

Gravitational-wave amplitudes for compact binaries in eccentric orbits at the third post-Newtonian order: Tail contributions and post-adiabatic corrections

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We compute the *tail* contributions to the gravitational-wave mode amplitudes for compact binaries in eccentric orbits at the third post-Newtonian order of general relativity. We combine them with the already available instantaneous pieces and include the *post-adiabatic* corrections required to fully account for the effects of radiation-reaction forces on the motion. We compare the resulting waveform in the small eccentricity limit to the circular one, finding perfect agreement.

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I. INTRODUCTION

In recent years the discoveries of gravitational waves (GW) by the LIGO and Virgo Collaborations have opened a new window to the Universe [1–10]. KAGRA will join the global GW detector network in 2019 [11] and LIGO-India in 2025 [12], improving source localization and parameter estimation [13], while LISA Pathfinder's exceptional performance [14] – showing that the LISA mission is feasible – and maturing pulsar timing arrays [15] mark the beginning of multiwavelength, multi-band GW astronomy.

Compact binary systems are the most prominent sources for the present and future GW observatories. So far these events have been analyzed using quasi-circular GW templates, as radiation-reaction effects tend to circularize the orbits [16, 17] for prototypical sources. For such systems one can thus assume that by the time the binary enters the sensitivity band of current ground-based detectors the eccentricity will be close to zero. However, there are a number of astrophysical scenarios in which binary systems could have moderate eccentricities when entering the sensitivity band of ground-based detectors [18–24]. Recently, there have been studies showing that triple interactions among black holes can produce coalescing binaries with moderate eccentricities (~ 0.1) when entering the LIGO band [25–27] or large eccentricities (~ 0.9) when entering the LISA band [28]. This has major implications on how to distinguish between binary black hole (BBH) formation channels [29] and motivates the development of waveforms valid for nonzero eccentricities.

There has been great effort to model GWs of eccentric binary systems. One usually employs the quasi-Keplerian parametrization [30, 31] to describe the conservative binary orbits. The phasing description, developed in Refs. [32, 33] and discussed in great detail for low-eccentricity binaries in Ref. [34], efficiently incorporates the effects of radiation reaction, describing the bi-

nary dynamics on three different timescales: the orbital timescale, the periastron precession timescale, and the radiation-reaction timescale. In addition, the secular evolution of the orbital elements has been completed at the third post-Newtonian (3PN) order in Refs. [35–37], including hereditary effects. Using this, several waveform models have been developed in the past years [38–48], for both nonspinning and spinning binaries.

In this paper, we extend the work in Ref. [49] by computing the tail contributions to the GW amplitudes for compact binaries in eccentric orbits at the third post-Newtonian level. Combining our tail results with the instantaneous ones, we then incorporate post-adiabatic corrections [32–34] to get a complete waveform including radiation-reaction effects valid during the early inspiral of the binary system. We present all our results in modified harmonic (MH) gauge in terms of the post-Newtonian parameter $\bar{x} = (Gm\bar{\omega}/c^3)^{2/3}$, where G denotes the gravitational constant, c the speed of light, m the total mass of the binary, and $\bar{\omega}$ the adiabatic orbital frequency (see Sec. V), as well as a certain time eccentricity $\bar{e} = \bar{e}_t$ associated with the PN-accurate quasi-Keplerian parametrization. To calculate the complicated tail integrals, we work within a low-eccentricity expansion and express everything in terms of the mean anomaly l and the phase angle λ , which accounts for the periastron advance. Compared to the results in Ref. [49], ours will thus not be valid for arbitrary eccentricities. Moreover, they will need to be completed by the memory contributions, which we will tackle in a follow-up paper [50].

This paper is structured as follows: In Sec. II we quickly review the basics of spherical harmonic decomposition and recall how to connect the radiative multipole moments to the actual source moments. We also review the conservative 3PN-accurate quasi-Keplerian parametrization [31]. In Sec. III, we discuss how to incorporate post-adiabatic corrections [32, 33] into this description. In Sec. IV, we are then in a position to calculate the various tail integrals appearing in the source

multipole moments. In Sec. V, we combine these results with the instantaneous ones and introduce post-adiabatic corrections. We also compare our results to the circular waveforms in Ref. [51]. Finally, in Sec. VI, we give a brief summary of our work. Throughout this paper we mostly present results up to $\mathcal{O}(e)$, though expressions up to $\mathcal{O}(e^6)$ for all tail and post-adiabatic modes will be listed in a supplemental *Mathematica* file [52].

II. CONSTRUCTION OF THE WAVEFORM FOR COMPACT BINARIES IN ECCENTRIC ORBITS

A. Polarizations and spherical-mode decomposition

The gravitational waves emitted by an isolated system near future radiative infinity are encoded in the transverse-traceless (TT) projection h_{ij}^{TT} of the deviation of the space-time metric $g_{\mu\nu}$ from a flat metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, in a radiative-type Cartesian-like coordinate grid $X^\mu = (cT, \mathbf{X})$, at order $1/R$, where $R = |\mathbf{X}|$ denotes the Euclidean distance of the vector \mathbf{X} to the origin. It is convenient to chose this origin at the center of mass of the full system and to introduce the standard spherical coordinates (Θ, Φ) associated with the so-defined Cartesian frame, for which the relation $X^i = R(\cos \Phi \sin \Theta, \sin \Phi \sin \Theta, \cos \Theta)$ holds. The radiative property of this frame ensures that a null geodesic going through the origin at time T_R will reach an observer with position \mathbf{X} at time $T = T_R + R/c$. If $\mathbf{N}(\Theta, \Phi) = \mathbf{X}/R$ denotes the unit direction of that observer, the plane span by the vectors $\mathbf{P}(\Theta, \Phi)$ and $\mathbf{Q}(\Theta, \Phi)$ belonging to some arbitrary direct orthonormal triad $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ must be transverse to the direction of propagation of wave rays.

