$ .

**Theorem 2.1.** *Let  $X$  be an irreducible normal affine variety over  $\mathbb{C}$ . Suppose  $X$  is equivariantly contractible as a topological  $\Gamma$ -space, and  $x_0 \in X$  is the fixed point for the action of  $\Gamma$ . Assume  $X$  has a dense orbit  $O$ . Then there exists a  $\Gamma$ -equivariant isomorphism  $E_G \cong X \times_{\rho} E_G(x_0)$  of algebraic principal  $G$ -bundles.*

*Proof.* Note that  $X$  is a reduced Stein space.

First we claim that the existence of an algebraic  $\Gamma$ -equivariant product bundle structure of  $E_G$  is equivalent to the existence of an analytic  $\Gamma$ -equivariant product bundle structure.

To prove the above claim, assume that there exists a  $\Gamma$ -equivariant complex analytic isomorphism  $\phi : X \times E_G(x_0) \longrightarrow E_G$ . Fix an element  $e \in E_G(x_0)$ . The action of  $\Gamma$  on  $E_G(x_0)$  is given by

$$\gamma(eg) = e\rho(\gamma)g \quad \text{for all } \gamma \in \Gamma, g \in G,$$

where

$$\rho : \Gamma \longrightarrow G$$

is a holomorphic (hence, algebraic) homomorphism of groups. Define an analytic section  $s$  of  $E_G$  by  $s(x) = \phi(x, e)$ . Then we have

$$\gamma s(x) = s(\gamma x)\rho(\gamma).$$

This implies that

$$s(\gamma x) = \gamma s(x)\rho(\gamma^{-1}). \tag{2.1}$$

Let  $x$  be a closed point in the dense orbit  $O \subset X$  of  $\Gamma$ . Since the action of  $\Gamma$  and the homomorphism  $\rho$  are both algebraic, it follows from (2.1) that  $s$  is an algebraic section of  $E_G$  over  $O$ . By applying Lemma 2.2,  $s$  is regular on  $X$ .

Secondly, by a corollary of the Homotopy Theorem, [7, p. 341], the question of existence of an analytic  $\Gamma$ -equivariant product bundle structure reduces to the question of existence of a topological  $K$ -equivariant product bundle structure.

Finally,  $E_G$  is a Cartan  $(K \times G(\mathbb{R}))$ -space in the sense of [18]. Therefore, it admits a slice at any point  $e' \in E_G$ . This ensures the existence of a topological  $K$ -equivariant trivialization of  $E_G$  near any  $x \in X$  (cf. Proposition 8.10 of [3]); see also [14, Corollary 2.11]. Then we may apply the equivariant homotopy principle (cf.

Theorem 8.15 of [3]) to obtain a topological  $K$ -equivariant product bundle structure for  $E_G$  over  $X$ .  $\square$

**Lemma 2.2.** *Let  $X = \text{Spec}(A)$  be a complex affine variety. Assume that  $A$  is a normal domain. Then a rational function  $\phi$  on  $X$  is regular if it is continuous in the complex analytic topology.*

*Proof.* Let  $K$  be the quotient field of  $A$ , and let  $\phi \in K$  be the given rational function. Let  $\mathfrak{p}$  be a height 1 prime ideal in  $A$ . Since  $A$  is a normal domain, the local ring  $A_{\mathfrak{p}}$  is a discrete valuation ring. Let  $v_{\mathfrak{p}}$  be its valuation. Since  $\phi$  has a continuous extension on  $X$ , we have  $v_{\mathfrak{p}}(\phi) \geq 0$ . Hence  $\phi \in A_{\mathfrak{p}}$  for every height 1 prime ideal  $\mathfrak{p}$ . Since  $A$  is normal we have  $A = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ , see [4, Corollary 11.4] or [1, Corollary 5.22]. Hence  $\phi$  is regular on  $X$ .  $\square$

The above results lead to a classification of equivariant principal bundles over arbitrary toric varieties, generalizing the classification in the nonsingular case in [2]. Note that every toric variety is normal (cf. [5, p. 29]).

