

Ladder operators for isospectral oscillators

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Abstract

We present, for the isospectral family of oscillator Hamiltonians, a systematic procedure for constructing raising and lowering operators satisfying any prescribed ‘distorted’ Heisenberg algebra (including the q -generalization). This is done by means of an operator transformation implemented by a shift operator. The latter is obtained by solving an appropriate partial isometry condition in the Hilbert space. Formal representations of the non-local operators concerned are given in terms of pseudo-differential operators. Using the new annihilation operators, new classes of coherent states are constructed for isospectral oscillator Hamiltonians. The corresponding Fock-Bargmann representations are also considered, with specific reference to the order of the entire function family in each case.

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I. INTRODUCTION

Generalized coherent states of different kinds have been defined and investigated for some years now, particularly in quantum optics and quantum measurement theory.¹⁻⁶ Recently, two different kinds of coherent states (CS) have been constructed^{7,8} in relation to a one-parameter ($\lambda \in \mathbb{R}$) family of Hamiltonians \widetilde{H} that are isospectral companions of the standard oscillator Hamiltonian $H = a^\dagger a$. (The latter corresponds to the limit $|\lambda| \rightarrow \infty$.) For each finite value of λ , the eigenstates of \widetilde{H} consist of a zero-energy state $|\theta_0\rangle$ and a set of states $\{|\theta_n\rangle\}$ with eigenvalues $n = 1, 2, \dots$, the *latter* being obtained from the oscillator states $|n\rangle$ ($n = 0, 1, \dots$) by a non-unitary transformation (see Eq. (2.5) below). Fernandez *et al.*⁷ have constructed coherent states that are eigenstates of an annihilation operator associated with $\{|\theta_n\rangle, n \geq 1\}$. However, these states are not coherent states in the group-theoretic or Perelomov sense,⁹ since they are not obtained by the action of a unitary ‘displacement’ operator on the base state $|\theta_1\rangle$. On the other hand, Kumar and Khare⁸ have shown that \widetilde{H} itself is essentially unitarily equivalent to H . This fact enables them to obtain coherent states simply by a unitary transformation of the standard oscillator CS. These states are thus coherent states in the group-theoretic sense in addition to being annihilation operator eigenstates. As may be anticipated, they reduce to the standard oscillator CS in the limit $|\lambda| \rightarrow \infty$. Subsequently, Fernandez *et al.*¹⁰ have generalized their earlier work⁷ by introducing a real parameter w as follows: the commutator of the lowering and raising operators in the $\{|\theta_n\rangle, n \geq 0\}$ basis is a diagonal operator given by $\text{diag}(0, w, 1, 1, 1, \dots)$. This has been termed a ‘distorted’ Heisenberg algebra. In the special case $w = 1$, the CS constructed are also CS in the group-theoretic sense.

A number of interesting problems arise, which we address in this paper: the generalization of the algebra to encompass an arbitrary diagonal operator as the basic commutator, including the corresponding q -generalization; the development of a systematic procedure to find raising and lowering operators in this case; and the construction of the corresponding CS (which are *not* unitarily equivalent to the standard oscillator CS). Our strategy is to find a suitable operator transformation that takes the original ladder operators to the required new ones. The transformation is implemented by means of a *shift operator* that is constructed so as to satisfy appropriate partial isometry conditions dictated by the algebra imposed. We also derive representations for the new raising and lowering operators in terms of pseudo-differential operators, to enable one to compare their structure explicitly with their original counterparts. Finally, we examine the Fock-Bargmann (or entire analytic function) representation of states based on the CS obtained in various

cases, with particular reference to the order of the entire function families concerned.

II. NOTATION AND REVIEW

For ready reference, we recapitulate briefly the relevant parts of Refs. 7, 8 and 10. Given $H = a^\dagger a$ with $[a, a^\dagger] = 1$, we have $H|n\rangle = n|n\rangle$ ($n = 0, 1, \dots$). In the position representation, $a = 2^{-1/2}(x + d)$ and $a^\dagger = 2^{-1/2}(x - d)$, where $d = d/dx$. Operators b and b^\dagger are then defined, given in the position representation by

$$b = 2^{-\frac{1}{2}}(x + d + \phi(x)) \quad , \quad b^\dagger = 2^{-\frac{1}{2}}(x - d + \phi(x)) \quad , \quad (2.1)$$