The transverse-traceless projection h_{ij}^{TT} can be uniquely decomposed into symmetric trace-free (STF) radiative mass-type (U_L) and current-type (V_L) multipole moments as:

$$h_{ij}^{\text{TT}} = \frac{4G}{c^2 R} \mathcal{P}_{ijab}(\mathbf{N}) \sum_{\ell=2}^{\infty} \frac{1}{c^\ell \ell!} \left\{ N_{L-2} U_{abL-2} - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} V_{b)dL-2} \right\} \Big|_{T_R} + \mathcal{O}\left(\frac{1}{R^2}\right). \quad (1)$$

Here $\mathcal{P}_{ijab} = \mathcal{P}_{ia}\mathcal{P}_{jb} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{ab}$, with $\mathcal{P}_{ij} = \delta_{ij} - N_i N_j$, is the TT projection operator. The waveform is usually projected on the transverse symmetric basis $e_{ij}^+ = \frac{1}{2}(P_i P_j - Q_i Q_j)$, $e_{ij}^\times = P_i Q_j$,

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \begin{pmatrix} e_{ij}^+ \\ e_{ij}^\times \end{pmatrix} h_{ij}^{\text{TT}}, \quad (2)$$

the resulting components being referred to as the plus and cross polarizations, respectively. Equivalently the complex basis formed by the vector $\mathbf{m} = (\mathbf{P} + i\mathbf{Q})/\sqrt{2}$

of spin weight 2 and its complex conjugate $\overline{\mathbf{m}}$ of spin weight -2 can be used. From the transverse trace-free character of the waveform, it follows that

$$h = h_+ - ih_\times = h_{ij}^{\text{TT}} \overline{\mathbf{m}}^i \mathbf{m}^j. \quad (3)$$

From now on we shall assume that the vector \mathbf{m} is proportional to $\mathbf{m}_S = (\partial\mathbf{N}/\partial\theta + i\sin^{-1}\theta \partial\mathbf{N}/\partial\phi)/\sqrt{2}$ so that the functions adapted to the spherical decomposition of the spin -2 quantity h are the usual spin-weighted spherical harmonics of weight -2 , which will be denoted by $Y_{-2}^{\ell m}(\Theta, \Phi)$. In our conventions, they are given by

$$Y_{-2}^{\ell m}(\Theta, \Phi) = \sqrt{\frac{2\ell+1}{4\pi}} d_2^{\ell m}(\Theta) e^{im\Phi}, \quad (4a)$$

$$d_2^{\ell m} = \sum_{k=k_{\min}}^{k_{\max}} \frac{(-1)^k}{k!} \times \frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2)!}}{(k-m+2)!(\ell+m-k)!(\ell-k-2)!} \times \left(\cos \frac{\Theta}{2}\right)^{2\ell+m-2k-2} \left(\sin \frac{\Theta}{2}\right)^{2k-m+2}, \quad (4b)$$

with $k_{\min} = \max(0, m-2)$ and $k_{\max} = \min(\ell+m, \ell-2)$. Thus, the gravitational waveform may be decomposed into spherical modes $h^{\ell m}$ as

$$h_+ - ih_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\Theta, \Phi). \quad (5)$$

The spherical harmonic modes $h^{\ell m}$ can be written in terms of the radiative mass-type ($U^{\ell m}$) and current-type ($V^{\ell m}$) multipole moments,

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}} \left(U^{\ell m} - \frac{i}{c} V^{\ell m} \right), \quad (6)$$

with the inverse relations

$$U^{\ell m} = -\frac{Rc^{\ell+2}}{\sqrt{2}G} (h^{\ell m} + (-1)^m \overline{h}^{\ell-m}), \quad (7a)$$

$$V^{\ell m} = -\frac{Rc^{\ell+3}}{\sqrt{2}iG} (-h^{\ell m} + (-1)^m \overline{h}^{\ell-m}). \quad (7b)$$

The radiative moments ($U^{\ell m}$, $V^{\ell m}$) are actually related to the STF radiative moments (U_L , V_L) by

$$U^{\ell m} = \frac{4}{\ell!} \sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \alpha_L^{\ell m} U_L, \quad (8a)$$

$$V^{\ell m} = -\frac{8}{\ell!} \sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}} \alpha_L^{\ell m} V_L, \quad (8b)$$

where the $\alpha_L^{\ell m}$ denote a set of constant STF tensors that connect the basis of spherical harmonics $Y^{\ell m}(\Theta, \Phi)$ to

the set of STF tensors $N_{\langle L \rangle}$ as

$$N_{\langle L \rangle}(\Theta, \Phi) = \sum_{m=-\ell}^{\ell} \alpha_L^{\ell m} Y^{\ell m}(\Theta, \Phi), \quad (9a)$$

$$Y^{\ell m}(\Theta, \Phi) = \frac{(2\ell+1)!!}{4\pi\ell!} \bar{\alpha}_L^{\ell m} N^{\langle L \rangle}(\Theta, \Phi). \quad (9b)$$

They can be calculated through

$$\alpha_L^{\ell m} = \int d\Omega N_{\langle L \rangle} \bar{Y}^{\ell m}, \quad (10)$$

and are given explicitly in Eq. (2.12) of Ref. [53].

Remarkably, for planar binaries, there exists a mode separation [54, 55] such that $h^{\ell m}$ is completely determined by mass-type radiative multipole moments $U^{\ell m}$ for $\ell + m$ even and by current-type radiative multipole moments $V^{\ell m}$ for $\ell + m$ odd, hence

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}} U^{\ell m} \quad \text{if } \ell + m \text{ is even,} \quad (11a)$$

$$h^{\ell m} = \frac{iG}{\sqrt{2}Rc^{\ell+3}} V^{\ell m} \quad \text{if } \ell + m \text{ is odd.} \quad (11b)$$

Let us finally specify the choice of the Cartesian frame and polarization vectors in the case of interest where the source is a binary system of pointlike objects with bound orbits, since this choice will fully set the amplitude modes computed in the present paper. We adopt the same conventions as in Ref. [51]. In the absence of spin, the orbits stay in a plane. The vector \mathbf{e}_3 is taken to be the unit normal vector orienting the sense of the motion positively. For the polarization vector \mathbf{P} , we pick the unit vector pointing towards the ascending node $\mathbf{N} \times \mathbf{e}_3$, with \mathbf{N} representing the direction of the Earth observer. Therefore, we can also make it coincide with \mathbf{e}_1 . To complete the triads \mathbf{e}_a and $(\mathbf{N}, \mathbf{P}, \mathbf{Q})$ we pose $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ and

$\mathbf{Q} = \mathbf{N} \times \mathbf{P}$. Notice that, by construction, \mathbf{N} belongs to the plane spanned by $\{\mathbf{e}_2, \mathbf{e}_3\}$. Its spherical coordinates, in terms of the inclination of the binary ι , are thus $(\Theta = \iota, \Phi = \pi/2)$.

B. Multipole moments

From Eqs. (6–8), we see that we need to relate the U_L and V_L to the actual source. In the multipolar post-Minkowskian (MPM) post-Newtonian (PN) formalism, the radiative moments (U_L, V_L) are functionals of six sets of source moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$. The relations between the radiative moments and the source moments have been obtained at the 3PN order and are listed in Ref. [51], Eqs. (5.4–5.11).

