

Information Spreading in Dynamic Networks under Oblivious Adversaries*

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Abstract

We study the problem of all-to-all information exchange, also known as *gossip*, in dynamic networks controlled by an adversary that can modify the network arbitrarily from one round to another, provided that the network is always connected. In the gossip problem, there are n tokens arbitrarily distributed among the n network nodes, and the goal is to disseminate all the n tokens to every node. Our focus is on *token-forwarding* algorithms, which do not manipulate tokens in any way other than storing, copying, and forwarding them. Gossip can be completed in linear time in any static network, but an important and basic open question for dynamic networks is the existence of a distributed protocol that can do significantly better than an easily achievable bound of $O(n^2)$ rounds.

In previous work, it has been shown that under adaptive adversaries—those that have full knowledge and control of the topology in every round and also have knowledge of the distributed protocol including its random choices—every token forwarding algorithm requires $\Omega(n^2 / \log n)$ rounds to complete. In this paper, we study oblivious adversaries, which differ from adaptive adversaries in one crucial aspect—they are *oblivious to the random choices* made by the protocol. We consider RAND-DIFF, a natural distributed algorithm in which neighbors exchange a token chosen uniformly at random from the difference of their token sets. Previous work has shown that starting from a distribution in which each node has a random constant fraction of the tokens, RAND-DIFF completes in $\tilde{O}(n)$ rounds. In contrast, we show that a polynomial slowdown is inevitable under more general distributions: we present an $\tilde{\Omega}(n^{3/2})$ lower bound for RAND-DIFF under an oblivious adversary. We also present an $\tilde{\Omega}(n^{4/3})$ lower bound under a stronger notion of oblivious adversary for a class of randomized distributed algorithms—*symmetric knowledge-based algorithms*—in which nodes make token transmission decisions based entirely on the sets of tokens they possess over time. On the positive side, we present a centralized algorithm that completes gossip in $\tilde{O}(n^{3/2})$ rounds with high probability, under any oblivious adversary. We also show an $\tilde{O}(n^{5/3})$ upper bound for RAND-DIFF in a restricted class of oblivious adversaries, which we call *paths-respecting*, that may be of independent interest.

1 Introduction

In a dynamic network, nodes (processors/end hosts) and communication links can appear and disappear over time. The networks of the current era are inherently dynamic. Modern communication networks (e.g., Internet, peer-to-peer, ad-hoc networks and sensor networks) and information networks (e.g., the Web, peer-to-peer networks and on-line social networks), and emerging technologies such as drone swarms are dynamic networked systems that are larger and more complex than ever before. Indeed, many such networks

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are subject to continuous structural changes over time due to sleep modes, channel fluctuations, mobility, device failures, nodes joining or leaving the system, and many other factors [25, 32, 5, 14, 18, 9, 30]. Therefore the formal study of algorithms for dynamic networks have gained much popularity in recent years and many of the classical problems and algorithms for static networks were extended to dynamic networks. During the past decade, new dynamic network models have been introduced to capture specific applications [10, 9, 25, 16, 18, 31], and the last few years have witnessed a burst of research activity on broadcasting, flooding, random-walk based, and gossip-style protocols in dynamic networks [26, 21, 20, 13, 35, 7, 6, 3, 4, 12, 19, 2, 15, 11, 34, 8, 22, 27].

Our paper continues this effort and studies a fundamental problem of information spreading, called *k-gossip*, on dynamic networks. In *k-gossip* (also referred to as *k-token dissemination*), k distinct pieces of information (tokens) are initially present in some nodes, and the problem is to disseminate all the tokens to all the nodes, under the constraint that one token can be sent on an edge per round of synchronous communication. This problem is a fundamental primitive for distributed computing; indeed, solving *n-gossip*, where each node starts with exactly one token, allows any function of the initial states of the nodes to be computed, assuming the nodes know n [26]. This problem was analyzed for static networks by Topkis [36], and was first studied on dynamic networks for general k in [26], and previously for the special case of one token and a random walk in [6].

In this paper, we consider *token-forwarding* algorithms, which do not manipulate tokens in any way other than storing, copying, and forwarding them. Token-forwarding algorithms are simple and easy to implement, typically incur low overhead, and have been widely studied (e.g. see [29, 33]). In any n -node *static* network, a simple token-forwarding algorithm that pipelines tokens up a rooted spanning tree, and then broadcasts them down the tree completes *k-gossip* in $O(n + k)$ rounds [36, 33]; this is tight since $\Omega(n + k)$ is a trivial lower bound due to bandwidth constraints. A central question motivating our study is whether a linear or near-linear bound is achievable for *k-gossip* on dynamic networks. It is important to note that algorithms that *manipulate* tokens, e.g., network coding based algorithms, have been shown to be efficient in dynamic settings [21], but are harder to implement and incur a large overhead in message sizes.

Several models have been proposed for dynamic networks in the literature ranging from stochastic models [6, 12] to weak and strong adaptive adversaries [26]. In this paper we consider one of the most basic models known as the *oblivious adversary* [6] or the *evolving graph* model [24, 17, 16, 34]. In this model, the adversary is unaware of any random decisions of the algorithm/protocol and must fix the sequence of graphs before the algorithm starts. The oblivious adversary can choose an arbitrary set of communication links among the (fixed set) of nodes for each round, with the only constraint being that the resulting communication graph is connected in each round. Formally, *oblivious adversary* fixes an infinite sequence of connected graphs $\mathcal{G} = G_1, G_2, \dots$ on the same vertex set V ; in round t , the algorithm operates on graph G_t . The adversary knows the algorithm, but is unaware of the outcome of its random coin tosses.

The oblivious adversary model captures worst-case dynamic changes that may occur independent of the algorithm's (random) actions. On the other hand, an *adaptive* adversary can choose the communication links in every round — depending on the actions of the algorithm — and is much stronger. Indeed, strong lower bounds are known for these adversaries [15, 23]: in particular, for the *strongly adaptive adversary*¹, there exists a $\tilde{\Theta}(nk)$ lower bound² for *k-gossip*, essentially matching the trivial upper bound of $O(nk)$.

The main focus of this paper is on closing the gap for the complexity of *k-gossip* under an oblivious adversary between the straightforward upper bound of $O(nk)$ and the trivial lower bound of $\Omega(n + k)$. In particular, can we achieve an upper bound of the form $\tilde{\Theta}(n + k)$? In fact, it is not even clear whether there even exists a *centralized* algorithm that can do significantly better than the naive bound of $O(nk)$.

¹In each round of the strongly adaptive adversary model, each node first chooses a token to *broadcast* to all its neighbors, and then the adversary chooses a connected network for that round with the knowledge of the tokens chosen by each node.

²The notation $\tilde{\Omega}$ hides polylogarithmic factors in the denominator and \tilde{O} hides polylogarithmic factors in the numerator.

The starting point of our study is RAND-DIFF, a simple local randomized algorithm for k -gossip. In each round of RAND-DIFF, along every existing edge (u, v) at that round, u sends a token selected uniformly at random from the difference between the set of tokens held by u and that held by node v , if such a token exists. Note that in RAND-DIFF, a node is aware of the tokens that its neighbours have and therefore RAND-DIFF guarantees progress, i.e., exchange of a missing token along *every* edge where such a progress is possible. Moreover, by using randomization it tries to keep the entropy of token distribution as high as possible in the presence of an adversary. RAND-DIFF is optimal for static networks, while for dynamic networks under an oblivious adversary, it completes k -gossip in $\tilde{O}(n+k)$ rounds for certain initial token distributions which take any token-forwarding algorithm $\tilde{\Omega}(nk)$ rounds under adaptive adversaries [15]³.

1.1 Our Contributions

We present lower and upper bounds for information spreading under the oblivious adversary model.

Lower Bound for RAND-DIFF. We show that RAND-DIFF requires $\tilde{\Omega}(n^{\frac{3}{2}})$ rounds to complete n -gossip under an oblivious adversary with high probability⁴ (Section 2). Our proof shows that even an oblivious adversary can block RAND-DIFF using a sophisticated strategy that prevents some tokens from reaching certain areas of the network. Although the adversary is unaware of the algorithm’s random choices, the adversary can exploit the randomization of the algorithm to act against its own detriment.

Lower bound for symmetric knowledge-based algorithms. We use the technical machinery developed for the RAND-DIFF lower bound to attack a broad class of randomized k -gossip algorithms called *symmetric knowledge-based (SKB) algorithms*, which are a subclass of the knowledge-based class introduced in [26] (Section 2.3). In any round, the token sent by a node in a knowledge-based algorithm is based entirely on the set of tokens it possesses over time; an SKB algorithm has the additional constraint that if two tokens first arrived at the node at the same time, then their transmission probabilities are identical. SKB algorithms are quite general in the sense that each node can use any probabilistic function that may depend on the node’s identity and the current round number to decide which token to send in a round. Indeed, this offers an attractive algorithmic feature that does not exist in RAND-DIFF: exploitation of information on the history of token arrivals. We show that this may not help achieve a near-linear bound: any SKB algorithm for n -gossip requires $\tilde{\Omega}(n^{\frac{4}{3}})$ rounds whp, under a stronger kind of oblivious adversary, which is also allowed to add tokens from the universe of n tokens to any node in any round.

We do not know whether either of the above lower bounds is tight. Our bounds do raise some intriguing questions: Can n -gossip be even solved in $O(n^{2-\epsilon})$ rounds (for some constant $\epsilon > 0$) rounds by any algorithm? Are there restricted versions of the oblivious adversary that are more amenable to distributed algorithms? We present two upper bound results that partially answer these questions.

Upper bound for RAND-DIFF under restricted oblivious adversaries. We introduce a new model for dynamic networks which restricts the oblivious adversary in the extent and location of dynamics it can introduce (Section 3). In the *paths-respecting* model, we assume that in each round, the dynamic network is a subgraph of an underlying *infrastructure graph* \mathcal{N} ; furthermore, for every pair (s, d) of nodes in \mathcal{N} , there exists a set N_{sd} of simple vertex-disjoint paths from s to d in \mathcal{N} such that in any round the adversary can remove at most $N_{sd} - 1$ edges from these paths. The paths-respecting model is quite general and of independent interest in modeling and analyzing protocols for dynamic networks.⁵ A basic special case of the paths-respecting model is one where \mathcal{N} is a λ -vertex-connected graph and the adversary fails at most $\lambda - 1$

³Actually, [15] shows the $O(n \text{ polylog}(n))$ bound applies even for a weaker protocol called SYM-DIFF, where the token exchanged between two neighbouring nodes is a random token from the *symmetric difference* of the token sets of the two nodes.

⁴Throughout, by “with high probability” or *whp*, we mean with probability at least $1 - 1/n^c$, where the constant c can be made sufficiently large by adjusting other parameters in the analysis.

⁵Indeed, an infrastructure-based model captures many real-world scenarios involving an underlying communication network with dynamics restricted to the network edges. This is unlike the case of a general oblivious adversary where the graph can change arbitrarily from round to round.