and the condition $bb^\dagger = aa^\dagger$ imposed. This implies that the function $\phi(x)$ satisfies the Riccati equation

$$\phi' + 2x\phi + \phi^2 = 0 \quad (2.2)$$

where $\phi' = d\phi/dx$. Equation (2.2) has the one-parameter family of solutions

$$\phi_\lambda(x) = \frac{e^{-x^2}}{(\lambda + \int_0^x dy e^{-y^2})} \quad (2.3)$$

where $|\lambda| > \sqrt{\pi}/2$ (to avoid a singular ϕ_λ). Now consider the Hamiltonian (actually, a family of Hamiltonians, parametrized by $\lambda \in (-\infty, \infty)$)

$$\widetilde{H} = b^\dagger b. \quad (2.4)$$

In order to keep the notation simple, the λ -dependence of b , b^\dagger and \widetilde{H} has not been indicated explicitly. Although $bb^\dagger = aa^\dagger$ (that is, the supersymmetric partners of H and \widetilde{H} are identical to each other), $b^\dagger b \neq a^\dagger a$. It is straightforward to check that the states

$$|\theta_n\rangle = n^{-\frac{1}{2}} b^\dagger |n-1\rangle \quad (n = 1, 2, \dots) \quad (2.5)$$

are eigenstates of the Hamiltonian \widetilde{H} with eigenvalues $n = 1, 2, \dots$. The *ground state* $|\theta_0\rangle$ of \widetilde{H} is obtained by requiring that

$$b|\theta_0\rangle = 0. \quad (2.6)$$

The eigenstates $\{|\theta_n\rangle, n = 0, 1, \dots\}$ form a complete orthonormal set of states in the Hilbert space \mathcal{H} spanned by the Fock basis $\{|n\rangle\}$. It will be useful to write $\mathcal{H}_0 = \text{span } |0\rangle$, $\mathcal{H}_1 = \text{span } \{|n\rangle, n \geq 1\}$, $\widetilde{\mathcal{H}}_0 = \text{span } |\theta_0\rangle$ and $\widetilde{\mathcal{H}}_1 = \text{span } \{|\theta_n\rangle, n \geq 1\}$, so that $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 = \widetilde{\mathcal{H}}_0 + \widetilde{\mathcal{H}}_1$. (For easy identification, we shall use a tilde for spaces and operators pertaining to the isospectral case and drop the tilde for their counterparts relating to the standard oscillator.) It is evident that H and \widetilde{H} are isospectral, and that

$\widetilde{H} \rightarrow H$ in the limit $|\lambda| \rightarrow \infty$. However, since $b^\dagger b \neq a^\dagger a$, the commutator $[b, b^\dagger] \neq 1$. Thus b^\dagger and b are *not* the appropriate raising and lowering operators for \widetilde{H} . Fernandez *et al.*⁷ consider the lowering operator (in their notation)

$$A = b^\dagger a b \quad (2.7)$$

which annihilates not only $|\theta_0\rangle$ but also $|\theta_1\rangle$; one finds

$$A|\theta_0\rangle = 0, \quad A|\theta_n\rangle = (n-1)n^{\frac{1}{2}}|\theta_{n-1}\rangle. \quad (2.8)$$

Similarly,

$$A^\dagger|\theta_0\rangle = 0, \quad A^\dagger|\theta_n\rangle = n(n+1)^{\frac{1}{2}}|\theta_{n+1}\rangle. \quad (2.9)$$

These relations help us understand how the Hilbert space \mathcal{H} breaks up in a natural fashion into the direct sum $\widetilde{\mathcal{H}}_0 + \widetilde{\mathcal{H}}_1$. As A is an annihilation operator in $\widetilde{\mathcal{H}}_1$, its eigenstates, satisfying $A|z; \theta_1\rangle = z|z; \theta_1\rangle$ ($z \in \mathbb{C}$), may be regarded as coherent states. They are given by⁷

$$|z; \theta_1\rangle = (\text{const.}) \sum_{n=0}^{\infty} \frac{z^n}{n! \sqrt{(n+1)!}} |\theta_{n+1}\rangle \quad (2.10)$$

However, since $[A, A^\dagger] \neq 1$ even in the subspace $\widetilde{\mathcal{H}}_1$, the state $|z; \theta_1\rangle$ cannot be obtained by a unitary transformation of the form $\exp(zA^\dagger - \bar{z}A)$ acting on the ground state $|\theta_1\rangle$. Therefore $|z; \theta_1\rangle$ is not a CS in the group-theoretic sense, nor can *generalized* coherent states be built up from it by the standard procedure – namely, by using $|z; \theta_1\rangle$ as the ground state of an appropriate Hamiltonian obtained by a unitary transformation of \widetilde{H} .