We can split the the expressions for the radiative moments into two parts, namely the instantaneous and the hereditary parts:

$$U_L = U_L^{\text{inst}} + U_L^{\text{hered}}. \quad (12)$$

The instantaneous contributions only depend on the state of the source at a given retarded time, while the hereditary parts depend on, and thus require knowledge of, the entire past history of the source. At leading order, the instantaneous parts of the radiative moments are directly related to the source moments as

$$U_L^{\text{inst}}(t_r) = I_L^{(\ell)}(t_r) + \mathcal{O}(c^{-3}), \quad (13a)$$

$$V_L^{\text{inst}}(t_r) = J_L^{(\ell)}(t_r) + \mathcal{O}(c^{-3}), \quad (13b)$$

with t_r denoting here a “dummy” variable. Corrections from the gauge moments (W_L, X_L, Y_L, Z_L) enter at higher orders. In this work, we will focus on the hereditary tail contributions. For a complete treatment of the instantaneous contributions, we refer to Ref. [49].

To the desired accuracy, the hereditary contributions to the radiative moments are given by

$$U_{ij}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12} \right] I_{ij}^{(4)}(t_r - \tau) - \frac{2G}{7c^5} \int_{-\infty}^{t_r} d\tau I_{a\langle i}^{(3)}(\tau) I_{j\rangle a}^{(3)}(\tau) + 2 \left(\frac{GM}{c^3} \right)^2 \int_0^\infty d\tau \left[\ln^2\left(\frac{\tau}{2\tau_0}\right) + \frac{57}{70} \ln\left(\frac{\tau}{2\tau_0}\right) + \frac{124627}{44100} \right] I_{ij}^{(5)}(t_r - \tau) + \mathcal{O}(c^{-7}), \quad (14a)$$

$$U_{ijk}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{97}{60} \right] I_{ijk}^{(5)}(t_r - \tau) + \frac{G}{c^5} \int_{-\infty}^{t_r} d\tau \left[-\frac{1}{3} I_{a\langle i}^{(3)}(\tau) I_{jk\rangle a}^{(4)}(\tau) - \frac{4}{5} \epsilon_{ab\langle i} I_{ja}^{(3)}(\tau) J_{k\rangle b}^{(3)}(\tau) \right] + \mathcal{O}(c^{-6}), \quad (14b)$$

$$U_{ijkl}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{59}{30} \right] I_{ijkl}^{(6)}(t_r - \tau) + \frac{2G}{5c^3} \int_{-\infty}^{t_r} d\tau I_{\langle ij}^{(3)}(\tau) I_{kl\rangle}^{(3)}(\tau) + \mathcal{O}(c^{-5}), \quad (14c)$$

$$U_{ijklm}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{232}{105} \right] I_{ijklm}^{(7)}(t_r - \tau) + \frac{20G}{21c^3} \int_{-\infty}^{t_r} d\tau I_{\langle ij}^{(3)}(\tau) I_{klm\rangle}^{(4)}(\tau) + \mathcal{O}(c^{-4}), \quad (14d)$$

$$V_{ij}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{7}{6} \right] J_{ij}^{(4)}(t_r - \tau) + \mathcal{O}(c^{-6}), \quad (15a)$$

$$V_{ijk}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{5}{3} \right] J_{ijk}^{(5)}(t_r - \tau) + \mathcal{O}(c^{-5}), \quad (15b)$$

$$V_{ijkl}^{\text{hered}}(t_r) = \frac{2GM}{c^3} \int_0^\infty d\tau \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{119}{60} \right] J_{ijkl}^{(6)}(t_r - \tau) + \mathcal{O}(c^{-4}), \quad (15c)$$

where $M = m(1 - \nu x/2) + \mathcal{O}(c^{-4})$ is the Arnowitt-Deser-Misner (ADM) mass of the source, $m = m_1 + m_2$ the total mass, $\nu = m_1 m_2 / m^2$ the symmetric mass ratio, and τ_0 an arbitrary length scale originally introduced in the MPM formalism. None of the other moments contributes to the hereditary part of the waveform (1) at 3PN order, since

$$U_{L>5}^{\text{hered}} = \mathcal{O}(c^{-3}), \quad (16a)$$

$$V_{L>4}^{\text{hered}} = \mathcal{O}(c^{-3}). \quad (16b)$$

In the above hereditary contributions, there are two different types of integrals: those with logarithms and those without. The logarithmic integral in the first line of Eq. (14a) is called the tail integral while the one on the second line is the tails-of-tails integral. On the other hand, the integral without a logarithmic kernel is the memory integral. Note that there are no memory contributions to the radiative current moments V_L . Physically, wave tails come from the scattering of the linear waves, generated by the matter source, off the space-time curvature due to the total ADM mass of the isolated system. It is a (power of) monopole-wave interaction effect with a

weak past dependence. By contrast, the memory pieces of the waves are produced by the effective stress-energy tensor of the source radiation itself. It is a wave-wave interaction effect with a strong past dependence [56].

The expressions for the source moments (I_L , J_L) in terms of the binary separation r , its time derivative \dot{r} , the polar angle ϕ of the relative position, and its derivative $\dot{\phi}$ are now required. Observing Eqs. (14–15), we note that I_{ij} , J_{ij} and I_{ijk} are needed to an accuracy of 1PN, while all other multipole moments are only needed to leading Newtonian order. The relevant expressions are listed in Ref. [36] using standard harmonic (SH) coordinates. The logarithms appearing at 3PN order in the SH gauge can, however, be transformed away in appropriate modified harmonic coordinates, as demonstrated Sec. IV B of Ref. [36]. For the hereditary parts, this will not make any difference, as we shall only need relative 1PN-accurate expressions for certain (I_L , J_L), but, when adding up instantaneous terms from Ref. [49] to our hereditary parts, we shall always work within the MH gauge. The binary separation vector will be represented by $x^i \equiv r n^i$, whereas $v^i = dx^i/dt$ will stand for the relative velocity. The expressions relevant for the calculation of the hereditary parts are

$$I_{ij} = \nu m \left(A_1 x_{\langle ij \rangle} + A_2 \frac{r\dot{r}}{c^2} x_{\langle i} v_{j \rangle} + A_3 \frac{r^2}{c^2} v_{\langle ij \rangle} \right) + \mathcal{O}(c^{-7}), \quad (17a)$$

$$I_{ijk} = -\nu m \Delta \left(B_1 x_{\langle ijk \rangle} + B_2 \frac{r\dot{r}}{c^2} x_{\langle ij} v_{k \rangle} + B_3 \frac{r^2}{c^2} x_{\langle i} v_{jk \rangle} \right) + \mathcal{O}(c^{-6}), \quad (17b)$$