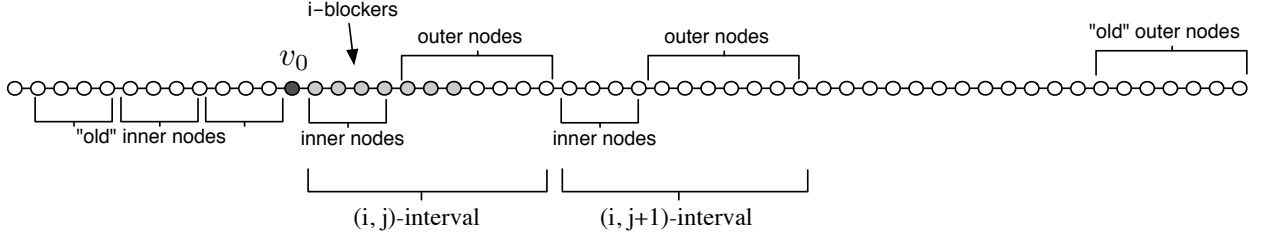


Figure 1: The dynamic line network for the lower bound for RAND-DIFF

edges in each round. Even for this special case, it is not obvious how to design fast distributed algorithm for n -gossip. In Section 3, we also present examples in this model where the adversary can remove a constant fraction of the edges of an infrastructure graph. We show that RAND-DIFF completes n -gossip in $\tilde{O}(n^{5/3})$ rounds under the *paths-respecting model* (Section 3). From a technical standpoint, this result is the most difficult one in this paper; it relies on a novel delay sequence argument, which may offer a framework for other related routing and information dissemination algorithms in dynamic networks.

A $\min\{nk, \tilde{O}((n+k)\sqrt{n})\}$ centralized algorithm for k -gossip. Finally, we present a centralized algorithm (cf. Section 4) that completes k -gossip in $\min\{nk, \tilde{O}((n+k)\sqrt{n})\}$ rounds (and hence n -gossip in $\tilde{O}(n^{3/2})$ rounds) whp, under an oblivious adversary. This answers the main open question affirmatively, albeit in the *centralized* setting. This result provides the first *sub-quadratic* token dissemination schedule in a dynamic network controlled by an oblivious adversary. One of the key ingredients of our algorithm is a load balancing routine that is of independent interest: n tokens are at a node, and the goal is to distribute these tokens among the n nodes, without making any copies of the tokens. This load balancing routine is implemented in a centralized manner; its complexity in the distributed setting under an oblivious adversary, however, is open. We believe that our centralized algorithm is a step towards designing a possible subquadratic-round fully distributed algorithm under an oblivious adversary.

2 An $\tilde{\Omega}(n^{1.5})$ lower bound for RAND-DIFF

In this section, we show that there exists an oblivious adversary under which RAND-DIFF takes $\tilde{\Omega}(n^{3/2})$ rounds to complete n -gossip whp. We will establish this result in two stages. In the first stage, we will introduce a more powerful class of adversaries, which we refer to as *invasive* adversaries. Like an oblivious adversary, an invasive adversary can arbitrarily change the graph connecting the nodes in each round, subject to the constraint that the network is connected. In addition, an invasive adversary can add, to each node, an arbitrary set of tokens from the existing universe of tokens. Similar to an oblivious adversary, an invasive adversary needs to specify, for each round, the network connecting the nodes as well as the tokens to add to each node, in advance of the execution of the gossip algorithm.

In Section 2.1, we will first show that there exists an invasive adversary under which RAND-DIFF takes $\tilde{\Omega}(n^{3/2})$ rounds to complete n -gossip whp. In Section 2.2, we will simulate the token addition process using RAND-DIFF and extend the lower bound claim to oblivious adversaries.

At a very high level, we use a dedicated set of m tokens, for a suitable choice of m , to block progress of an arbitrary token for a number of rounds super-linear in m . A judicious repetition of this process, together with appropriate network dynamics, and a careful setting of parameters then yields the desired lower bound.

2.1 Lower bound under an invasive adversary

Our invasive adversary proceeds in $\sqrt{n}/(2 \log n)$ *phases*, each phase consisting of $\Omega(n)$ rounds, divided into *segments* of \sqrt{n} rounds each. Throughout the process, the network is always a line, you can refer to Fig. 1 throughout the description on the network. We build this line network by attaching two line networks

– which we refer to as *left* and *right* lines – each of which has the same designated source node v_0 at one of its ends. The size of the left line keeps growing with time, while the size of the right line shrinks with time. After the end of each segment, we move $\log n$ nodes closest to v_0 in the right line to the left line, so the size of the left line at the start of segment j of phase i is exactly $((i-1)\frac{\sqrt{n}}{3} + j-1)\log n$.

At the start of each phase, we label the nodes in the right line (other than the source v_0) as v_1 through v_p (where p is the number of nodes in the right line at that time). For any j , we refer to set $\{v_l : 2(j-1)\sqrt{n} \leq l < 2j\sqrt{n}\}$ as the (i, j) -interval. We refer to the first $\log n$ nodes of the (i, j) -interval as the (i, j) -inner nodes, and the remaining $2\sqrt{n} - \log n$ nodes as the (i, j) -outer nodes.

Initially, v_0 has all of the n tokens and every other node has no token. We arbitrarily partition the *tokens* into \sqrt{n} groups of \sqrt{n} tokens each. We use B_i to denote the i th group, and refer to any token in B_i , $1 \leq i \leq \sqrt{n}/(2\log n)$ as an *i -blocker* since the adversary will use the tokens in B_i in phase i to impede the progress of tokens not in $\cup_{j \leq i} B_j$. Let $M(u)$ denote the set of tokens in node u at any time.

At a high level, our adversary operates as follows. Throughout segment j of phase i , the adversary keeps the line unchanged. At the start of segment j , the adversary adds randomly chosen subsets of tokens from B_i to the \sqrt{n} nodes of (i, j) -interval which are the \sqrt{n} consecutive nodes adjacent to v_0 from the right. We argue that this action ensures that in subsequent $\varepsilon\sqrt{n}$ rounds, no token outside the set $\cup_{i' \leq i} B_{i'}$ makes it to an (i, j) -outer node. Since in each phase the adversary uses the same set of \sqrt{n} tokens, namely B_i , as “blockers”, it can continue this for $\Omega(\sqrt{n}/\log n)$ phases, and ensure that whp, no token in, say $B_{\sqrt{n}}$, has reached the right line in $\Omega(n^{3/2}/\log n)$ rounds. We now formally describe how our adversary operates.

Phase i , $1 \leq i \leq \sqrt{n}/(2\log n)$:

- **Segment j , $1 \leq j \leq \sqrt{n}/3$:** The network is a line, that has two parts. The first part is the left line with v_0 at one end, connected to all the (i', j') -inner nodes, where either *A*: $i' < i$ or *B*: $i' = i$ and $j' < j$. The second part is a line with v_0 at one end connected to (i, j') -intervals in sequence, $j' \geq j$, followed by (i, j') -outer nodes, $j' < j$.
 - **Pre-Segment Insertion:** For each token τ in B_i and each node v among the first \sqrt{n} nodes of (i, j) -interval nodes: adversary inserts τ in v independently with probability $1/2$.
 - **Run:** Execute RAND-DIFF for $\varepsilon\sqrt{n}$ rounds of segment j .
 - **Post-Segment Shifting:** The adversary moves the (i, j) -inner nodes to the left line, and the (i, j) -outer nodes to the right end of the line and connect the $(i, j+1)$ -interval to v_0 .
- **Post-Phase Insertion:** For every node in the right line, the adversary inserts any token missing from B_i .

Lemma 2.1. *In every round of phase i and segment j , for any of two adjacent nodes u and v on the (i, j) -inner nodes, the probability that $|M(u) - M(v)|$ is less than $\sqrt{n}/16$ is at most $e^{-\Omega(\sqrt{n})}$.*

Proof. Let X be the random variable denoting the number of tokens node u has but node v does not have, at the start of segment j . Clearly, X equals $\sum_{\tau \in B_i} I_\tau$, where I_τ is the indicator variable for token τ ; I_τ is 1 if u has token τ and v does not have τ ; otherwise it is 0. Using linearity of expectation, we obtain $E[X] = \sum_{\tau \in B_i} E[I_\tau]$. Since the adversary adds each token to each node with probability of $1/2$ independently, we have $E[I_\tau] = 1/4$ and $E[X] = \sqrt{n}/4$. Using a standard Chernoff bound argument, we obtain that the probability that $X \leq \sqrt{n}/8$ is $e^{-\Omega(\sqrt{n})}$. During the remainder of segment j , since each node has two neighbors on the line, node v may receive at most $2\varepsilon\sqrt{n}$ new tokens. Thus, $|M(u) - M(v)|$ is at least $\sqrt{n}/8 - 2\varepsilon\sqrt{n}$ whp (for $\varepsilon \leq 1/32$, this difference is at least $\sqrt{n}/16$). \square

Lemma 2.2. *In segment j of phase i , the probability that any token in $\cup_{i' > i} B_{i'}$ reaches an (i, j) -outer node is at most $1/n^9$.*

Proof. Let α be an arbitrary token in the set $\cup_{i' > i} B_{i'}$. By Lemma 2.1, the probability that at an arbitrary round token α is sent from one node to its adjacent node on (i, j) -interval is at most $16/\sqrt{n}$. The probability

that token α goes further than $\log n$ steps during segment j (which is $\varepsilon\sqrt{n}$ rounds) is at most $\left(\frac{\varepsilon\sqrt{n}}{\log n}\right)\left(\frac{16}{\sqrt{n}}\right)^{\log n}$, which is $O(1/n^{10})$. Now using union bound, we obtain that the probability that any token in $\cup_{i'>i}B_{i'}$ reaches any i -outer node is at most $n/n^{10} = 1/n^9$. \square

Lemma 2.3. *At the end of phase i , the set of tokens in any node $\neq v_0$ in the right line is $\cup_{i'\leq i}B_{i'}$ whp.*

Proof. The proof is by induction on i . For convenience, we set the induction base case to be $i = 0$ and assume B_0 is the empty set. So the base case, at the start of the algorithm, is trivial since initially every node other than v_0 has no tokens. For the induction step, we consider phase $i > 0$. Let R_i denote the set of nodes in the right line at the end of phase i . We first observe that $R_i \subseteq R_{i-1}$. By the induction hypothesis, it follows that the token set at every node in R_i at the end of phase $i - 1$ is precisely $\cup_{i'<i}B_{i'}$. Furthermore, the adversary guarantees that every node in R_i has all tokens from B_i at the end of phase i .

It remains to prove that no token from $\cup_{i'>i}B_{i'}$ arrives at any node in R_i during phase i . Our proof is by contradiction. Let v be the first node in R_i to receive a token τ from $\cup_{i'>i}B_{i'}$ in phase i . Since v is first such node, it received τ from v_0 or from an (i, j) -inner node since R_i is the union of the sets of all (i, j) -outer nodes. Now, v can be connected to such an (i, j) -inner node only during segment j . By Lemma 2.2, however, no (i, j) -outer node receives a token from $\cup_{i'>i}B_{i'}$ whp. \square

Theorem 2.1. *Under the invasive adversary defined above, whp, RAND-DIFF requires $\Omega(n^{3/2}/\log n)$ rounds to complete n -gossip.*

Proof. Each phase consists of $\sqrt{n}/3$ segments, with each segment having $\varepsilon\sqrt{n}$ rounds. So the total number of rounds after $\sqrt{n}/(2\log n)$ phases is $\Omega(n^{3/2}/\log n)$. We obtain that after $\sqrt{n}/(2\log n)$ phases, the size of the left line is at most $n/2$, implying that the right line has $\Omega(n)$ nodes. By Lemma 2.3, whp, every node in the right line is missing at least one token, completing the proof of the theorem. \square

2.2 Lower bound under an oblivious adversary

In this section, we extend the lower bound established in Section 2.1 to oblivious adversaries. Thus, the adversary can no longer insert tokens into the network nodes; the pre-segment insertion and post-phase insertion steps of the adversary of Section 2.1 are no longer permitted. We simulate these two steps using RAND-DIFF and a judicious use of (oblivious) network dynamics.