In contrast to the foregoing, Kumar and Khare⁸ have shown that b and b^\dagger can be written in the form

$$b = aU^\dagger, \quad b^\dagger = Ua^\dagger \quad (2.11)$$

where U is unitary ($UU^\dagger = U^\dagger U = 1$) in the full space \mathcal{H} , and formally determined the matrix elements of U in the basis $\{|n\rangle\}$. Therefore $bb^\dagger = aa^\dagger$, as required. Further, $\widetilde{H} = b^\dagger b = UHU^\dagger$, so that \widetilde{H} and H are actually unitarily equivalent. Writing $\tilde{a} = UaU^\dagger$, one has $\widetilde{H} = \tilde{a}^\dagger \tilde{a}$ where $[\tilde{a}, \tilde{a}^\dagger] = 1$. The eigenstates $|\theta_n\rangle$ of \widetilde{H} are just the unitary transforms of those of H , *i.e.*,

$$|\theta_n\rangle = U|n\rangle \quad \text{and} \quad U^\dagger|\theta_n\rangle = |n\rangle, \quad n = 0, 1, \dots \quad (2.12)$$

. The operator \tilde{a} only annihilates the vacuum $|\theta_0\rangle$. The state

$$|\alpha; \theta_0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\theta_n\rangle \quad (\alpha \in \mathbb{C}) \quad (2.13)$$

is both an annihilation operator eigenstate and a CS in the Perelomov sense. It is readily verified that (i) $|\alpha; \theta_0\rangle = U |\alpha\rangle$ where $|\alpha\rangle$ is the standard oscillator CS; (ii) $|\alpha; \theta_0\rangle$ is obtainable by a unitary displacement operator acting on $|\theta_0\rangle$; and (iii) $|\alpha; \theta_0\rangle \rightarrow |\alpha\rangle$ in the limit $|\lambda| \rightarrow \infty$. Generalized CS may be constructed from $|\alpha; \theta_0\rangle$ in essentially the same manner as in the case of $|\alpha\rangle$. Viewed in this manner, the isospectral case is simply a unitary transformation of the standard oscillator problem, and nothing new is gained by considering it in preference to the latter.

In order to have ladder operators in $\widetilde{\mathcal{H}}_1$ whose commutator is equal to unity, and also to generalize this algebra to some extent, Fernandez *et al.* have subsequently defined¹⁰ a new lowering operator C_w (in their notation) according to

$$C_w = b^\dagger f(H) a b. \quad (2.14)$$

Here, the form of $f(H)$ is deduced from the requirement that the commutator $[C_w, C_w^\dagger]$ be equal to $\text{diag}(w, 1, 1, \dots)$ in $\widetilde{\mathcal{H}}_1$, where w is any positive number. (The operators C_w and C_w^\dagger continue to annihilate the subspace $\widetilde{\mathcal{H}}_0$.) It is found that

$$f(H) = \frac{1}{H+1} \left(\frac{H+w}{H+2} \right)^{\frac{1}{2}}. \quad (2.15)$$

It is evident that $\text{diag}(w, 1, 1, \dots)$ is not the most general diagonal form possible for the basic commutator, although it can be shown to be the only consistent one once the specific form given by Eq. (2.14) is assumed for the lowering operator. As we shall see, a procedure of wider applicability is to look for an operator transformation on \tilde{a} of the type $\tilde{S}^\dagger \tilde{a} \tilde{S}$. In the next section, we develop a systematic procedure to find the transformation \tilde{S} , and from it the ladder operators *for any prescribed diagonal form of their commutator*, including the case of the q -generalization of the isospectral oscillator.

III. SHIFT OPERATORS AND OPERATOR TRANSFORMATIONS

We wish to implement transformations on the ladder operators \tilde{a} and \tilde{a}^\dagger so as to obtain a lowering operator

$$\tilde{a}_1 = \tilde{S}^\dagger \tilde{a} \tilde{S} \quad (3.1)$$

and a hermitian conjugate raising operator

$$\tilde{a}_1^\dagger = \tilde{S}^\dagger \tilde{a}^\dagger \tilde{S} \quad (3.2)$$

with the following properties: both operators must annihilate $|\theta_0\rangle$ (so that $\widetilde{\mathcal{H}}_0$ is the kernel of the transformation); and their commutator is a prescribed diagonal operator in $\widetilde{\mathcal{H}}_1$.