$$I_{ijkl} = \nu m (1 - 3\nu) x_{\langle ijkl \rangle} + \mathcal{O}(c^{-5}), \quad (17c)$$

$$I_{ijklm} = -\nu m \Delta (1 - 2\nu) x_{\langle ijklm \rangle} + \mathcal{O}(c^{-4}), \quad (17d)$$

$$J_{ij} = -\nu m \Delta \left(C_1 \epsilon_{ab\langle i} x_{j \rangle a} v_b + C_2 \frac{r\dot{r}}{c^2} \epsilon_{ab\langle i} v_{j \rangle b} x_a \right) + \mathcal{O}(c^{-6}), \quad (18a)$$

$$J_{ijk} = \nu m (1 - 3\nu) \epsilon_{ab\langle i} x_{jk \rangle a} v_b + \mathcal{O}(c^{-5}), \quad (18b)$$

$$J_{ijkl} = -\nu m \Delta (1 - 2\nu) \epsilon_{ab\langle i} x_{jkl \rangle a} v_b + \mathcal{O}(c^{-4}), \quad (18c)$$

where $\Delta = (m_1 - m_2)/m$ is the mass difference ratio and the constants A_i , B_i , and C_i read

$$A_1 = 1 + \frac{1}{c^2} \left[v^2 \left(\frac{29}{42} - \frac{29\nu}{14} \right) + \frac{Gm}{r} \left(-\frac{5}{7} + \frac{8\nu}{7} \right) \right], \quad (19a)$$

$$A_2 = -\frac{4}{7} + \frac{12\nu}{7}, \quad (19b)$$

$$A_3 = \frac{11}{21} - \frac{11\nu}{7}, \quad (19c)$$

$$B_1 = 1 + \frac{1}{c^2} \left[v^2 \left(\frac{5}{6} - \frac{19\nu}{6} \right) + \frac{Gm}{r} \left(-\frac{5}{6} + \frac{13\nu}{6} \right) \right], \quad (19d)$$

$$B_2 = -(1 - 2\nu), \quad (19e)$$

$$B_3 = 1 - 2\nu, \quad (19f)$$

$$C_1 = 1 + \frac{1}{c^2} \left[v^2 \left(\frac{13}{28} - \frac{17\nu}{7} \right) + \frac{Gm}{r} \left(\frac{27}{14} + \frac{15\nu}{7} \right) \right], \quad (19g)$$

$$C_2 = \frac{5}{28}(1 - 2\nu). \quad (19h)$$

C. Quasi-Keplerian parametrization

The expressions in Eqs. (17–18) in terms of the variables $(r, \dot{r}, \phi, \dot{\phi})$ are the most general ones. Now, when calculating the tail integrals, we should replace the latter quantities by their actual analytic time evolution for eccentric orbits. At the third post-Newtonian order, the conservative orbital dynamics of compact binaries in eccentric orbits is specified by providing the following generalized quasi-Keplerian parametrization [31] for the dynamical variables r and ϕ :

$$r = a_r (1 - e_r \cos u), \quad (20a)$$

$$\begin{aligned} \phi - \phi_0 &= (1 + k)v + (f_{4\phi} + f_{6\phi}) \sin(2v) \\ &\quad + (g_{4\phi} + g_{6\phi}) \sin(3v) + i_{6\phi} \sin(4v) \\ &\quad + h_{6\phi} \sin(5v), \end{aligned} \quad (20b)$$

$$\text{where } v = 2 \arctan \left[\left(\frac{1 + e_\phi}{1 - e_\phi} \right)^{1/2} \tan \frac{u}{2} \right]. \quad (20c)$$

An interesting feature in the above equations is the presence of different eccentricity parameters e_r and e_ϕ , introduced in such a way that the parametrization looks “Keplerian”. The parameter k is nothing but the periastron advance per orbital revolution. The parameters a_r , e_r , and e_ϕ are the PN-accurate semi-major axis and the radial and angular eccentricities, while $f_{4\phi}$, $f_{6\phi}$, $g_{4\phi}$, $g_{6\phi}$, $i_{6\phi}$, and $h_{6\phi}$ are some orbital functions of the energy and angular momentum that enter at the 2PN and 3PN orders. The explicit expressions are available in Ref. [31].

The eccentric anomaly u is linked to the mean anomaly l through the 3PN-accurate Kepler equation

$$\begin{aligned} l &= u - e_t \sin u + (g_{4t} + g_{6t})(v - u) \\ &\quad + (f_{4t} + f_{6t}) \sin v + i_{6t} \sin(2v) + h_{6t} \sin(3v). \end{aligned} \quad (21)$$

Here, e_t is another eccentricity parameter, usually called the time eccentricity, and the functions g_{4t} , g_{6t} , f_{4t} , f_{6t} , i_{6t} , and h_{6t} are additional 2PN and 3PN orbital functions of the energy and angular momentum. Together, Eqs. (20) and (21) fully parametrize the conservative orbital dynamics of compact binaries on eccentric orbits.

Note that we choose to express all our equations in terms of the post-Newtonian parameter $x = (Gm\omega/c^3)^{2/3}$ and the time eccentricity $e = e_t$, with $\omega = (1 + k)n$ being the orbital frequency and $n = 2\pi/P$ the mean motion associated with the period P . In the next section, we shall introduce post-adiabatic corrections to this quasi-Keplerian description. We will then have to replace the parameters (x, e) with their slowly evolving counterparts (\bar{x}, \bar{e}) .

The appearance of the periastron precession at first post-Newtonian order introduces a double periodic motion on two timescales: the orbital timescale and the precession timescale. It is thus customary to split the phase ϕ into an angle λ that is linear in l and an oscillatory part $W(l)$ that is 2π -periodic in l [32, 57, 58]. This leads us to write

$$\phi = \lambda + W(l), \quad (22a)$$

$$\lambda = \phi_0 + (1 + k)l, \quad (22b)$$

$$\begin{aligned} W(l) &= (1 + k)(v - l) + (f_{4\phi} + f_{6\phi}) \sin(2v) \\ &\quad + (g_{4\phi} + g_{6\phi}) \sin(3v) + i_{6\phi} \sin(4v) \\ &\quad + h_{6\phi} \sin(5v), \end{aligned} \quad (22c)$$

with ϕ_0 denoting the initial polar angle at $u = 0$.