We now describe how to implement the token insertion process for RAND-DIFF with an oblivious adversary.

Pre-Segment Insertion: The pre-segment insertion step occurs at the beginning of every segment j of every phase i . Our implementation varies depending on whether j is 1 or greater than 1. Let $X_{i,j}$ denote the set of $n^{1/2}$ nodes in the (i, j) -interval that are nearest to v_0 .

- **Token insertion for first segment of phase i :** To implement token insertion, we add two new rounds to the first segment. In the first round, the adversary adds an edge from v_0 to each node in $X_{i,1}$ (we refer to these as *direct* edges). For the second round of the phase, the adversary removes the direct edges added above (except the one that connects v_0 to its right neighbor), adds back the edge from v_0 to its neighbor in the right line, and adds an edge between any two nodes in $X_{i,1}$, independently with probability $1/2$. The remainder of the network is unchanged from the previous round. The remainder of the first segment follows exactly the process of the invasive adversary, as in Section 2.1. Consistent with our notation for the invasive adversary, we define B_i to be the set of tokens that were transferred from v_0 to $X_{i,1}$ in the first round of phase i (the i -blockers).
- **Token insertion for remaining segments of phase i :** Between two consecutive segments j and $j + 1$, $j \geq 1$, of a phase i , we need a mechanism to transfer the set B_i of tokens introduced into the nodes in the set $X_{i,j}$ to the nodes in $X_{i,j+1}$. The oblivious adversary achieves this in two steps. First, in addition to

the line network, it forms a clique for $\Theta(\log n)$ rounds among all the outer nodes in $X_{i,j}$. Then, for one round, the adversary adds a biclique between all the outer nodes in $X_{i,j}$ and all nodes in $X_{i,j+1}$.

We now show that the token distributions inserted by the invasive adversary of Section 2.1 are achieved by the actions of the above oblivious adversary. We begin by showing in Lemmas 2.4 through 2.5 that the pre-segment insertion step for the first segment of each phase is faithfully implemented.

Lemma 2.4. *The number of tokens in B_i is $n^{1/2}(1 - o(1))$ whp.*

Proof. Let M_i denote the complement of set $\cup_{i' < i} B_{i'}$. For any token τ in M_i , let I_τ be an indicator random variable that is 1 if token τ is sent to a node in $X_{i,1}$ through any of the $n^{1/2}$ links from v_0 , and 0 otherwise. We thus have $E[I_\tau] = 1 - (1 - \frac{1}{|M_i|})^{n^{1/2}}$. Then, the expected size of B_i , that is, the expected number of different tokens that the nodes in X_i together collect in the first round of phase i is

$$|M_i| \left(1 - \left(1 - \frac{1}{|M_i|} \right)^{n^{1/2}} \right) = n^{1/2}(1 - o(1)),$$

since $|M_i|$ exceeds $n/2$ for every phase.

Though the I_τ 's, for different tokens τ , are not independent of one another, a standard application of the method of bounded differences and Azuma's inequality yields that the size of B_i is $n^{1/2}(1 - o(1))$ whp. \square

Lemma 2.5. *Suppose $|B_i|$ is $n^{1/2}(1 - o(1))$. Whp over the random choices in the first round, for any token τ in B_i , and any node v in $X_{i,1}$, τ is in v at the end of round with probability at least $1/2$ and at most some constant $p < 1$, independent of every other token in B_i .*

Proof. The probability that more than c copies of a token exist in $X_{i,1}$ after the end of the first round is at most $\binom{n^{1/2}}{c} (1/|M_i|)^c$, which can be made an arbitrarily small inverse-polynomial by setting c suitably high, since $|M_i| \geq n/2$.

Now, fix a node v in $X_{i,1}$ and a token τ in B_i . Let ℓ denote the number of copies of τ in the nodes of $X_{i,1}$ at the end of the first round of phase i . The probability that τ is in v at the end of the second round is at least $1/2$ (since $\ell \geq 1$) and at most $1 - 1/2^c$. \square

We now show that the pre-segment step for the remaining segments of each phase are implemented faithfully by the oblivious adversary.

Lemma 2.6. *After the $1 + \log n$ rounds introduced by the oblivious adversary between segments j and $j + 1$, the probability that a given token τ in B_i is at a given node v in $X_{i,j+1}$ is a positive constant in $(0, 1)$, independent of every other token in B_i .*

Proof. First, the clique over $\log n$ rounds guarantees that there is a coupon collector process for each outer node in $X_{i,j}$, so that whp, every outer node in $X_{i,j}$ has every token in B_i after the $\Theta(\log n)$ rounds. For the remainder of the proof, we assume that the preceding condition holds.

Fix v in $X_{i,j+1}$ and τ in B_i . In the next round, the probability that τ is sent to v is exactly $1 - (1 - 1/|B_i|)^{\Omega(\sqrt{n})}$. This probability is $e/(e - 1) \pm o(1)$ since $|B_i|$ is $\Omega(\sqrt{n}(1 - o(1)))$ whp and every outer node in $X_{i,j}$ has all of B_i whp before this round. This completes the proof of the desired claim. \square

Post-Phase Insertion: At the end of phase i , our oblivious adversary simulates the insertion process of the invasive adversary, using one round: in addition to the network edges present in the last round of the last segment of the phase, the adversary adds a clique over all nodes in the right line of G , excluding v_0 .

Finally, we show that the post-phase insertion completes correctly whp, and prove the main result.

Lemma 2.7. *Whp, every node in the right line has every token in B_i at the end of phase i .*

Proof. The proof is by induction on phases. For convenience, we set the base case to $i = 0$ with B_0 being the empty set; so the claim is trivially true. We now consider the induction step, which concerns phase i . Fix arbitrary token τ in B_i and an arbitrary u in the right line. We argue that if u does not have τ prior to the last round of phase i , then the probability that node u does not receive token τ in the last round is at most $e^{-\Omega(\sqrt{n})}$.

By Lemma 2.6, there are at least $\sqrt{n} - \log n$ nodes among the nodes in (i, j) -interval that receive τ with at least a constant probability in the pre-segment token insertion process. So whp, $\Omega(n)$ nodes in the right line have token τ before the last round of the phase is executed. Consider the clique among the nodes of the right line in the last round of the phase. By Lemma 2.2 and the induction hypothesis, the size of the difference of the sets of tokens in two neighboring nodes in the right line is at most $n^{1/2}$. So the probability that token τ is not sent by one of these $\Omega(n)$ nodes to a node missing τ is at most $1 - 1/\Theta(n^{1/2})$.

Since each link is independent, the probability that token τ is not sent through any of these links is at least $(1 - 1/n^{1/2})^{\Omega(n)} = e^{-\Omega(n^{1/2})}$. Applying a union bound, we obtain that the probability that a token from B_i is missing at any node in the right line at the end of phase i is at most $n^{3/2}e^{-\Omega(n^{1/2})}$, completing the proof of the desired claim. \square

Thus, we can claim the following theorem.

Theorem 2.2. *RAND-DIFF requires $\Omega(n^{3/2}/\log n)$ rounds whp under an oblivious adversary.*

2.3 Lower bound for symmetric knowledge-based algorithms

In this section, we present a lower bound for a broad class of randomized algorithms for gossip, called *symmetric knowledge-based (SKB)* algorithms. We first introduce some notation. For round t , we define $a_t : U \times V \rightarrow T$, where U is the universe of all tokens and V is the set of all nodes: if τ is at u at the start of round t , then $a_t(\tau, u)$ is the time that τ first arrived at u ; otherwise $a_t(\tau, u)$ is \perp .

Definition 2.1. *An SKB algorithm is specified by a collection of functions $P_{t,u} : U \rightarrow [0, 1]$, where $P_{t,u}(\tau)$ is the probability with which u sends τ to each of its neighbors in round t , satisfying the following properties:*

- **Token transmission:** *for any t , if $a_t(\tau, u) = \perp$, then $P_{t,u} = 0$, the different token sending events for a node in round t are mutually exclusive, and $\sum_{\tau \in U} P_{t,u}(\tau) \leq 1$.*
- **Symmetry:** *for any τ_1, τ_2 such that $a_t(\tau_1, u) = a_t(\tau_2, u)$, $P_{t,u}(\tau_1) = P_{t,u}(\tau_2)$.*

We note that the $P_{t,u}$ may differ arbitrarily from node to node and round to round. The symmetry property and the resulting dependence on the arrival times of tokens are the only constraint on the algorithm.

We now show that there exists an invasive adversary under which SKB takes $\Omega(\frac{n^{4/3}}{\log n})$ rounds to complete n -gossip whp. In order to block the progress of an arbitrary token, the adversary inserts a subset of m tokens, for a suitable choice of m , at the same time as that token reaches a node. We refer to this subset of tokens as a **Blocker Set**. A random selection of the blocker sets, a judicious repetition of this process, together with appropriate network dynamics, yields the desired lower bound.

At the start of the process, the invasive adversary takes $\frac{n}{2 \log n}$ of tokens arbitrarily, and forms $\frac{n^{2/3}}{2 \log n}$ blocker sets $B_{i,k}$ for $1 \leq i \leq \frac{n^{1/3}}{2 \log n}$ and $1 \leq k \leq n^{1/3}$, each consisting of $n^{1/3}$ tokens. Then, the adversary proceeds in $\frac{n^{1/3}}{\log n}$ phases, each phase consisting of $\Omega(n)$ rounds, divided into $n^{2/3}$ segments. Through phase i , the adversary uses blocker sets $B_{i,k}$ for $1 \leq k \leq n^{1/3}$.

Throughout the process, the network is always a line, consisting of three parts – which we refer to as *left*, *middle* and *right*. At the very beginning, the left part and right part are empty, and all nodes of the network are included in the middle part. The left most node of middle part is always called s and has all tokens in U .

The size of left line – nodes at the left side of node s keeps growing with time.

Phase i , $1 \leq i \leq \frac{n^{1/3}}{2 \log n}$: At this time, the left part has $(i - 1)n^{2/3} \log n$ nodes, then the adversary takes all the nodes at the right side of s as the nodes in the middle part, and makes the right part empty.

- **Segment j , $1 \leq j \leq n^{2/3}$:** Segment j is $n^{1/3}$ rounds. Let $v_1, v_2, \dots, v_{n^{1/3}}$ be the first $n^{1/3}$ nodes of middle part next to s , and call $v_1, \dots, v_{\log n}$ (i, j) -inner nodes and call $v_{\log n+1}, \dots, v_{n^{1/3}}$ (i, j) -outer nodes. During segment j , at round k ($1 \leq k \leq n^{1/3}$), the adversary inserts blocker set $B_{i,k}, B_{i,k-1}, \dots, B_{i,1}$ to nodes v_1, v_2, \dots, v_k respectively.
- **Post-Segment Shifting j :** The adversary takes the (i, j) -inner and (i, j) -outer nodes, then moves the first $\log n$ nodes of them to the left part, and moves rest of them, specifically $n^{1/3} - \log n$ nodes to the right part.