For the sake of clarity, we solve the problem first in the standard Fock basis, and then use the operator U to transform to the isospectral case.

Accordingly, we first seek to eliminate the subspace \mathcal{H}_0 from \mathcal{H} , and to find conjugate lowering and raising operators in \mathcal{H}_1 given by

$$a_1 = S^\dagger a S \quad , \quad a_1^\dagger = S^\dagger a^\dagger S, \quad (3.3)$$

whose commutator is equal to *unity* throughout the latter space. For this we need to find an operator S in \mathcal{H} that satisfies the *partial isometry*

$$SS^\dagger = 1, \quad S^\dagger S = 1 - |0\rangle\langle 0|. \quad (3.4)$$

It is easy to see that this is solved by the *shift* operator

$$S = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \quad (3.5)$$

(Note that this operator has the closed form $S = (1 + H)^{-1/2} a$ in \mathcal{H} .) It is easily verified that a_1 and a_1^\dagger satisfy

$$a_1 |0\rangle = a_1^\dagger |0\rangle = 0 \quad , \quad a_1 |1\rangle = 0. \quad (3.6)$$

Now consider the *general* ‘distorted’ Heisenberg algebra in which the commutator is specified (in the Fock basis) to be

$$[a_1, a_1^\dagger] = \text{diag}(0, w_1, w_2, \dots) = \sum_{n=1}^{\infty} w_n |n\rangle\langle n|, \quad (3.7)$$

where $\{w_n\}$ are real positive constants. S is now found by generalising the partial isometry condition, Eq. (3.4), to read

$$SS^\dagger = \sum_{n=0}^{\infty} c_n |n\rangle\langle n| \quad , \quad S^\dagger S = \sum_{n=0}^{\infty} c_n |n+1\rangle\langle n+1|, \quad (3.8)$$

where the coefficients c_n are to be determined. The deviation of the c_n ’s from unity is a measure of the extent to which the shift operator deviates from the exact partial isometry specified in Eqs. (3.4). Solving for S and S^\dagger in the form

$$S = \sum_{n=0}^{\infty} c_n^{\frac{1}{2}} |n\rangle\langle n+1| \quad , \quad S^\dagger = \sum_{n=0}^{\infty} c_n^{\frac{1}{2}} |n+1\rangle\langle n| \quad (3.9)$$

and using Eqs. (3.3) and (3.7), we obtain the recursion relations

$$c_0 c_1 = w_1 \quad , \quad (n+1) c_n c_{n+1} - n c_n c_{n-1} = w_{n+1} \quad (n = 1, 2, 3, \dots). \quad (3.10)$$

We may choose $c_0 = 1$ without loss of generality, and solve the recursion relations to get

$$c_n = \frac{(n-1)!!}{n!!} \frac{(w_1 + w_2 + \dots + w_n)!!}{(w_1 + w_2 + \dots + w_{n-1})!!}. \quad (3.11)$$

Here the notation $(w_1 + w_2 + \dots + w_n)!!$ stands for the product

$$(w_1 + w_2 + \dots + w_n) \times (w_1 + w_2 + \dots + w_{n-2}) \cdots$$

Substituting the values for c_n given above in Eq. (3.9) yields S and S^\dagger . The corresponding expressions for a_1 and a_1^\dagger are then obtained from Eqs. (3.3), and their commutator is guaranteed to satisfy Eq. (3.7).