To evaluate the various time integrals appearing in the tail contributions to the waveform, we will need explicit expressions for u and ϕ in terms of the angles l and λ . This can be achieved by solving the Kepler equation (21). We employ the method described in Ref. [59], which yields

$$u = l + \sum_{s=1}^{\infty} A_s \sin(sl), \quad (23a)$$

$$A_s = \frac{2}{s} J_s(se_t) + \sum_{j=1}^{\infty} \alpha_j \{J_{s+j}(se_t) - J_{s-j}(se_t)\}, \quad (23b)$$

where the constants α_j are some PN-accurate functions of the energy and angular momentum entering at the second post-Newtonian order. It remains to display an explicit expression for the 2π -periodic function $W(l)$ in

terms of l ,

$$W(l) = \sum_{s=1}^{\infty} \mathcal{W}_s \sin(sl), \quad (24a)$$

$$\begin{aligned} \mathcal{W}_s = & (1+k)B_s + (f_{4\phi} + f_{6\phi})\sigma_s^{2v} \\ & + (g_{4\phi} + g_{6\phi})\sigma_s^{3v} + i_{6\phi}\sigma_s^{4v} + h_{6\phi}\sigma_s^{5v}, \end{aligned} \quad (24b)$$

with the constants B_s and σ_s^{jv} given in Eqs. (C8) and (32b) of Ref. [59]. We finally find, expanding to $\mathcal{O}(x^3)$ and $\mathcal{O}(e)$,

$$\begin{aligned} u = & l + e \sin(l) + x^2 \left(-\frac{15}{2} + \frac{9\nu}{8} + \frac{\nu^2}{8} \right) e \sin(l) \\ & + x^3 \left(-55 + \frac{104593\nu}{1680} + \frac{3\nu^2}{4} + \frac{\nu^3}{24} \right) e \sin(l), \end{aligned} \quad (25a)$$

$$\begin{aligned} \phi = & \lambda + 2e \sin(l) + x(10 - \nu)e \sin(l) \\ & + x^2 \left(52 - \frac{235\nu}{12} + \frac{\nu^2}{12} \right) e \sin(l) \\ & + x^3 \left(292 + \left(-\frac{420131}{840} + \frac{287\pi^2}{32} \right) \nu \right. \\ & \left. + \frac{521\nu^2}{24} + \frac{\nu^3}{24} \right) e \sin(l). \end{aligned} \quad (25b)$$

We shall use these expressions to write the source multipole moments (I_L , J_L) in terms of l and λ .

III. PHASING OF THE ORBITAL ELEMENTS

So far, we used the conservative quasi-Keplerian description of the dynamics of nonspinning compact binaries. This analytic parametrization is possible due to the fact that the conservative problem admits four integrals of motion, or even two, when the problem is restricted to the orbital plane. In our case, those two integrals are encoded in the two intrinsic constants x and $e = e_t$. There also exist two extrinsic constants c_l and c_λ ,

$$l(t) = n(t - t_0) + c_l, \quad (26a)$$

$$\lambda(t) = (1+k)n(t - t_0) + c_\lambda, \quad (26b)$$

corresponding to the initial values of the two phase angles l and λ , respectively. We now move to include phasing effects due to energy and angular momentum loss into this quasi-Keplerian parametrization. An efficient description of the dynamics of nonspinning compact binaries with phasing is presented in Refs. [32, 33]. Following Ref. [60], they employ a method of *variation of constants* where the constants of motion of the conservative problem (x , e , c_l , c_λ) are treated as time-varying quantities. Specifically, the post-Newtonian parameter $x = x(t)$ and the time eccentricity $e = e(t)$ are now genuine functions of time, while the angles l and λ are given by

$$l(t) = \int_{t_0}^t n(t') dt' + c_l(t), \quad (27a)$$

$$\lambda(t) = \int_{t_0}^t [1 + k(t')] n(t') dt' + c_\lambda(t). \quad (27b)$$

To obtain the evolution of the functions $c_\alpha(t) = (x(t), e(t), c_l(t), c_\lambda(t))$, one starts from the PN-accurate equations of motion

$$\dot{\mathbf{x}} = \mathbf{v}, \quad (28a)$$

$$\dot{\mathbf{v}} = \mathcal{A}_0(\mathbf{x}, \mathbf{v}) + \mathcal{A}'(\mathbf{x}, \mathbf{v}), \quad (28b)$$

with \mathcal{A}_0 being the conservative and \mathcal{A}' the dissipative piece of the equations of motion. These equations are first solved neglecting the dissipative term \mathcal{A}' , leading to the conservative quasi-Keplerian description of Sec. II C. The full solution including radiation reaction is then found by varying the ‘‘constants’’ $c_\alpha(t)$, leading to differential equations of the form

$$\frac{dc_\alpha}{dt} = G_\alpha(l, c_\alpha). \quad (29)$$

One can then introduce a two-scale decomposition of all phase variables $c_\alpha(l)$ into a slow (radiation-reaction timescale) secular drift and a fast (orbital timescale) periodic oscillation as

$$c_\alpha(t) = \bar{c}_\alpha(t) + \tilde{c}_\alpha(t), \quad (30)$$

with

$$\frac{d\bar{c}_\alpha}{dl} = \bar{G}_\alpha(l, c_\alpha), \quad (31a)$$

$$\frac{d\tilde{c}_\alpha}{dl} = \tilde{G}_\alpha(l, c_\alpha) = G_\alpha(l, c_\alpha) - \bar{G}_\alpha(l, c_\alpha), \quad (31b)$$

\bar{G}_α and \tilde{G}_α here being the orbital averaged and oscillatory pieces of G_α . The secular evolution of the orbital elements (31a) can also be derived from the heuristic balance equations $\langle dE/dt \rangle = -\langle \mathcal{F} \rangle$ and $\langle dJ/dt \rangle = -\langle \mathcal{G} \rangle$, where \mathcal{F} is the energy flux and \mathcal{G} the angular momentum flux. This approach is discussed at the 3PN order in a series of papers [35–37], which notably take care of the hereditary contributions to the energy and angular momentum fluxes.