Lemma 2.8. *In phase i , segment j , no token in the set $\cup_{i' > i, 1 \leq k \leq n^{1/3}} B_{i',k}$ reaches (i, j) -outer nodes.*

Proof. Consider an arbitrary token τ in $\cup_{i' > i, 1 \leq k \leq n^{1/3}} B_{i',k}$. Since segment j is $n^{1/3}$ rounds, token τ can go at most $n^{1/3}$ far from s . From the definition of segment j , it follows that whenever a token τ reaches a node u in segment j , there is exactly one blocker set which is inserted by adversary at the same round to the same node u . This implies that $P_{t',u}(\tau) \leq \frac{1}{n^{1/3}}$, $t' \geq t$, assuming that arrival time of τ at u is t . The reason is that the algorithm cannot distinguish between a token that has been inserted by the adversary and a token that comes from the source. So the probability that token τ goes one edge further is at most $\frac{1}{n^{1/3}}$, and the probability that it goes beyond the (i, j) -inner nodes is at most

$$\sum_{q=\log n}^{n^{1/3}} \binom{n^{1/3}}{q} \left(\frac{1}{n^{1/3}}\right)^q \left(1 - \frac{1}{n^{1/3}}\right)^{(n^{1/3}-q)} \leq \binom{n^{1/3}}{\log n} \left(\frac{1}{n^{1/3}}\right)^{\log n} = o\left(\frac{1}{n^{10}}\right)$$

□

Lemma 2.9. *Let τ be an arbitrary token from set $\cup_{i' > p, 1 \leq k \leq n^{1/3}} B_{i',k}$. Then after $p = \frac{n^{1/3}}{2 \log n}$ phases, with high probability token τ has not reached any of (i, j) -outer nodes, for $1 \leq i \leq p$, $1 \leq j \leq n^{1/3}$.*

Proof. Using lemma 2.8, the probability that token τ reaches any (i, j) -outer node is as follows

$$\frac{n^{1/3}}{2 \log n} \times n^{2/3} \times \frac{1}{n^{10}} = o\left(\frac{1}{n^9}\right).$$

□

Theorem 2.3. *Under an invasive adversary, SKB requires $\Omega\left(\frac{n^{4/3}}{\log n}\right)$ rounds whp.*

3 Analysis of RAND-DIFF under a paths-respecting adversary

In this section, first we introduce a new model, the *paths-respecting* adversary, under which we show that RAND-DIFF completes n -gossip in $\tilde{O}(n^{5/3})$ rounds whp.

3.1 The paths-respecting model

In the paths-respecting model we assume that there is an underlying infrastructure network \mathcal{N} such that at the start of every round t , the network N_t laid out by the adversary is a subgraph of \mathcal{N} ; we refer to any edge in $\mathcal{N} - N_t$ as an *inactive* or *failed edge* in round t . Before presenting the model, we note that the assumption of an infrastructure network is essentially without loss of generality. For instance, it captures 1-interval connectivity, a central dynamic network model of Kuhn et al [26]: we can let \mathcal{N} be the complete graph and require that N_t be a connected subgraph of \mathcal{N} for each t .

Definition 3.1. *The paths-respecting model places some constraints on \mathcal{N} and the set of edges that the adversary can render inactive in any given round. In particular, we assume that for every pair (s, d) of nodes in \mathcal{N} , there exists a set N_{sd} of simple vertex-disjoint paths from s to d such that the total number of inactive edges of paths in N_{sd} in any round is at most $|N_{sd}| - 1$.*

Before analyzing the paths-respecting model, we present two examples. First, a natural special case of this model is one where \mathcal{N} is a λ -vertex-connected graph and the adversary fails at most $\lambda - 1$ edges in each round. If $\lambda = 2$, then a simple example is that of a ring network in which an arbitrary edge fails in each round. In this example, the adversary is significantly restricted in the number of total edges it can fail in a given round; yet, it is not obvious how a distributed token-forwarding algorithm can exploit this fact since for any pair of vertices, no specific path between the two may be active for more than n rounds over an interval of λn rounds. A radically different example of the paths-respecting model in which the adversary can fail a constant fraction of edges in each round is the following: \mathcal{N} consists of a set of r center vertices and a set of $n - r$ terminals, with an edge between each center and each other vertex. Any two vertices have at least $r - 1$ vertex-disjoint paths between them. An adversary can remove edges between $\lfloor (r - 2)/2 \rfloor$ of the centers and all the terminals – and hence, nearly half of the edges of the network – while satisfying the constraint that at most $r - 2$ edges are removed in any collection of $r - 1$ vertex-disjoint paths passing through the centers.

Our main result here is the analysis of RAND-DIFF in the paths-respecting model.

Theorem 3.1. *Under any n -node paths-respecting dynamic network, RAND-DIFF completes n -gossip in $O(n^{5/3} \log^3 n)$ rounds whp.*

Our proof of Theorem 3.1 proceeds in a series of arguments, beginning with a restricted version of the paths-respecting model, and successively relaxing the restriction until we have the result for the paths-respecting model. Fix a token τ , and source s that has τ at the start of round 0. Let d be an arbitrary node in the network. In our analysis, we focus our attention on the set N_{sd} of vertex-disjoint paths between s and d such that the total number of inactive edges of N_{sd} in any round is at most $|N_{sd}| - 1$. In Section 3.2, we analyze RAND-DIFF under the assumption that the lengths of all paths in N_{sd} are within a factor of two of one another, and the adversary fails at most one edge in any path. In Section 3.3, we drop the restrictions that at most one edge is inactive in any path and path lengths are near-uniform, and complete the proof of Theorem 3.1.

3.2 Near-uniform length paths and at most one inactive edge per path

Lemma 3.1. *Suppose there exists an integer $l > 0$ such that the length of each path in N_{sd} is in $[l, 2l)$. Further suppose that in addition to the conditions of the paths-respecting model, for every path in N_{sd} , the adversary can fail at most one edge in the path in any round. Then, the token τ is at d in $O(n^{5/3} \log n)$ rounds whp.*

The proof of Lemma 3.1 is a *delay sequence argument* that proceeds backwards in time. Delay sequence arguments have been extensively used in the analysis of routing algorithms [28]. A major technical challenge we face in our analysis, distinct from previous use of delay sequence arguments, is network dynamics. The number of possible dynamic networks, even subject to the paths-respecting model, is huge and our analysis cannot afford to account for them independent of the actions of the algorithm.

Pebbles. We consider a run of the algorithm for $T = cn^{5/3} \log n$ rounds, for a sufficiently large constant c . We show that the probability that d is missing any tokens at the end of round T is $1/\text{poly}(n)$. In our analysis, we use a notion of “pebbles moving along the paths”. Each pebble is a *certificate* for the event that a node is missing some tokens; in particular, each pebble has an associated set, which represents a subset of the tokens missing at the node where the pebble is located at that time. Note that since the analysis proceeds

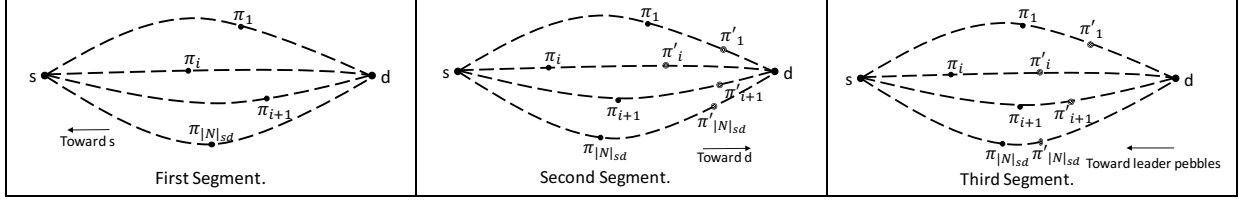


Figure 2: Illustrating the three segments in the analysis of RAND-DIFF

backwards in time, for any node (and any pebble) this set of missing tokens grows during the course of the analysis. We also remark that we use the notion of pebbles for analysis only; pebbles should not be confused with tokens being sent around the network by RAND-DIFF.

Pebble updates. The way a pebble moves along a path is as follows. Suppose a pebble π is located at node v with associated missing set M at the end of round t . Consider a neighbor u of v . We now consider cases depending on what happened in round t of RAND-DIFF along edge (u, v) . If (u, v) was made inactive by the adversary, then we do not gain any more information about missing tokens at v at the end of round $t - 1$. If (u, v) was active, however, and no token was sent along (u, v) , then we know that the set of tokens missing at u at the end of round $t - 1$ is a superset of M ; we can depict this case by having the pebble π , with its associated missing set M , move to u ; we call this a *pebble move*. On the other hand, if (u, v) was active and a token α was sent along (u, v) , then we can depict this case by setting the missing set of pebble π at the end of round $t - 1$ to be $M + \{\alpha\}$; we call this a *missing set update*. Thus, the set of missing tokens associated with a pebble is monotonically nondecreasing.

Phases and segments. Our analysis groups consecutive rounds into phases, starting from round T and proceeding backwards in time. At the start of each phase, we have a pebble located on each path in N_{sd} . We refer to these pebbles as *leader pebbles*; these start from d and proceed toward s during the course of the analysis. Let the leader pebble on the i th path of N_{sd} be labeled π_i , and the location of π_i at the start of round t be $v_i(t)$, and let $u_i(t)$ be the adjacent node to $v_i(t)$, on path i , on the side closer to s . Let L denote the set of leader pebbles. At time t , let $M_i(t)$ denote the missing set associated with the pebble π_i and $S_i(t)$ denote the set of tokens of node $u_i(t)$. Each phase is divided into three segments, which are described below.

We define the *total distance* of the pebble set at the start of any round t , $TD(t)$, to be the sum, over all i , of the distance between $v_i(t)$ and s ; note that since paths are vertex-disjoint, by definition, the $TD(t)$ at any instant is at most n ; our pebble movement process will ensure that the total distance measure is monotonically non-increasing with decreasing t .

Pebble Invariant. We maintain the invariant that at the start of any phase, the missing token sets that we associate with the pebbles are all identical. Consider the start of a phase in round t . Let M denote the set of missing tokens at each of the pebbles. At the beginning of the analysis, $t = T$, each of the $|N_{sd}|$ pebbles is located at node d , with the missing token set being the set of all tokens missing at d at the end of round T . Thus, the above invariant is satisfied at the start of the first phase.

First Segment. The first segment of any phase consists of $T_1 = \max\{l, \frac{n}{7}\}$ rounds. Recall that the length of each path is in $[l, 2l]$; since the paths are vertex-disjoint, the number of paths is at most n/l . If the number of times the leader pebbles move toward node s during the T_1 rounds is at least $T_1/2$, then we let the second and third segments be empty, end this phase and proceed to the next phase. In this case, to maintain the pebble invariant, for the beginning of next phase, we associate with each pebble the same set of missing tokens they had at the end of the last phase. The second and third segments for the other case are described below, after the analysis of the first segment. All segments are illustrated in Figure 2.