We go over now to the isospectral case. As the corresponding shift and ladder operators are merely the unitary transformations

$$\tilde{S} = U S U^\dagger, \quad \tilde{a}_1 = U a_1 U^\dagger, \quad \tilde{a}_1^\dagger = U a_1^\dagger U^\dagger, \quad (3.12)$$

we find

$$\tilde{S} = \sum_{n=0}^{\infty} c_n^{\frac{1}{2}} |\theta_n\rangle \langle \theta_{n+1}|, \quad (3.13)$$

where $c_0 = 1$ as already stated, and c_n is given by Eq. (3.11); and

$$\tilde{a}_1 = \sum_{n=1}^{\infty} (w_1 + w_2 + \dots + w_n)^{\frac{1}{2}} |\theta_n\rangle \langle \theta_{n+1}|, \quad (3.14)$$

$$\tilde{a}_1^\dagger = \sum_{n=1}^{\infty} (w_1 + w_2 + \dots + w_n)^{\frac{1}{2}} |\theta_{n+1}\rangle \langle \theta_n|. \quad (3.15)$$

It is easily verified from these expressions that $[\tilde{a}_1, \tilde{a}_1^\dagger]$ is $\text{diag}(0, w_1, w_2, \dots)$ in the *transformed* basis, *i.e.*,

$$[\tilde{a}_1, \tilde{a}_1^\dagger] = \sum_{n=1}^{\infty} w_n |\theta_n\rangle \langle \theta_n|, \quad (3.16)$$

as required (*cf.* Eq. (3.7)).

In order to see how our general expressions simplify in particular cases, it is convenient to use Eq. (2.5) and the usual representation of a in the Fock basis, to express Eq. (3.14) in the form

$$\tilde{a}_1 = b^\dagger \sum_{n=0}^{\infty} \left(\frac{W_{n+1}}{(n+1)(n+2)} \right)^{\frac{1}{2}} |n\rangle \langle n+1| b, \quad (3.17)$$

or in the equivalent form

$$\tilde{a}_1 = b^\dagger (H+1)^{-\frac{1}{2}} \left\{ \sum_{n=0}^{\infty} W_{n+1}^{\frac{1}{2}} |n\rangle \langle n+1| \right\} (H+1)^{-\frac{1}{2}} b, \quad (3.18)$$

where we have written

$$W_n = w_1 + w_2 + \dots + w_n \quad (3.19)$$

for brevity. It is clear from the above that, whenever $W_{n+1}^{1/2}$ can be written as some function of n , say $F(n)$, we obtain the closed-form expression

$$\tilde{a}_1 = b^\dagger (H+1)^{-1} f(H) a (H+1)^{-\frac{1}{2}} b. \quad (3.20)$$

If, further, $F(n)$ is a product of the form $(n+1)^{1/2} G(n) G(n+1)$, we obtain the *symmetric* form

$$\tilde{a}_1 = b^\dagger (H+1)^{-\frac{1}{2}} G(H) a G(H) (H+1)^{-\frac{1}{2}} b. \quad (3.21)$$

Corresponding expressions obtain, of course, for a_1^\dagger as well. A remark is in order here on the square-root operator occurring in these expressions. As $(1+H)^{-1}$ is a bounded positive definite operator in \mathcal{H} , it has a unique positive square root in that space, by virtue of the square-root lemma.¹¹ Following the procedure of Ref. 12, this can be defined rigorously in terms of the resolvent operator $(\xi+1+H)^{-1}$ as

$$(1+H)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty d\xi \xi^{-\frac{1}{2}} (1+\xi+H)^{-1}. \quad (3.22)$$

We are now ready to read off various special cases of the foregoing.

(i) First of all, if we set $w_n = w$ for *all* $n \geq 1$, the operator in curly brackets in Eq. (3.18) above is simply $w^{1/2} a$, and we get the symmetric expression

$$\tilde{a}_1 = w^{\frac{1}{2}} b^\dagger (H+1)^{-\frac{1}{2}} a (H+1)^{-\frac{1}{2}} b. \quad (3.23)$$

(ii) Next, we note that the ‘distorted’ Heisenberg algebra introduced in Ref. 10 corresponds to setting $w_1 = w$ and $w_n = 1$ for $n \geq 2$, so that $W_{n+1} = (w+n)$. Therefore \tilde{a}_1 reduces, in this special case, to a closed-form expression – namely, the operator C_w defined in Eqs. (2.14) and (2.15).