After the above procedure is applied, we have

$$x(t) = \bar{x}(t) + \tilde{x}(t), \quad (32a)$$

$$e(t) = \bar{e}(t) + \tilde{e}(t), \quad (32b)$$

$$c_l(t) = \bar{c}_l + \tilde{c}_l(t), \quad (32c)$$

$$c_\lambda(t) = \bar{c}_\lambda + \tilde{c}_\lambda(t), \quad (32d)$$

where \bar{c}_l and \bar{c}_λ are found to be true integration constants. The secular evolution of the orbital elements $\bar{n}(t)$, $\bar{k}(t)$, $\bar{x}(t)$, and $\bar{e}(t)$ is given in Sec. VI of Ref. [37]. At leading order, these equations reduce to the famous formulas by Peters and Mathews [16, 17]:

$$\frac{d\bar{x}}{dt} = \frac{c^3 \nu}{Gm} \frac{\bar{x}^5}{(1 - \bar{e}^2)^{7/2}} \left(\frac{64}{5} + \frac{584}{15} \bar{e}^2 + \frac{74}{15} \bar{e}^4 \right), \quad (33a)$$

$$\frac{d\bar{e}}{dt} = -\frac{c^3\nu}{Gm} \frac{\bar{e}\bar{x}^4}{(1-\bar{e}^2)^{5/2}} \left(\frac{304}{15} + \frac{121}{15}\bar{e}^2 \right). \quad (33b)$$

The periodic variations in Eqs. (32) can be computed from Eqs. (34) and (35) of Ref. [33] and are explicitly given in Eqs. (36). Note, though, that there is an error in the expressions for \tilde{c}_l and \tilde{c}_λ provided by Eqs. (36c) and (36d) of that paper. Indeed, the periodic variations \tilde{c}_l and \tilde{c}_λ refer to the zero-average oscillatory contributions to c_l and c_λ . They are found by integrating Eqs. (35) and then subtracting the orbital average, i.e., finding the

unique zero-average primitive, so that we are left with a purely oscillatory solution. Now, we find that, unfortunately, the explicit orbital averages of Eqs. (36c) and (36d) in Ref. [33] do not give zero. This is because the averaging of these terms is performed over the eccentric anomaly du , whereas the orbital averaging requires integrating temporal variations over an orbital period and, therefore, should be done using $dl = (1 - e \cos u)du$. We show below the corrected expressions for \tilde{c}_l and \tilde{c}_λ in terms of $e_t = \bar{e}$, $\xi = \bar{x}^{3/2}$ and $u = \bar{u}$, as they appear in Ref. [33]:

$$\tilde{c}_l = -\frac{2\xi^{5/3}\nu}{45e_t^2} \left\{ \frac{144e_t^2}{\chi} + \frac{18 - 258e_t^2}{\chi^2} + \frac{-56 + 92e_t^2 - 36e_t^4}{\chi^3} + \frac{105(1 - e_t^2)^2}{\chi^4} - \frac{1}{2(1 - e_t^2)^{1/2}} \left[134 - 339e_t^2 + 288e_t^2\sqrt{1 - e_t^2} \right] \right\} + \mathcal{O}(\xi^{7/3}), \quad (34a)$$

$$\tilde{c}_\lambda = \frac{2\xi^{5/3}\nu}{45e_t^2} \left\{ \left[\frac{18}{\chi^2} - \frac{56 - 36e_t^2}{\chi^3} + \frac{105(1 - e_t^2)}{\chi^4} \right] \sqrt{1 - e_t^2} - \frac{144e_t^2}{\chi} - \frac{18 - 258e_t^2}{\chi^2} + \frac{56 - 92e_t^2 + 36e_t^4}{\chi^3} - \frac{105(1 - e_t^2)^2}{\chi^4} - \frac{1}{2(1 - e_t^2)} \left[134 - 147e_t^2 + 288e_t^4 - (134 - 339e_t^2)\sqrt{1 - e_t^2} \right] \right\} + \mathcal{O}(\xi^{7/3}). \quad (34b)$$

Similarly, we split the angles l and λ into orbital averaged and oscillatory contributions

$$l(t) = \bar{l}(t) + \tilde{l}(t), \quad (35a)$$

$$\lambda(t) = \bar{\lambda}(t) + \tilde{\lambda}(t), \quad (35b)$$

with $\bar{l}(t)$ and $\bar{\lambda}(t)$ defined by

$$\bar{l}(t) = \int_{t_0}^t \bar{n}(t') dt' + \bar{c}_l, \quad (36a)$$

$$\bar{\lambda}(t) = \int_{t_0}^t [1 + \bar{k}(t')] \bar{n}(t') dt' + \bar{c}_\lambda. \quad (36b)$$

The oscillatory contributions \tilde{l} and $\tilde{\lambda}$ are calculated as in Eqs. (39) of Ref. [33],

$$\tilde{l}(\bar{l}) = \int \frac{\tilde{n}}{\bar{n}} dl + \tilde{c}_l(\bar{l}), \quad (37a)$$

$$\tilde{\lambda}(\bar{\lambda}) = \int \left[(1 + \bar{k}) \frac{\tilde{n}}{\bar{n}} + \tilde{k} \right] dl + \tilde{c}_\lambda(\bar{\lambda}), \quad (37b)$$

where $\tilde{k} = (\partial k / \partial n) \tilde{n} + (\partial k / \partial e_t) \tilde{e}_t$ denotes the periodic part of k and the integrals again mean the unique zero-average primitives. Equations (40) for \tilde{l} and $\tilde{\lambda}$ in Ref. [33] are erroneous, since they do not average to zero either. We list below the corrected expressions:

$$\begin{aligned} \tilde{l}(l) = & \frac{\xi^{5/3}\nu}{15(1 - e_t^2)^3} \left\{ (602 + 673e_t^2)\chi + (314 - 203e_t^2 - 111e_t^4) \ln \chi - (602 + 673e_t^2) + \frac{-98 + 124e_t^2 + 46e_t^4 - 72e_t^6}{\chi} \right. \\ & - \frac{105(1 - e_t^2)^3}{\chi^2} - \frac{1}{2} \left[432 + 444e_t^2 + 543e_t^4 - 144e_t^6 - (838 - 826e_t^2 - 12e_t^4)\sqrt{1 - e_t^2} + (628 - 406e_t^2 - 222e_t^4) \right. \\ & \left. \left. \times \ln \left(\frac{1 + \sqrt{1 - e_t^2}}{2} \right) \right] \right\} + \frac{\xi^{5/3}\nu}{5(1 - e_t^2)^{7/2}} (96 + 292e_t^2 + 37e_t^4) \int \left[2 \tan^{-1} \left(\frac{\beta_t \sin u}{1 - \beta_t \cos u} \right) + e_t \sin u \right] \chi du \\ & + \tilde{c}_l(l) + \mathcal{O}(\xi^{7/3}), \end{aligned} \quad (38a)$$

$$\tilde{\lambda}(l) = \tilde{l}(l) - \tilde{c}_l(l) + \tilde{c}_\lambda(l) + \mathcal{O}(\xi^{7/3}). \quad (38b)$$

The errors in Eqs. (36c), (36d), and (40) of Ref. [33],

though, do not affect the other equations of that work.

We refer to Appendix A for some integral relations necessary to compute the zero-average primitives.