Lemma 3.2 (Analysis of first segment). *Given any possible value for sets $S_i(t - T_1)$ and $M_i(t - T_1)$ of tokens, and pebble locations at round $t - T_1$ such that $\text{TD}(t - T_1)$ is greater than $\text{TD}(t) - T_1/2$, we have $\bigcup_i M_i(t - T_1) \setminus M = \Omega(T_1^{1/3}/\log n)$ whp.*

Proof. In each of the T_1 rounds of the first segment, there exists at least one path such that none of its edges are failed by the adversary. Therefore, in each round $t' \in [t - T_1, t]$ either at least one leader pebble is moved toward node s or at least one token is sent from $u_i(t')$ to $v_i(t')$ for some path i , resulting in a missing set update. If the number of times the leader pebbles move toward node s during the T_1 rounds is at least $T_1/2$, then we obtain that $\text{TD}(t - T_1)$ is at most $\text{TD}(t) - T_1/2$. So in the remainder of the proof, we assume that number of times that pebbles move toward node s is at most $T_1/2$. Thus, in at least in half of the rounds of the first segment, a token is sent from u_i to v_i for some i , resulting in a missing set update. We consider two separate cases:

- **case a.** Suppose there exists a path $p_j \in N_{sd}$ such that at least $T_1^{1/3}$ of the $T_1/2$ token exchanges happen on path j . In this case, we infer the following for the size of set of missing tokens of π_j during $[t - T_1, t]$:

$$|M_j(t - T_1) - M| \geq T_1^{1/3} \mid \bigcup_i M_i(t - T_1) - M| \geq T_1^{1/3},$$

completing the proof of the lemma in this case.

- **case b.** If there does not exist a path that has at least $T_1^{1/3}$ of the $T_1/2$ token exchanges, then there exists a set of paths P of size at least $T_1^{2/3}/2$ such that at least one instance of token exchange happens on each of paths in P . For each path in P consider the instance which happens latest according to time during this T_1 rounds. For path $p_i \in P$, let the latest instance of token exchange take place in time $t'_i \in [t - T, t]$. From the definition of RAND-DIFF, node $u_i(t'_i)$ sends a token α_i to $v_i(t'_i)$ from $S_i(t'_i) - M_i(t'_i)$ chosen uniformly at random; this implies that $S_i(t'_i) \cap M_i(t'_i) \neq \emptyset$. Note that $M_i(t') = M$ for $(t' > t'_i)$ since t'_i is the latest time in segment 1, that an instance of token exchange has happened. We also know that at $M \cup \{\alpha_i\} \subseteq M_i(t'_i)$ and $\alpha_i \notin M$.

We group the set P of paths into at most $\log n$ categories: path p_i is in category C_j if $S_i(t'_i) \cap M_i(t'_i)$ is in range $[2^{j-1}, 2^j)$. One of these categories, say C_j , has $\Omega(T_1^{2/3}/\log(n))$ paths. In the following, we show that $\bigcup_i \{\alpha_i\} = \Omega(T_1^{1/3}/\log^2(n))$ whp. Let $k = 2^{j-1}$. We say that a token has *high frequency* if it appears in more than $ck \ln n$ different $S_i(t'_i) \cap M_i(t'_i)$, for a constant $c > 0$ to be specified later; otherwise, we call it as a token with *low frequency*. If a token α has high frequency, then α will be sent by some u_j whp, since

$$\Pr\{\text{None of the nodes that has } \alpha \text{ send it}\} \leq \left(1 - \frac{1}{2k}\right)^{ck \ln n} \leq \frac{1}{n^{c/2}}$$

Every $S_i(t'_i) \cap M_i(t'_i)$ needs to have at least one token with low frequency; otherwise, all its tokens will be sent by some u_j whp. Any token with low frequency will be picked $O(\log n)$ times whp, since the probability of sending each token in a given round is at most $1/k$, and a token with low frequency by definition appears in at most $ck \ln n$ of the sets. Thus, by a Chernoff bound, the number of distinct tokens sent by $S_i(t'_i) - M_i(t'_i)$ for all i which $p_i \in P$ is at least $\Omega\left(\frac{T_1^{2/3}}{k \log^2 n}\right)$ whp, by just considering tokens with low frequencies.

If a particular $S_i(t'_i) \cap M_i(t'_i)$ has at least $k/2$ tokens with high frequency, then the number of distinct tokens selected is at least $k/2$; if there is no such case among any of i 's, then by the above calculation using low frequency tokens, we obtain that the number of distinct tokens selected is at least

$\Omega(T_1^{2/3}/\log^2 n)$ whp. We thus obtain that the number of distinct tokens sent is $\Omega(\max\{\frac{k}{2}, \frac{T_1^{2/3}}{k \log^2 n}\})$, which is minimized for $k = \sqrt{2}T_1^{1/3}/\log n$, yielding a bound of $\Omega(\frac{T_1^{1/3}}{\log n})$. Hence, we obtain that $\bigcup_i M_i(t - T_1) - M = \Omega(T_1^{1/3}/\log n)$ whp, completing the proof of the lemma. \square

Second segment. The second segment consists of T_2 rounds, where T_2 may vary from phase to phase. At the start of the second segment we introduce a new pebble π'_i at each node v_i , with associated missing token set M_i . These pebbles proceed toward d , updating their missing token sets as they move toward d . Let t_m denote the time that it takes the last pebble π'_m to arrive at d . Then, the missing token set at d at this time is a superset of the union, over i , of M_i , which has at least $\Omega(T_1^{1/3}/\log(n))$ tokens more than M whp, by Lemma 3.2.

We consider two cases. If t_m is at most $2T_1^{1/3}$, then we set $T_2 = t_m$, calling this an end to the second segment. Otherwise, we consider two subcases. In the first subcase, the number of tokens in the missing set associated with the pebble that arrived last is at least $|M| + (t_m - T_1)/2$. In this subcase, we set $T_2 = t_m$, calling this an end to the second segment. We obtain that the number of missing tokens at d is at least $|M| + (t_m - T_1)/2$.

In the second subcase, the number of tokens in the missing set associated with the pebble that arrived last is less than $|M| + (t_m - T_1)/2$. This implies that the pebble π'_m was blocked for at least $(t_m - T_1)/2$ steps on its way to d , which in turn implies that the pebble π_m has increased its missing token set size by at least $\frac{(t_m - T_1)}{2} - p$ tokens. We now create a new copy of pebble π_m at v_m (note that v_m may have changed since the start of the second segment) and send this pebble again toward d . Again, we consider the time it takes for this new pebble to reach d . If this is within $\Theta(T_1)$, or the arriving pebble gained tokens at least a constant fraction of the time spent, we end the second segment. Otherwise, π_m has gained tokens at least a constant fraction of the time spent, in which case we repeat this argument.

We continue this until either the missing set of pebble π_m is the set of all tokens, or we find that d is missing tokens whose size is $|M| + \Omega(T_2)$. It is easy to argue that under RAND-DIFF, every node receives at least one token in $O(n)$ rounds, so the first of the two possibilities cannot happen. We thus have the following lemma.

Lemma 3.3 (Analysis of the second segment). *If T_2 is the time taken for the second segment, then the set of missing tokens at d at time $t - T_1 - T_2$ has size at least $|M| + \Omega(T_1^{1/3}/\log(n))$ if T_2 is $O(T_1)$, and at least $|M| + \Omega(T_2)$ otherwise. \square*

Third Segment. In the third segment, we send pebbles from d to v_i along each path i , so that the pebbles end up at the same position as the start of the phase. Again, we consider the last time t'_m at which the pebble on path say m reaches v_m . If t'_m is $O(\max\{T_1, T_2\})$, then we terminate the third segment and the phase. In this case, by Lemma 3.3, we obtain at the end of this phase that for each i , the pebble at v_i has missing set that has increased by size either $\Omega(T_1^{1/3}/\log(n))$, if the time of the phase is $O(T_1)$, or by at least a constant fraction times the length of the phase.

If t'_m is $\Omega(\max\{T_1, T_2\})$, we find that during this phase, the missing set of tokens at v_m has increased by at least a constant fraction times the length of the phase so far (but possibly not at other nodes v_i). As in the second segment, we send a pebble from v_m back to d ; we repeat the argument, always having a node whose number of missing tokens exceeds $|M|$ by a number that is at least a constant fraction times the current duration of the phase. This cannot go on for more than linear number of rounds, so it ends in the situation where for all i , the pebble at v_i has missing set that has increased by size either $\Omega(T_1^{1/3}/\log(n))$, if the time of the phase is $O(T_1)$, or by at least a constant fraction times the length of the phase.

Lemma 3.4 (Analysis of third segment). *If T_3 is the time taken for the third segment, then each of the pebbles at v_i has an associated missing set M' of tokens, where $|M'|$ is at least $|M| + \Omega(T_1^{1/3}/\log(n))$ if $T_2 + T_3$ is $O(T_1)$, and at least $|M| + \Omega(T_2 + T_3)$ otherwise. \square*

We are now ready to complete the proof of Lemma 3.1. Consider any phase of length L . If the phase ends after the first segment (hence, has length T_1), we have a decrease in the total distance measure by half the number of rounds in the phase. Otherwise, by Lemma 3.4, we have whp that the number of missing tokens associated with the pebbles at the end of the phase increases by at least $T_1^{1/3}/\log n$, if $L = O(T_1)$, and $\Omega(T_1)$, otherwise. Hence, in a phase, either the rate of decrease of total distance per round is at least $1/2$, or the rate of increase of the number of missing tokens is at least $1/(n^{2/3} \log n)$. Since the total distance measure is initially n and is always nonnegative, and the number of missing tokens is initially 1 and is at most n , it follows that T is $O(n^{5/3} \log n)$ whp, completing the proof of Lemma 3.1.

3.3 Removing restriction on path lengths and inactive edges per path

We first extend the claim of the preceding section to the case where the adversary can fail an arbitrary number of edges in any path of N_{sd} , subject to the constraint imposed by the paths-respecting model that the number of inactive edges in N_{sd} is at most $|N_{sd}| - 1$. We continue to make the assumption of near-uniform path lengths. In a round, call a path *active* if none of its edges is failed, *1-inactive* if exactly one of its edges is inactive, and *dead* if more than one of its edges are inactive. Since the adversary can fail at most $|N_{sd}| - 1$ edges among $|N_{sd}|$ disjoint paths, it follows that the number of active paths is at least one more than the number of dead paths. This is the only constraint we place on the adversary that we analyze in this section: the number of active paths is at least one more than the number of dead paths.

Lemma 3.5. *Suppose there exists an integer $l > 0$ such that the length of each path in N_{sd} is in $[l, 2l)$. Further assume that the number of dead paths is in $[a, 2a)$ for some a , in each round. Then, under RAND-DIFF, τ is at d in $O(n^{5/3} \log n)$ rounds whp.*

Proof. We consider the run of RAND-DIFF for $T = c'n^{5/3} \log n$ rounds, where c' is a sufficiently large constant. We use the probabilistic method [1] to show that in the time interval $[1, T]$, there exists a set R of $cn^{5/3} \log n$ (not necessarily consecutive) rounds in which there is a subset P of paths such that none of the paths from P is dead and at least one of paths in P is active in each round in R . (Here, the constant c can be made sufficiently large by choosing c' appropriately.) We then invoke Lemma 3.1 to establish the desired claim.

All that remains is to establish the existence of P and R as required above. We choose a set P of paths by picking each path independently with probability $\frac{1}{a+1}$. From the assumption of the lemma, we know that any round has at most $2a$ number of dead paths. Therefore, the probability that none of these dead paths are in P is at least $(1 - \frac{1}{a+1})^{2a} \geq \frac{1}{e^2}$. By assumption, we have that in any round there exists at least $a + 1$ active paths. Therefore, the probability that there is at least one active path among paths in P in round r is at least $1 - (1 - \frac{1}{a+1})^{a+1} \geq 1 - \frac{1}{e}$.