(iii) Again, if we choose $w_n = n$ itself – *i.e.*, we have the algebra

$$[\tilde{a}_1, \tilde{a}_1^\dagger] = U H U^\dagger = \tilde{H} = b^\dagger b, \quad (3.24)$$

we get

$$\tilde{a}_1 = 2^{-\frac{1}{2}} b^\dagger (H+1)^{-\frac{1}{2}} a b. \quad (3.25)$$

(iv) On the other hand, if we set $w_1 = w$ and $w_n = 0$ for $n \geq 2$, we get

$$\tilde{a}_1 = w^{\frac{1}{2}} b^\dagger (H+1)^{-1} a (H+1)^{-\frac{1}{2}} b. \quad (3.26)$$

(v) Finally, the q -generalization¹³ of the isospectral case is immediately found by setting $w_n = q^n$: this implies that

$$[\tilde{a}_1, \tilde{a}_1^\dagger] = q^{\tilde{H}}, \quad (3.27)$$

which is a transformed, equivalent, version of a q -commutation relation between a conjugate pair of ladder operators. We obtain in this case the result

$$\tilde{a}_1 = q^{\frac{1}{2}} b^\dagger (H+1)^{-1} \left(\frac{1-q^{H+1}}{1-q} \right)^{\frac{1}{2}} a (H+1)^{-\frac{1}{2}} b. \quad (3.28)$$

It is readily seen that Eq. (3.28) reduces to Eq. (3.23) (with $w=1$) in the limit $q \rightarrow 1$.

IV. REPRESENTATION IN TERMS OF PSEUDO-DIFFERENTIAL OPERATORS

The expressions obtained in the preceding section involve various non-local operators, for which it is appropriate to exhibit suitable explicit representations. Kernels in the position basis for the integral operators represented by \tilde{a}_1 and \tilde{a}_1^\dagger may of course be written down at once from Eqs. (3.14) and (3.15) in terms of the wavefunction $\tilde{\psi}_n(x) = \langle x | \theta_n \rangle$ and its complex conjugate. *Formal* representations for the operators concerned in a form that is an extension of the standard ones $a = 2^{-1/2}(x+d)$, $a^\dagger = 2^{-1/2}(x-d)$ may be given in terms of pseudo-differential operators,^{14,15} *i.e.*, operators of the form

$$\sum_{n=0}^N u_n(x) d^n + \sum_{n=0}^{\infty} u_{-n}(x) d^{-n}$$

where N is a finite positive integer (and $d \equiv d/dx$). Here d^{-1} is the antiderivative, defined recursively through

$$d^{-1}(f \cdot) = \sum_{n=0}^{\infty} (-1)^n f^{(n)} d^{-1-n}(\cdot) \quad (4.1)$$

where $f^{(n)}$ is the n th derivative of $f(x)$. This leads to the basic relationship

$$[d^{-r}, f] = \sum_{n=1}^{\infty} (-1)^n \binom{n+r-1}{n-1} f^{(n)} d^{-n-r}. \quad (4.2)$$

With the help of the foregoing we can find expansions for the operators $(d^2 + u(x))^{1/2}$ and $(d^2 + u(x))^{-1/2}$ in powers of d^{-1} . In particular, we find

$$\begin{aligned} (1 + H)^{-\frac{1}{2}} &= -2^{\frac{1}{2}} i (d^2 - x^2 - 1)^{-\frac{1}{2}} \\ &= -2^{\frac{1}{2}} i \left\{ d^{-1} + \frac{1}{2}(1 + x^2) d^{-3} - \frac{3}{2} x d^{-4} + \dots \right\}. \end{aligned} \quad (4.3)$$

Using these results, formal expansions for \tilde{a}_1 and \tilde{a}_1^\dagger may be derived in those cases discussed in Sec. 3 in which closed-form expressions obtain for these operators. In particular, corresponding to the algebra¹⁰ $[\tilde{a}_1, \tilde{a}_1^\dagger] = \text{diag}(w, 1, 1, \dots)$ in $\widetilde{\mathcal{H}}_1$, for which \tilde{a}_1 is the operator C_w given by Eqs. (2.14) and (2.15), we get the following expansions (quoted up to $O(d^{-2})$ for brevity):

$$2^{\frac{1}{2}} \tilde{a}_1 = x + d - (w - 2 - \phi_\lambda') d^{-1} + [x(2 - w) + \phi_\lambda \phi_\lambda' + x \phi_\lambda' + 2 \phi_\lambda] d^{-2} + \dots \quad (4.4)$$

and

$$2^{\frac{1}{2}} \tilde{a}_1^\dagger = x - d + (w - 2 - \phi_\lambda') d^{-1} + [x(2 - w) - x \phi_\lambda' - \phi_\lambda \phi_\lambda'] d^{-2} + \dots \quad (4.5)$$

Further, we find

$$\tilde{a}_1 \tilde{a}_1^\dagger = \frac{1}{2} (-d^2 + x^2 + 2w - 3) - \phi_\lambda', \quad (4.6)$$

while in the space span $\{|\theta_n\rangle, n \geq 2\}$ we also have

$$\tilde{a}_1^\dagger \tilde{a}_1 = \frac{1}{2} (-d^2 + x^2 + 2w - 5) - \phi_\lambda'. \quad (4.7)$$

When $w = 1$, Eq. (4.8) is valid in all of $\widetilde{\mathcal{H}}_1$.