We finally give expressions for the oscillatory contributions \tilde{x} , \tilde{e} , \tilde{l} , and $\tilde{\lambda}$ in terms of the slowly evolving variables \bar{x} , \bar{e} , and \bar{l} . We list here the expressions to $\mathcal{O}(\bar{e}^2)$:

$$\begin{aligned} \tilde{x}(t) = & \nu \bar{x}^{7/2} \bar{e} \left[80 \sin(\bar{l}) + \frac{1436}{15} \bar{e} \sin(2\bar{l}) \right. \\ & \left. + \bar{e}^2 \left(\frac{4538}{15} \sin(\bar{l}) + \frac{6022}{45} \sin(3\bar{l}) \right) \right] \\ & + \mathcal{O}(\bar{x}^{9/2}), \end{aligned} \quad (39a)$$

$$\begin{aligned} \tilde{e}(t) = & -\nu \bar{x}^{5/2} \left[\frac{64}{5} \sin(\bar{l}) + \frac{352}{15} \bar{e} \sin(2\bar{l}) \right. \\ & \left. + \bar{e}^2 \left(\frac{1138}{15} \sin(\bar{l}) + \frac{358}{9} \sin(3\bar{l}) \right) \right] \\ & + \mathcal{O}(\bar{x}^{7/2}), \end{aligned} \quad (39b)$$

$$\begin{aligned} \tilde{l}(t) = & -\nu \bar{x}^{5/2} \left[\frac{64}{5\bar{e}} \cos(\bar{l}) + \frac{352}{15} \cos(2\bar{l}) \right. \\ & \left. + \bar{e} \left(\frac{1654}{15} \cos(\bar{l}) + \frac{358}{9} \cos(3\bar{l}) \right) \right. \\ & \left. + \bar{e}^2 \left(\frac{694}{15} \cos(2\bar{l}) + \frac{1289}{20} \cos(4\bar{l}) \right) \right] \\ & + \mathcal{O}(\bar{x}^{7/2}), \end{aligned} \quad (39c)$$

$$\begin{aligned} \tilde{\lambda}(t) = & -\nu \bar{x}^{5/2} \left[\frac{296}{3} \bar{e} \cos(\bar{l}) + \frac{199}{5} \bar{e}^2 \cos(2\bar{l}) \right] \\ & + \mathcal{O}(\bar{x}^{7/2}). \end{aligned} \quad (39d)$$

These results agree with Eqs. (4.9) of Ref. [34], except two constant terms in $\tilde{l}(t)$ and $\tilde{\lambda}(t)$, due to the already mentioned incorrect average. Indeed, all our results are purely oscillatory, zero-average functions and thus correctly describe the periodic post-adiabatic corrections.

Given the waveform in terms of the conservative quasi-Keplerian parametrization, one can then include post-adiabatic effects by making the simple substitutions

$$x \rightarrow \bar{x} + \tilde{x}, \quad (40a)$$

$$e \rightarrow \bar{e} + \tilde{e}, \quad (40b)$$

$$l \rightarrow \bar{l} + \tilde{l}, \quad (40c)$$

$$\lambda \rightarrow \bar{\lambda} + \tilde{\lambda}. \quad (40d)$$

As all of the periodic (tilde) contributions are of relative 2.5PN order compared to the slowly evolving (bar) parts, we only have to make these substitutions at leading Newtonian and 0.5PN order in the $h^{\ell m}$ to be accurate to 3PN order. In all higher-order terms, we can simply replace the variables (x, e, l, λ) by their secular evolving parts $(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda})$.

Note that Eq. (20b) gives the relation between the geometrical phase ϕ and the angles l and λ . We can rewrite this relation in terms of the slowly evolving angles \bar{l} and $\bar{\lambda}$ and find

$$\phi = \lambda + W(l) = \bar{\lambda} + \bar{W}(\bar{l}) + \tilde{\lambda} + (\tilde{v} - \tilde{l}), \quad (41)$$

where $\bar{W}(\bar{l})$ is given by Eq. (22c), but with all quantities on the RHS replaced with their secular evolving parts, and the periodic variation \tilde{v} of the true anomaly is given by

$$\begin{aligned} \tilde{v} = & \frac{\partial \bar{v}}{\partial \bar{u}} \bar{u} + \frac{\partial \bar{v}}{\partial \bar{e}} \bar{e} \\ = & \frac{\sqrt{1 - \bar{e}^2}}{1 - \bar{e} \cos \bar{u}} \bar{u} + \frac{\sin \bar{u}}{\sqrt{1 - \bar{e}^2} (1 - \bar{e} \cos \bar{u})} \bar{e}. \end{aligned} \quad (42)$$

Expanded to $\mathcal{O}(\bar{x}^3)$ and $\mathcal{O}(\bar{e})$ this finally gives us

$$\begin{aligned} \phi = & \bar{\lambda} + 2\bar{e} \sin(\bar{l}) + \bar{x}(10 - \nu)\bar{e} \sin(\bar{l}) \\ & + \bar{x}^2 \left(52 - \frac{235\nu}{12} + \frac{\nu^2}{12} \right) \bar{e} \sin(\bar{l}) \\ & - \bar{x}^{5/2} \nu \left(\frac{128}{5} + \frac{888}{5} \bar{e} \cos(\bar{l}) \right) \\ & + \bar{x}^3 \left(292 + \left(-\frac{420131}{840} + \frac{287\pi^2}{32} \right) \nu \right. \\ & \left. + \frac{521\nu^2}{24} + \frac{\nu^3}{24} \right) \bar{e} \sin(\bar{l}). \end{aligned} \quad (43)$$

This is very similar to Eq. (25b), but with the quantities on the RHS replaced by their slowly evolving parts and with additional terms at 2.5PN order.

IV. HEREDITARY CONTRIBUTIONS

A. Tail integrals

Note that tail effects start appearing at 1.5PN order, and thus post-adiabatic corrections to those will only enter the waveform at 4PN order and beyond. We can thus neglect any radiation-reaction effects in this section and only consider the conservative problem. At the end, we can then replace all variables (x, e, l, λ) with their slowly evolving counterparts $(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda})$ to get the secular evolving amplitudes.