Since paths in P are picked independently, the probability that in a certain round, no path in P is dead and at least one path in P is active is at least $\frac{1}{e^2}(1 - \frac{1}{e})$. Thus, the expected number of rounds that have no dead paths and have at least one active path in P is $c' \frac{e-1}{e^3} n^{5/3} \log n$. For $c' \frac{e-1}{e^3} \geq c$, there exists a set P of paths and set R of at least $cn^{5/3} \log n$ rounds such that:

- for all p in P and r in R , p is either active or 1-inactive during round r .
- for each r in R , there exists at least one path p in P such that p is active in round r .

This establishes the existence of P and R as desired, and completes the proof of the lemma. \square

Now, we extend Lemma 3.5 by removing the constraint on the number of dead paths.

Lemma 3.6. *Suppose there exists an integer $l > 0$ such that the length of each path in N_{sd} is in $[l, 2l)$. Then, in the paths-respecting model, using RAND-DIFF, the token τ is at d in $O(n^{5/3} \log^2 n)$ rounds whp.*

Proof. We divide the $c'n^{5/3} \log^2 n$ rounds into $\log n$ classes, where the i th class consists of rounds in which the number of dead paths is in the interval $[2^i, 2^{i+1})$. By simple averaging, we obtain that there is at least one class with at least $c'n^{5/3} \log n$ rounds, and the number of dead paths in any round in this class is in $[2^i, 2^{i+1})$ for some integer $i > 0$. We now apply Lemma 3.5 to establish the desired claim. \square

We now complete the proof of Theorem 3.1, restated below, by removing the assumption of near-uniform path lengths in the paths of N_{sd} . This is a standard argument in which we incur another multiplicative factor of $\log n$ in our bound.

Theorem 3.2. *In the paths-respecting model, RAND-DIFF completes gossip in $O(n^{5/3} \log^3(n))$ rounds whp.*

Proof. Let τ be any token located at a source node s at the start of round 0, and let d be any other node. Consider the set N_{sd} of paths from s to d with the property that the number of edges which fail in paths of N_{sd} in any round is at most $|N_{sd}| - 1$.

We divide these $|N_{sd}|$ paths into $\log n$ groups such that group i ($1 \leq i \leq \log n$) includes paths of length between 2^i and $2^{i+1} - 1$. Let λ_i show the number of paths in group i . By simple averaging, we obtain that in any round there is a group i of paths such the number of edges removed by the adversary from that group is at most $\lambda_i - 1$ in that round. Then if we consider run of algorithm for $T = c'n^{5/3} \log^3 n$ rounds, there is a group i of paths such that for at least $c'n^{5/3} \log^2 n$ rounds the number of edges removed by the adversary from that group in each of these rounds is at most $\lambda_i - 1$. We now apply Lemma 3.6 to derive that d receives token τ in at most T rounds whp.

Since the above claim holds for each token τ , a union bound yields us that RAND-DIFF completes gossip in $O(n^{5/3} \log^3 n)$ rounds whp. \square

4 Centralized k -gossip in $\min\{nk, \tilde{O}((n+k)\sqrt{n})\}$ rounds

In this section we present a centralized algorithm that completes k -gossip in $\tilde{O}((n+k)\sqrt{n})$ rounds against any oblivious adversary. Since k -gossip can be completed in nk rounds by separately broadcasting each token over n rounds, this yields a bound of $\min\{nk, \tilde{O}((n+k)\sqrt{n})\}$ on centralized k -gossip using token forwarding.

We begin by arguing that a $\tilde{O}(n^{3/2})$ -round algorithm for n -gossip implies a $\tilde{O}((n+k)\sqrt{n})$ -round algorithm for k -gossip. We first make the assumption that k is a multiple of n . When k is not a multiple of n , then we add distinct dummy tokens to make the total number of tokens a multiple of n . Given that the bound we seek is at least linear in $n+k$, this maintains the asymptotic complexity of the bound. When k is less than n , we introduce $n-k$ dummy tokens. When k exceeds n , we group the k tokens into $\lfloor k/n \rfloor$ sets of n tokens and one set of less than n tokens. With this reduction, it is easy to see that an $\tilde{O}(n^{3/2})$ -round n -gossip algorithm implies an $\tilde{O}((n+k)\sqrt{n})$ -round k -gossip for arbitrary k .

We present our centralized algorithm for n -gossip in two parts. We first solve a special case of n -gossip – n -broadcast – in which all the tokens are located in one node. We then extend the claim to arbitrary initial distributions of the n tokens. We start by introducing two useful subroutines: *random load balancing* and *greedy token exchange*.

4.1 Random load balancing and greedy token exchange

In the random load balancing subroutine, we have a set F of nodes, each of which contains the same set T of at least n items (each item is a copy of some token), and a set R of nodes such that $F \cup R$ is the set of all n nodes. The goal is to distribute the items among nodes in R such that the following properties hold at the end of the subroutine: (B1) each item in T is in exactly one node in R ; (B2) every node has either $\lfloor |T|/|R| \rfloor$ or $\lceil |T|/|R| \rceil$ items; (B3) the set X of items placed at any subset $S \subseteq R$ of nodes is drawn uniformly at random from the collection of all subsets of T of size $|X|$.

LoadBalance(F, T, R): Assign a rank to each item in T using a random permutation. In round i , $i \in [|T|]$:

1. Identify a node $v \in R$ that has been distributed fewer than $\lfloor |T|/|R| \rfloor$ items yet, and is closest to a node in F , say v_0 , among all such nodes in R .
2. Let P denote a shortest path from v_0 to v . Let ℓ be the number of edges in P , and let (v_{j-1}, v_j) , $0 \leq j < \ell$, denote the j th edge in P ; so $v_\ell = v$. Then, v_0 sends item of rank i to v_1 ; in parallel, for every edge (v_{j-1}, v_j) , $1 \leq j < \ell$, v_{j-1} sends an arbitrary item it received earlier in this subroutine to v_j .

Lemma 4.1. *The subroutine **LoadBalance**(F, T, R) completes in $|T|$ rounds and satisfies the properties (B1), (B2), and (B3).*

Proof. The number of rounds taken by the subroutine is by construction. Property (B1) is satisfied since no copies of items are made. In each round, the number of items placed at exactly one node in R increases by 1, while the number of items at other nodes remains the same. So the total number of items placed at the nodes in R at the end of $|T|$ rounds is exactly $|T|$. Furthermore, no node in R receives more than $\lceil |T|/|R| \rceil$ items; this establishes property (B2). Finally, property (B3) is satisfied since the items are placed in order of a random permutation. \square

The greedy token exchange is a one round subroutine in which the goal is to maximize the number of new tokens received at each node in that round.

GreedyExchange: Fix a round. For each node v , let $S(v)$ be the set of tokens that node v has at the start of the round. Let N_v denote the set of neighbors of v . Let U_v be the set $\cup_{u \in N_v} S(u) \setminus S(v)$. For each node v , we perform the following operations. Construct a bipartite graph H_v , in which one side is the set N_v , and the other side is the set U_v . For each $u \in N_v$ and $\tau \in U_v$, there is a link between u and τ if token $\tau \in S(u)$. Compute a maximum bipartite matching M_v in H_v . If (τ, u) is in M_v , then u sends token τ to v .

Lemma 4.2. *In each round, the subroutine **GreedyExchange** maximizes, for each node v , number of new tokens that can be added to the node in that round.*

Proof. Since each node can send a distinct token on each of its incident edges, the problem of maximizing the number of distinct tokens received by a node is independent of the same problem for a different node. By construction of the bipartite graph, for every possible set S of tokens arriving at a node v , we have a bipartite matching M_v in H_v such that for every $\tau \in S$, there exists an edge (τ, u) for some u in N_v . Similarly, every bipartite matching M_v corresponds to a valid set of token transfers to v in the network at that round. Thus, **GreedyExchange** maximizes, for each node v , number of new tokens that can be added to the node in that round. \square

4.2 n -broadcast

We now present a $\tilde{\Theta}(n^{3/2})$ -round algorithm for n -broadcast, where all tokens are located initially in a single node. The algorithm consists of $O(\log n)$ stages. Let U denote the set of all n tokens. We now describe each stage. Call a node *full* if it has all of the n tokens at the start of the stage, and *non-full* otherwise. Let R denote the set of non-full nodes at the start of the stage, and let $r = |R|$. The stage consists

of $\Theta(\sqrt{n} \log n)$ identical phases. Each phase consists of a sequence of steps divided into two segments: distribution and exchange.

1. **Distribution segment:** Distribute the n tokens among the non-full nodes R in the network, as evenly as possible, in n rounds by running **LoadBalance**(F, R, U).
2. **Exchange segment:** Starting with the distribution of tokens as specified in the preceding distribution segment; i.e., each full node has all tokens, and each non-full node has exactly the tokens distributed in the above segment, run n rounds of **GreedyExchange** maximizing the total number of new tokens received by the nodes in each round.

Lemma 4.3. *If R is the set of non-full nodes at the start of a stage, then during any phase of the stage, the sum, over all nodes in R , of the number of tokens received by the nodes is $\Omega(|R|\sqrt{n})$.*

Proof. Fix a stage and a phase of the stage. Consider the following initial distribution of tokens at the start of the phase: each full node at the start of the stage has all of the tokens, while each non-full node has no token. At the start of any round, we use *configuration* to refer to the set of tokens that a node has at the start of the round, starting from the preceding initial token distribution.

Fix a round of the exchange segment. We first consider the case in which there exists a round in which the number of different configurations at the start of the round, m , is less than \sqrt{n} . We number the m configurations arbitrarily from 1 to m , and let n_i denote the number of nodes in the i th configuration. By properties (B1) and (B2) of Lemma 4.1, after the distribution segment, each non-full node started with $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ distinct tokens that are unique among all non-full nodes. Therefore, in any set of ℓ nodes that have the same configuration, every node has at least $\ell \lfloor n/r \rfloor$ tokens. Thus, the sum, over each node, of the number of tokens in the node is at least $\lfloor n/r \rfloor \sum_{i=1}^m n_i^2$, where $\sum_i n_i = r$ and $m \leq \sqrt{n}$. Under these conditions, $\sum_{i=1}^m n_i^2$ is minimized when each $n_i = r/m$, yielding the number of tokens received by the non-full nodes to be at least $\lfloor n/r \rfloor m \cdot r^2/m^2 = \lfloor n/r \rfloor r^2/m$. If $r > n/2$, then we have $\lfloor n/r \rfloor r^2/m = r^2/m \geq r\sqrt{n}/2$ since $m < \sqrt{n}$. If $r \leq n/2$, then we have $\lfloor n/r \rfloor r^2/m \geq (n/r - 1)r^2/m \geq r\sqrt{n} - r\sqrt{n}/2 = r\sqrt{n}/2$ since $m < \sqrt{n}$. We thus have the desired claim that the sum, over all nodes in R , of the number of tokens received by the node is $\Omega(|R|\sqrt{n})$.

We next argue that if m is at least \sqrt{n} , then either the number of new token arrivals in the round, over all non-full nodes, is $\Omega(\sqrt{n})$, or the total number of tokens already received by the non-full nodes in this phase is $\Omega(r\sqrt{n})$. Let C_1 through C_m denote the m configurations in this round. We construct an auxiliary graph in which each vertex is a configuration, and we have an edge between vertices C_i and C_j if there is an edge between a node having configuration C_i and a node having configuration C_j in this round. Note that since the network in each round is connected, so is the auxiliary graph.