Finally, we remark that in the limit $|\lambda| \rightarrow \infty$, $\phi_\lambda(x)$ and its derivatives vanish, $U \rightarrow 1$, $b \rightarrow a$, $\widetilde{\mathcal{H}}_1 \rightarrow \mathcal{H}_1$, $\tilde{a}_1 \rightarrow a_1$, etc., in all the foregoing expressions.

V. COHERENT STATES

With the operators $\tilde{a}_1, \tilde{a}_1^\dagger$ in $\widetilde{\mathcal{H}}_1$ and their commutation relation at hand (Eqs. (3.14)-(3.16)), we may construct coherent states (CS) in the usual manner, *i.e.*, as eigenstates of \tilde{a}_1 built upon the base state $|\theta_1\rangle$. We find, for every $\zeta \in \mathbb{C}$,

$$\tilde{a}_1 |\zeta; \theta_1\rangle = \zeta |\zeta; \theta_1\rangle \quad (5.1)$$

where the normalized eigenstate is given by

$$|\zeta; \theta_1\rangle = h^{-\frac{1}{2}} (|\zeta|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} d_n^{\frac{1}{2}} \zeta^n |\theta_{n+1}\rangle \quad (5.2)$$

with

$$d_0 = 1, \quad d_n = (W_1 \cdots W_n)^{-1}. \quad (5.3)$$

The normalization factor in Eq. (5.2) is defined by

$$h(|\zeta|^2) = \sum_{n=0}^{\infty} d_n |\zeta|^{2n}. \quad (5.4)$$

A question of interest is the kind of Fock-Bargmann or entire function representation of states^{9,16} in $\widetilde{\mathcal{H}}_1$ associated with the CS constructed above – specifically, the *order*¹⁷ of the class of entire functions involved. A state $|\Psi\rangle$ of finite norm is represented by the analytic function

$$\Psi(\zeta) = \sum_{n=0}^{\infty} d_n^{\frac{1}{2}} \langle \theta_{n+1} | \Psi \rangle \zeta^n. \quad (5.5)$$

As the state $|\zeta; \theta_1\rangle$ in Eq. (5.2) is normalized to unity, we have $|\Psi(\zeta)| \leq \|\Psi\| |h|^{1/2}$. Hence the asymptotic behavior of $\Psi(\zeta)$ is decided by that of $h^{1/2}$ which, in turn, is controlled by the growth properties of the sequence $\{d_n\}$. In particular, the *order* of the class of entire functions to which $\Psi(\zeta)$ belongs is given by

$$\rho = \lim_{n \rightarrow \infty} \frac{2n \log n}{\log (W_1 \cdots W_n)}. \quad (5.6)$$

In the special cases (i)-(v) considered at the end of Sec. 3, this leads to the following results: In Case (i), $(W_1 \cdots W_n) = w^n n!$, while in Case (ii)¹⁰ it is $\Gamma(n+w)/\Gamma(n)$. In both cases it follows easily from Eq. (5.6) that $\rho = 2$, *i.e.*, the order is exactly the same as it is for the Fock-Bargmann representation corresponding to the standard oscillator CS. In Case (iii), $(W_1 \cdots W_n) = 2^{-n} n!(n+1)!$, leading to $\rho = 1$. All these cases are covered by the following generalization: if the coefficient w_n of the algebra $[\tilde{a}_1, \tilde{a}_1^\dagger] = \text{diag}(w_1, w_2, \dots)$ has the leading asymptotic behavior n^ν for large n , with $\nu > -1$, then $(W_1 \cdots W_n) \sim n^{n(1+\nu)}$, which leads to the result

$$\rho = 2(1+\nu)^{-1} \quad (5.7)$$

for the order of the corresponding entire function representation of $\widetilde{\mathcal{H}}_1$. Cases (i) and (ii) correspond to $\nu = 0$, while $\nu = 1$ in Case (iii). In Case (iv), $h = w(w - |\zeta|^2)^{-1}$, so that the representation is not in terms of entire functions at all.