**Corollary 2.3.** *Let  $X$  be an affine toric variety. Then a torus equivariant algebraic principal  $G$ -bundle over  $X$  admits an equivariant product bundle structure.*

*Proof.* This follows immediately from Theorem 2.1 if  $X$  has a fixed point of the torus action. In the general case, the proof is similar to the proof of [2, Lemma 2.8]. Let  $T$  denote the dense torus of  $X$ . The idea is to write  $X = Y \times O$  and  $T = H \times K$  where  $H, K$  are sub-tori of  $T$  that act on  $Y, O$  respectively, such that  $Y$  is an affine toric variety which is  $H$ -equivariantly contractible with dense a  $H$ -orbit, and  $K$  acts freely, transitively on  $O$ . The variety  $X = Y \times O$  is endowed with the diagonal action of  $H \times K$ . Let  $E_G$  be a  $T$ -equivariant principal  $G$ -bundle over  $X$ . By Theorem 2.1, there exists an  $H$ -equivariant section (often called semi-equivariant in the literature) of  $E_G$  over  $Y$ . Then one extends this section to a  $T$ -equivariant section of  $\mathcal{E}$  over  $X = Y \times O$  by using the lift to  $E_G$  of the free, transitive action of  $K$  on  $O$ .  $\square$

Now let  $X$  be a toric variety corresponding to a fan  $\Xi$ . Let  $T$  be the dense torus of  $X$ . For each cone  $\sigma \in \Xi$ , let  $T_{\sigma}$  denote the stabilizer of the  $T$ -orbit corresponding to  $\sigma$ . For each  $\sigma$ , fix a projection homomorphism  $\pi_{\sigma} : T \rightarrow T_{\sigma}$  which restricts to the identity map on  $T_{\sigma}$ .

**Definition 2.4.** Let  $\Xi^*$  denote the set of maximal cones in  $\Xi$ . An admissible collection  $\{\rho_{\sigma}, P(\tau, \sigma)\}$  consists of a collection of homomorphisms

$$\{\rho_{\sigma} : T \rightarrow G \mid \sigma \in \Xi^*\}$$

and a collection of elements  $\{P(\tau, \sigma) \in G \mid \tau, \sigma \in \Xi^*\}$  satisfying the following conditions:

- (1)  $\rho_{\sigma}$  factors through  $\pi_{\sigma} : T \rightarrow T_{\sigma}$ .
- (2) For every pair  $(\tau, \sigma)$  of maximal cones,  $\rho_{\tau}(t)P(\tau, \sigma)\rho_{\sigma}^{-1}(t)$  extends to a  $G$ -valued regular algebraic function over  $X_{\sigma} \cap X_{\tau}$ .
- (3)  $P(\sigma, \sigma) = 1_G$  for all  $\sigma$ .

- (4) For every triple  $(\tau, \sigma, \delta)$  of maximal cones, the cocycle condition  $P(\tau, \sigma)P(\sigma, \delta)P(\delta, \tau) = 1_G$  holds.

Two such admissible collections  $\{\rho_\sigma, P(\tau, \sigma)\}$  and  $\{\rho'_\sigma, P'(\tau, \sigma)\}$  are equivalent if the following hold:

- (i) For every  $\sigma$  there exists an element  $g_\sigma \in G$  such that  $\rho'_\sigma = g_\sigma^{-1}\rho_\sigma g_\sigma$ .  
(ii) For every pair  $(\tau, \sigma)$ ,  $P'(\tau, \sigma) = g_\tau^{-1}P(\tau, \sigma)g_\sigma$ , where  $g_\sigma$  and  $g_\tau$  are as in (i) above.

Using Corollary 2.3, the following classification theorem can be proved in a similar manner as Theorem 3.2 of [2].

**Theorem 2.5.** *Let  $X$  be a toric variety. Let  $G$  an affine algebraic group. Then the isomorphism classes of algebraic  $T$ -equivariant principal  $G$ -bundles on  $X$  are in one-to-one correspondence with equivalence classes of admissible collections  $\{\rho_\sigma, P(\tau, \sigma)\}$ .*

The following is then obtained exactly as Corollary 3.4 of [2].

**Corollary 2.6.** *If  $G$  is a nilpotent group, then an algebraic  $T$ -equivariant principal  $G$ -bundle over a toric variety admits an equivariant reduction of structure group to a maximal torus of  $G$ . In particular, if  $G$  is unipotent then the bundle is trivial with trivial  $T$ -action.*

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