Let \mathcal{T} denote an arbitrary spanning tree in this auxiliary graph. Consider the set \mathcal{S} of stars \mathcal{T} formed by the edges in either the odd levels of \mathcal{T} or the even levels of \mathcal{T} , whichever is greater. The number of edges in \mathcal{S} is at least $\sqrt{n}/2$. Let S be a star in \mathcal{S} with configuration C_0 as the root and C_i as the i th leaf. Each edge (C_0, C_i) in \mathcal{S} corresponds to an edge, say (u_i, v_i) , where u_i and v_i hold configurations C_0 and C_i , respectively, at the start of this round. Furthermore, all the v_i s are distinct nodes, while the u_i 's may not be distinct.

We call an edge (u_i, v_i) *bad* if $C_0 \subset C_i$ and C_0 contains all of the tokens that were at any node in configuration C_i in the initial distribution of the Exchange segment; otherwise, we call the edge good. Note that if an edge (u_i, v_i) is good then we have two cases: if C_0 is not a subset of C_i , then we can transfer a token in $C_0 \setminus C_i$ from u_i to v_i ; if C_0 does not contain all of the tokens that were at any node in configuration C_i in the initial distribution of the Exchange segment, then we can transfer a token from v_i to u_i that is distinct from any other token that can be sent from $v_j \neq v_i$ to u_i . It thus follows that if the number of distinct u_i 's, over all the stars in \mathcal{S} , is at least $\sqrt{n}/4$ or the number of good edges is at least $\sqrt{n}/4$, then

we can identify a token transfer along the edges (u_i, v_i) such that the total number, over all the nodes in the stars, of the distinct tokens received by the node in the round is at least $\sqrt{n}/4$. By Lemma 4.2, the exchange segment guarantees that the number of token transfers in this round is at least $\sqrt{n}/4$.

It remains to consider the case where the number of distinct u_i 's is less than $\sqrt{n}/4$ and the number of good edges is less than $\sqrt{n}/4$. In this case, let u be a node in a root configuration C_0 of star S that has bad edges to ℓ nodes, say v_1 through v_ℓ , with configurations C_1 through C_ℓ , respectively. By the definition of bad edges, we obtain that u has all the tokens that all the nodes with configuration C_1 through C_ℓ had in their initial distributions. Since C_1 through C_ℓ is a superset of C_0 , it follows that if x is the number of tokens in $\cup_{0 \leq i \leq \ell} C_i$, then each of the nodes having any of configurations C_0 through C_ℓ has at least x tokens.

We consider the following partition of all the nodes of the graph. Let S be a star in \mathcal{S} with root C_0 , the i th leaf being given by C_i , and network edge (u_i, v_i) corresponding to the auxiliary graph edge (C_0, C_i) . We define a group of the partition to be the union of set of the nodes with configuration C_0 and the union, over all i such that (u_i, v_i) is a bad edge, of the set of nodes with configuration C_i . If (u_i, v_i) is a good edge, we have the set of nodes with configuration C_i form their own group. Since the total number of distinct u_i 's, over all the stars in \mathcal{S} , is at most $\sqrt{n}/4$ and the number of good edges is at most $\sqrt{n}/4$, the above procedure partitions all the nodes into at most \sqrt{n} groups such that in each group, every node has all of the tokens that every node in its group had in the initial token distribution of the Exchange step. Using the same calculation as the first case above, we obtain that in this case the total useful token exchange already achieved is $\Omega(r\sqrt{n})$. \square

Lemma 4.4. *After $\Omega(\sqrt{n} \log n)$ phases starting from a set R of r non-full nodes, there exist at least $r/3$ nodes in R that receive all tokens whp.*

Proof. Suppose the number of non-full nodes remains at least $2r/3$ after $c\sqrt{n} \log n$ phases, where c is an arbitrary constant whose value will be set later in the proof. Then, by Lemma 4.3, the sum, over all nodes, of the total number of tokens received at the node during these phases is at least $2crn(\log n)/3$. Note that since each phase is implemented starting from an initial distribution in which every node not in R has all tokens, while every node in R has no tokens, the set of tokens received by a node in a phase may intersect the set of tokens received by the same node in another phase.

Since a full node does not receive any tokens during the distribution and exchange segments, and any node receives at most n tokens in a phase, we obtain from an averaging argument that at least $r/3$ non-full nodes each receives at least $cn(\log n)/2$ tokens in this stage; otherwise, the total number of token exchanges in $c\sqrt{n} \log n$ phases is less than $(2r/3)cn(\log n)/2 + (r/3)cn(\log n) = 2crn(\log n)/3$, a contradiction.

Consider any node v that receives at least $2cn(\log n)/3$ tokens, taken over all the $c\sqrt{n} \log n$ phases in this stage; note that while the tokens received in a phase are distinct, these tokens are not necessarily distinct across phases. Suppose v receives p_i tokens in phase i . By property (B3) of Lemma 4.1, the p_i tokens distributed to v in phase i are drawn uniformly at random from the set of all tokens. Therefore, by a standard coupon collector argument, we obtain that if c is sufficiently large, v has all of the n tokens with high probability, and thus becomes full after $c\sqrt{n} \log n$ phases. This completes the proof that at least $r/3$ nodes that were non-full at the start of the stage become full after $c\sqrt{n} \log n$ phases. \square

Theorem 4.1. *The n -broadcast problem completes in $O(n^{3/2} \log^2 n)$ rounds whp.*

Proof. By Lemma 4.4, we obtain that there exist at least $n/3$ nodes that have received all tokens after $O(n^{3/2} \log n)$ rounds whp. The remaining problem is that of disseminating the n tokens among $2n/3$ non-full nodes. Applying Lemma 4.4 repeatedly $O(\log n)$ times completes the proof of the theorem. \square

4.3 n -gossip

Our centralized algorithm for arbitrary n -gossip instances is as follows.

Consolidation stage: (a) For each token i , in sequence: for \sqrt{n} rounds, every node holding token i broadcasts token i (i.e., flooding of token i); (b) Identify a set S of $\tilde{O}(\sqrt{n})$ nodes such that every token is in some node in S ; arbitrarily assign each token to a node in S that has the token.

Distribution stage: Each node in S makes \sqrt{n} copies of each of its allocated tokens, for a total of $n^{3/2}$ tokens in all, including copies. If any node in S has a token multiset of fewer than n tokens, then it adds dummy tokens to the multiset to make it of size n . Let T_u denote the multiset of tokens at u . For each node u in S , we ensure that each node receives a distinct random token from the multiset of u : **LoadBalance**($\{u\}, V, T_u$).

Exchange stage: Maximize the number of token exchanges in each round by repeatedly calling **GreedyExchange**, until some node, say s , has at least $n - c\sqrt{n} \log n$ tokens, for a constant c that is chosen sufficiently large. If n -gossip is not yet completed, then: (a) Run n -broadcast with source s to complete the dissemination of the $n - c\sqrt{n} \log n$ tokens at s ; (b) Run at most $c\sqrt{n} \log n$ separate broadcasts, spanning n rounds, disseminating the remaining at most $c\sqrt{n} \log n$ tokens to all nodes.

Lemma 4.5. *The consolidation stage takes $n^{3/2}$ rounds, at the end of which we can find a set S of at most $O(\sqrt{n} \log n)$ nodes that together contain all of the tokens whp.*

Proof. The running time of the consolidation stage is immediate, since each token broadcast period consists of \sqrt{n} rounds.

Next, consider a set S of $c\sqrt{n} \log n$ nodes selected uniformly at random from the set of all nodes. The probability that for a given token τ , S does not include any of the at least \sqrt{n} nodes that have τ after the consolidation phase is at most

$$\begin{aligned} \binom{n - \sqrt{n}}{|S|} / \binom{n}{|S|} &= \frac{(n - \sqrt{n}) \cdots (n - \sqrt{n} - |S| + 1)}{n \cdots n - |S| + 1} \\ &\leq \left(1 - \frac{\sqrt{n}}{n - |S| + 1}\right)^{|S|} \\ &\leq \left(1 - \frac{2}{\sqrt{n}}\right)^{c\sqrt{n} \log n} \\ &\leq 1/\text{poly}(n), \end{aligned}$$

for n sufficiently large, and c a sufficiently large constant. By applying a union bound over the n tokens, we get the desired claim. \square

Lemma 4.6. *Consider any phase of the distribution stage, in which node v distributes its multiset T_v of tokens among the n nodes. Each node receives at least $\lfloor |T_v|/n \rfloor$ tokens from T_v , and the tokens arriving at a node are a subset of T_v drawn uniformly at random from T_v .*

Proof. Since each leader node assigns a random rank to each token copy it has, by property (B3) of Lemma 4.1, the token distribution process ensures that each node receives tokens of random ranks. Furthermore, by properties (B1) and (B2) of Lemma 4.1, each node receives $\lfloor |T_v|/n \rfloor$ or $\lceil |T_v|/n \rceil$ tokens. \square

Lemma 4.7. *After $O(n^{3/2} \log n)$ rounds of the exchange stage, we have a node that has at least $n - \sqrt{n} \log n$ tokens whp.*

Proof. Define a *configuration* to be the set of tokens that a node has at any time. Let m denote the number of distinct configurations that are present at the start of any round. If m is at least $\sqrt{n}/(c \log n)$, for a constant $c > 0$ chosen suitably large later, then as in Lemma 4.3, we argue that the sum, over all nodes, of the number of distinct tokens received by the node in the round is at least $\sqrt{n}/(c \log n)$ (this follows from Lemma 4.2, which establishes that **GreedyExchange** maximizes the number of tokens exchanged in any given round). We can be in this case for at most $cn^{3/2} \log n$ rounds since each node receives at most n distinct tokens.

In the second case, that is, where m is at most $\sqrt{n}/(c \log n)$, there exist at least $\sqrt{n} \log n$ nodes that have the same set of tokens. We now argue that any set of $c\sqrt{n} \log n$ nodes together have at least $n - c\sqrt{n} \log n$ tokens, for c chosen suitably large.

Fix a set X of $c\sqrt{n} \log n$ nodes. Consider the i th phase of the distribution stage in which tokens from a multiset set T_v are distributed from a node v . Let α_i be the number of tokens in $\Gamma \cap T_v$. By Lemma 4.6, the nodes in X together receive a subset of $\lfloor |T_v|/n \rfloor |X|$ tokens, chosen uniformly at random from the multiset T_v . Note that every token in T_v has \sqrt{n} copies in T_v and $|T_v|$ is at least n ; therefore, the probability that none of these α_i distinct tokens in $\Gamma \cap T_v$ are in X is at most

$$\begin{aligned} \frac{\binom{|T_v| - \sqrt{n}\alpha_i}{\lfloor |T_v|/n \rfloor |X|}}{\binom{|T_v|}{\lfloor |T_v|/n \rfloor |X|}} &\leq \left(1 - \frac{\lfloor |T_v|/n \rfloor |X|}{|T_v| - \lfloor |T_v|/n \rfloor |X|}\right)^{\lfloor |T_v|/n \rfloor c\sqrt{n} \log n} \\ &\leq \left(1 - \frac{\sqrt{n}\alpha_i}{|T_v| - \lfloor |T_v|/n \rfloor c\sqrt{n} \log n}\right)^{\lfloor |T_v|/n \rfloor c\sqrt{n} \log n} \\ &\leq \left(1 - \frac{\sqrt{n}\alpha_i}{|T_v|}\right)^{c|T_v| \log(n)/\sqrt{n}} \\ &\leq e^{-c\alpha_i \log(n)/2}. \end{aligned}$$

(In the second last inequality, we use the fact that $|T_v|$ is at least n , which implies $\lfloor |T_v|/n \rfloor \leq |T_v|/(2n)$. In the last inequality, we use the fact that $1 - x \leq e^{-x}$ for $0 \leq x < 1$.)