For the q -generalization, Case (v), $(W_1 \cdots W_n)$ is a ' q -factorial' given by

$$q(q+q^2) \cdots (q+q^2+\cdots+q^n) = q^n (q-1)^{(1-n)} (q^2-1) \cdots (q^n-1). \quad (5.8)$$

When $q < 1$, it is readily established that the series defining h has a finite radius of convergence ($|\zeta| = q^{1/2}$), as in Case (iv). When $q = 1$, of course, we recover Case (i) or (ii) with $w = 1$, so that $\rho = 2$. On the other hand, when $q > 1$, $(W_1 \cdots W_n)$ has the leading asymptotic behaviour $q^{n^2/2}$, which implies that $\rho = 0$.

The CS considered above have been constructed as annihilation operator eigenstates. In the special case $w_n = 1$ for all $n \geq 1$, they are also CS in the group-theoretic sense. Each coefficient c_n in Eq. (3.11) is now equal to unity, and the shift operator \tilde{S} obeys the exact partial isometry condition

$$\tilde{S} \tilde{S}^\dagger = 1, \quad \tilde{S}^\dagger \tilde{S} = 1 - |\theta_0\rangle \langle \theta_0|. \quad (5.9)$$

We then have $[\tilde{a}_1, \tilde{a}_1^\dagger] = 1$ in $\widetilde{\mathcal{H}}_1$, and the Hamiltonian $\widetilde{H}_1 = \tilde{a}_1^\dagger \tilde{a}_1$ with eigenstates $|\theta_n\rangle$ and eigenvalues $(n-1)$, $n = 1, 2, \dots$, is the exact analog in $\widetilde{\mathcal{H}}_1$ of the Hamiltonian $H = a^\dagger a$ in \mathcal{H} . The transformation from (a, a^\dagger) to $(\tilde{a}_1, \tilde{a}_1^\dagger)$ is a true operator map, in the sense that any function $\phi(a, a^\dagger) \rightarrow \phi(\tilde{a}_1, \tilde{a}_1^\dagger) = \tilde{S}^\dagger \phi(a, a^\dagger) \tilde{S}$. The coherent states

$$|\zeta; \theta_1\rangle = \exp\left(-\frac{|\zeta|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!}} |\theta_{n+1}\rangle \quad (\zeta \in \mathbb{C}) \quad (5.10)$$

are normalized eigenstates of the annihilation operator \tilde{a}_1 . They are *also* CS in the Perelomov sense, for we have $|\zeta; \theta_1\rangle = D(\zeta, \bar{\zeta})|\theta_1\rangle$, where the displacement operator

$$D(\zeta, \bar{\zeta}) = \exp(\zeta \tilde{a}_1^\dagger - \bar{\zeta} \tilde{a}_1) \quad (5.11)$$

is unitary in $\widetilde{\mathcal{H}}_1$. Moreover, as in the case of the standard oscillator CS, the state $|\zeta; \theta_1\rangle$ is the ground state of the displaced Hamiltonian $D\widetilde{H}_1D^\dagger$. The excited states $|\zeta; \theta_n\rangle$ ($n \geq 2$) of this Hamiltonian, *i.e.*, the generalized coherent states corresponding to $|\zeta; \theta_1\rangle$, are obtained *either* by operating on $|\zeta; \theta_1\rangle$ by the transformed raising operator $(D\widetilde{a}_1^\dagger D^\dagger)^{n-1}$, *or* by displacing the corresponding base state $|\theta_n\rangle$:

$$|\zeta; \theta_n\rangle = (D\widetilde{a}_1^\dagger D^\dagger)^{n-1}|\zeta; \theta_1\rangle = D|\theta_n\rangle, (n = 2, 3, \dots). \quad (5.12)$$

Thus, these coherent states and generalized CS have in $\widetilde{\mathcal{H}}_1$ the same standing as their standard oscillator counterparts have in \mathcal{H} . They therefore constitute the closest analogs of the latter for the isospectral family of oscillators in $\widetilde{\mathcal{H}}_1$, while remaining a distinctly new (unitarily inequivalent) class of coherent states. However, the generalized algebra of Eq. (3.16) leads, as we have seen, to an even wider variety of possible coherent states with interesting properties.

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