Since the choice of the random permutation in each phase of the distribution stage is independent of the choice in any other phase, we obtain that the probability that none of the tokens in Γ are in X is at most $\prod_i e^{-c\alpha_i \log(n)/2} \leq e^{-c \sum_i \alpha_i \log(n)/2} = e^{-c^2 \sqrt{n} \log^2(n)/2}$. The number of different choices for X and Γ is $\binom{n}{c\sqrt{n} \log n}^2 \leq (e\sqrt{n}/(c \log n))^{c\sqrt{n} \log n} \leq e^{c \ln 2 \sqrt{n} \log^2 n}$. Applying a union bound, we achieve a high probability bound on the event that there exists a node with at least $n - c\sqrt{n} \log n$ tokens. \square

Theorem 4.2. *Our centralized algorithm completes in $O(n^{3/2} \log^2 n)$ rounds, whp.*

Proof. The consolidation stage takes $O(n^{3/2} \log n)$ rounds. The distribution stage takes $O(n^{3/2})$ rounds. The exchange stage takes $O(n^{3/2} \log^2 n)$ rounds for the n -broadcast (by Theorem 4.1) and $O(n^{3/2} \log n)$ rounds for broadcasting the last $O(\sqrt{n} \log n)$ tokens. The high probability successful completion follows from Lemma 4.7, the correctness of n -broadcast (Theorem 4.1), and part (b) of the exchange stage. \square

5 Concluding remarks

Our work has focused on the basic question of whether there exists a fully distributed n -gossip protocol that runs in sub-quadratic time, i.e., $O(n^{2-\epsilon})$ rounds (for some positive constant ϵ), or even faster $O(n \text{ polylog } n)$ rounds, under an oblivious adversary. We showed that somewhat surprisingly, RAND-DIFF, a potentially strong candidate for a fast distributed algorithm, has a $\tilde{\Omega}(n^{3/2})$ lower bound. Moreover, for symmetric knowledge-based algorithms (SKB), we showed a lower bound of $\tilde{\Omega}(n^{4/3})$ under invasive adversaries, a stronger version of oblivious adversaries. We complemented these results with two upper bounds. First, we showed that RAND-DIFF can complete n -gossip in subquadratic time — $\tilde{O}(n^{5/3})$ — under a restricted oblivious adversary which has to respect some infrastructure-based path constraints. We believe the analysis of RAND-DIFF, in fact, extends to the more efficient SYM-DIFF protocol as well, and also to more general path-respecting adversaries. Second, we presented a centralized algorithm that achieves a $\tilde{O}(n^{3/2})$ bound using under any oblivious adversary.

Our work leaves several intriguing open problems and directions for future research: Is there a hybrid of RAND-DIFF and a knowledge-based algorithm that can achieve sub-quadratic complexity? What is the best

bound for n -gossip achieved by centralized token-forwarding? Explore paths-respecting and related models further to gain a better understanding of network dynamics from a practical standpoint.

References

- [1] ALON, N., AND SPENCER, J. H. *The Probabilistic Method*. Wiley, New York, NY, 1991.
- [2] AUGUSTINE, J., MOLLA, A. R., MORSY, E., PANDURANGAN, G., ROBINSON, P., AND UPFAL, E. Storage and search in dynamic peer-to-peer networks. In *SPAA (2013)*, G. E. Blelloch and B. Vöcking, Eds., ACM, pp. 53–62.
- [3] AUGUSTINE, J., PANDURANGAN, G., AND ROBINSON, P. Fast byzantine agreement in dynamic networks. In *ACM Symposium on Principles of Distributed Computing, PODC '13, Montreal, QC, Canada, July 22-24, 2013* (2013), pp. 74–83.
- [4] AUGUSTINE, J., PANDURANGAN, G., ROBINSON, P., ROCHE, S., AND UPFAL, E. Enabling robust and efficient distributed computation in dynamic peer-to-peer networks. In *FOCS (2015)*.
- [5] AUGUSTINE, J., PANDURANGAN, G., ROBINSON, P., AND UPFAL, E. Towards robust and efficient computation in dynamic peer-to-peer networks. In *ACM-SIAM SODA (2012)*, pp. 551–569.
- [6] AVIN, C., KOUCKÝ, M., AND LOTKER, Z. How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). In *ICALP-08, Automata, Languages and Programming (2008)*, pp. 121–132.
- [7] BAUMANN, H., CRESCENZI, P., AND FRAIGNIAUD, P. Parsimonious flooding in dynamic graphs. In *Proceedings of the 28th ACM symposium on Principles of distributed computing (2009)*, ACM, pp. 260–269.
- [8] BAUMANN, H., CRESCENZI, P., AND FRAIGNIAUD, P. Parsimonious flooding in dynamic graphs. *Distributed Computing* 24, 1 (2011), 31–44.
- [9] BOLLOBÁS, B., AND RIORDAN, O. The diameter of a scale-free random graph. *Combinatorica* 24, 1 (2004), 5–34.
- [10] BRODER, A., KUMAR, R., MAGHOUL, F., RAGHAVAN, P., RAJAGOPALAN, S., STATA, R., TOMKINS, A., AND WIENER, J. Graph structure in the web. *Computer Networks* 33, 1-6 (2000), 309–320.
- [11] CASTEIGTS, A., FLOCCHINI, P., QUATTROCIOCCHI, W., AND SANTORO, N. Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems* 27, 5 (2012), 387–408.
- [12] CLEMENTI, A. E., MACCI, C., MONTI, A., PASQUALE, F., AND SILVESTRI, R. Flooding time in edge-markovian dynamic graphs. In *PODC '08: Proceedings of the twenty-seventh ACM symposium on Principles of distributed computing* (New York, NY, USA, 2008), ACM, pp. 213–222.
- [13] CLEMENTI, A. E. F., MONTI, A., PASQUALE, F., AND SILVESTRI, R. Broadcasting in dynamic radio networks. *J. Comput. Syst. Sci.* 75, 4 (2009), 213–230.
- [14] COOPER, C., AND FRIEZE, A. Crawling on simple models of web graphs. *Internet Mathematics* 1 (2003), 57–90.

- [15] DUTTA, C., PANDURANGAN, G., RAJARAMAN, R., SUN, Z., AND VIOLA, E. On the complexity of information spreading in dynamic networks. In *SODA (2013)*, S. Khanna, Ed., SIAM, pp. 717–736.
- [16] FERREIRA, A. Building a reference combinatorial model for manets. *Network, IEEE* 18, 5 (2004), 24–29.
- [17] FERREIRA, A., GOLDMAN, A., AND MONTEIRO, J. On the evaluation of shortest journeys in dynamic networks. In *Sixth IEEE International Symposium on Network Computing and Applications (NCA 2007) (2007)*, pp. 3–10.
- [18] FLAXMAN, A., FRIEZE, A. M., AND UPFAL, E. Efficient communication in an ad-hoc network. *J. Algorithms* 52, 1 (2004), 1–7.
- [19] GEORGIU, C., GILBERT, S., GUERRAOU, R., AND KOWALSKI, D. R. On the complexity of asynchronous gossip. In *PODC '08: Proceedings of the twenty-seventh ACM symposium on Principles of distributed computing* (New York, NY, USA, 2008), ACM, pp. 135–144.
- [20] GUREVICH, M., AND KEIDAR, I. Correctness of gossip-based membership under message loss. In *PODC '09: Proceedings of the 28th ACM symposium on Principles of distributed computing* (New York, NY, USA, 2009), ACM, pp. 151–160.
- [21] HAEUPLER, B. Analyzing Network Coding Gossip Made Easy. *Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing, STOC (2011)*, pp 293–302.
- [22] HAEUPLER, B., AND KARGER, D. Faster information dissemination in dynamic networks via network coding. In *Proceedings of the 30th annual ACM SIGACT-SIGOPS symposium on Principles of distributed computing* (2011), ACM, pp. 381–390.
- [23] HAEUPLER, B., AND KUHN, F. Lower bounds on information dissemination in dynamic networks. In *Distributed Computing - 26th International Symposium, DISC 2012, Salvador, Brazil, October 16-18, 2012. Proceedings* (2012), pp. 166–180.
- [24] JARRY, A., AND LOTKER, Z. Connectivity in evolving graph with geometric properties. In *DIALM-POMC '04: Proceedings of the 2004 Joint Workshop on Foundations of Mobile Computing* (New York, NY, USA, 2004), ACM Press, pp. 24–30.
- [25] KEMPE, D., KLEINBERG, J., AND KUMAR, A. Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences* 64, 4 (2002), 820–842.
- [26] KUHN, F., LYNCH, N., AND OSHMAN, R. Distributed computation in dynamic networks. In *STOC '10: Proceedings of the 42nd ACM symposium on Theory of computing* (New York, NY, USA, 2010), ACM, pp. 513–522.
- [27] KUHN, F., OSHMAN, R., AND MOSES, Y. Coordinated consensus in dynamic networks. In *Proceedings of the 30th annual ACM SIGACT-SIGOPS symposium on Principles of distributed computing* (2011), ACM, pp. 1–10.
- [28] LEIGHTON, F. T. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, and Hypercubes*. Morgan-Kaufmann, San Mateo, CA, 1991.
- [29] LEIGHTON, F. T. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes*. Morgan Kaufmann, 1991.

- [30] LIBEN-NOWELL, D., NOVAK, J., KUMAR, R., RAGHAVAN, P., AND TOMKINS, A. Geographic routing in social networks. *Proceedings of the National Academy of Sciences of the United States of America* 102, 33 (2005), 11623–11628.
- [31] O’DELL, R., AND WATTENHOFER, R. Information dissemination in highly dynamic graphs. In *DIALM-POMC* (2005), pp. 104–110.
- [32] PANDURANGAN, G. Distributed algorithmic foundations of dynamic networks. In *Structural Information and Communication Complexity - 21st International Colloquium, SIROCCO 2014, Takayama, Japan, July 23-25, 2014. Proceedings* (2014), pp. 18–22.
- [33] PELEG, D. *Distributed Computing: A Locality-Sensitive Approach*. SIAM, 2000.
- [34] SARMA, A. D., MOLLA, A. R., AND PANDURANGAN, G. Fast distributed computation in dynamic networks via random walks. In *DISC* (2012), M. K. Aguilera, Ed., vol. 7611 of *Lecture Notes in Computer Science*, Springer, pp. 136–150.
- [35] SARWATE, A., AND DIMAKIS, A. The impact of mobility on gossip algorithms. In *INFOCOM 2009, IEEE* (2009), IEEE, pp. 2088–2096.
- [36] TOPKIS, D. M. Concurrent broadcast for information dissemination. *IEEE Trans. Softw. Eng.* 11 (1985), 1107–1112.