We now employ the quasi-Keplerian parametrization introduced in Sec. II C. As we use the two angles l and λ to parameterize the orbital motion, time derivatives of the source multipole moments (I_L, J_L) can be calculated as

$$\frac{d}{dt} = n \left(\frac{d}{dl} + (1+k) \frac{d}{d\lambda} \right). \quad (44)$$

We use a low-eccentricity expansion to simplify expressions, so we expand everything in powers of both x and e . Inserting Eqs. (25) into the source multipole moments (17–18), and substituting those into the radiative moments (14–15) we can then easily calculate the spherical harmonic modes in terms of l and λ . We find, e.g., for the dominant h_{tail}^{22} mode

$$h_{\text{tail}}^{22} = \frac{8Gm\nu}{c^2 R} x^{5/2} \sqrt{\frac{\pi}{5}} \frac{x^{3/2} c^3}{Gm}$$

$$\times \int_0^\infty d\tau e^{-2i(\lambda-\lambda(\tau))} \left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12} \right] \\ \times \left[-8 + e \left(\frac{3}{2} e^{i(l-l(\tau))} - \frac{81}{2} e^{-i(l-l(\tau))} \right) \right], \quad (45)$$

where $l(\tau) = n\tau$ and $\lambda(\tau) = (1+k)n\tau$ and where we restrict ourselves to the leading post-Newtonian order and $\mathcal{O}(e)$. All other modes can be calculated similarly and be given as integrals over past history. These integrals can then be solved using the standard formulas

$$\int_0^\infty d\tau e^{-i\omega\tau} = -\frac{i}{\omega}, \quad (46a) \\ \int_0^\infty d\tau e^{-i\omega\tau} \ln\left(\frac{\tau}{2\tau_0}\right) = \\ -\frac{1}{\omega} \left(\frac{\pi}{2} \text{sign}(\omega) - i [\ln(2|\omega|\tau_0) + \gamma_E] \right), \quad (46b)$$

$$\int_0^\infty d\tau e^{-i\omega\tau} \ln^2\left(\frac{\tau}{2\tau_0}\right) = \\ -\frac{i}{\omega} \left(\frac{\pi^2}{6} + \left(\frac{\pi}{2} \text{sign}(\omega) - i [\ln(2|\omega|\tau_0) + \gamma_E] \right)^2 \right). \quad (46c)$$

Note that for terms of the form $\int d\tau e^{-i(\alpha l(\tau) + \beta \lambda(\tau))} [\dots]$ we have $\omega = n(\alpha + (1+k)\beta)$.

We are now able to give the tail contributions to the spherical harmonic modes in terms of the parameters x , $e = e_t$ and the angles ϕ and l . The modes have the following structure:

$$h_{\text{tail}}^{\ell m} = \frac{8Gm\nu}{c^2 R} x \sqrt{\frac{\pi}{5}} e^{-im\phi} H_{\text{tail}}^{\ell m}. \quad (47)$$

The various contributions to, e.g., the H_{tail}^{22} mode are given to $\mathcal{O}(e)$ by

$$(H_{\text{tail}}^{22})_{1.5\text{PN}} = x^{3/2} \left(2\pi + 6i \ln\left(\frac{x}{x'_0}\right) + e \left\{ e^{-il} \left[\frac{11\pi}{4} + \frac{27i}{2} \ln\left(\frac{3}{2}\right) + \frac{33i}{4} \ln\left(\frac{x}{x'_0}\right) \right] \right. \right. \\ \left. \left. + e^{il} \left[\frac{13\pi}{4} + \frac{3i}{2} \ln(2) + \frac{39i}{4} \ln\left(\frac{x}{x'_0}\right) \right] \right\} \right), \quad (48a)$$

$$(H_{\text{tail}}^{22})_{2.5\text{PN}} = x^{5/2} \left(\pi \left(-\frac{107}{21} + \frac{34\nu}{21} \right) + \left(-\frac{107i}{7} + \frac{34i\nu}{7} \right) \ln\left(\frac{x}{x'_0}\right) \right. \\ \left. + e \left\{ e^{il} \left[-\frac{9i}{2} + \pi \left(\frac{229}{168} + \frac{61\nu}{42} \right) + \left(\frac{473i}{28} - \frac{3i\nu}{7} \right) \ln(2) + \left(\frac{229i}{56} + \frac{61i\nu}{14} \right) \ln\left(\frac{x}{x'_0}\right) \right] \right. \right. \\ \left. \left. + e^{-il} \left[-\frac{27i}{2} + \pi \left(-\frac{1081}{168} + \frac{137\nu}{42} \right) + \left(\frac{27i}{4} + 9i\nu \right) \ln\left(\frac{3}{2}\right) \right. \right. \right. \\ \left. \left. \left. + \left(-\frac{1081i}{56} + \frac{137i\nu}{14} \right) \ln\left(\frac{x}{x'_0}\right) \right] \right\} \right), \quad (48b)$$

$$(H_{\text{tail}}^{22})_{3\text{PN}} = x^3 \left(-\frac{515063}{22050} + \frac{428i\pi}{105} + \frac{2\pi^2}{3} + \left(-\frac{428}{35} + 12i\pi \right) \ln\left(\frac{x}{x'_0}\right) - 18 \ln^2\left(\frac{x}{x'_0}\right) \right. \\ \left. + e \left\{ e^{-il} \left[-\frac{515063}{7200} + \frac{749i\pi}{60} + \frac{49\pi^2}{24} + \left(-\frac{2889}{70} + \frac{81i\pi}{2} \right) \ln\left(\frac{3}{2}\right) - \frac{81}{2} \ln^2\left(\frac{3}{2}\right) \right. \right. \right. \\ \left. \left. \left. + \left(-\frac{749}{20} + \frac{147i\pi}{4} - \frac{243}{2} \ln\left(\frac{3}{2}\right) \right) \ln\left(\frac{x}{x'_0}\right) - \frac{441}{8} \ln^2\left(\frac{x}{x'_0}\right) \right] \right. \right. \right. \\ \left. \left. \left. + e^{il} \left[-\frac{14936827}{352800} + \frac{3103i\pi}{420} + \frac{29\pi^2}{24} + \left(-\frac{107}{70} + \frac{3i\pi}{2} \right) \ln(2) + \frac{3}{2} \ln^2(2) \right. \right. \right. \right. \\ \left. \left. \left. + \left(-\frac{3103}{140} + \frac{87i\pi}{4} - \frac{9}{2} \ln(2) \right) \ln\left(\frac{x}{x'_0}\right) - \frac{261}{8} \ln^2\left(\frac{x}{x'_0}\right) \right] \right\} \right). \quad (48c)$$

Here, x'_0 is related to the arbitrary constant τ_0 by

$$x'_0 = \left(\frac{Gm}{c^3} \frac{e^{11/12 - \gamma_E}}{4\tau_0} \right)^{2/3}. \quad (49)$$

We list expressions for all $h_{\text{tail}}^{\ell m}$ modes in a supplemental *Mathematica* file.

B. Memory integrals

The nonlinear memory effect arises from the nonlogarithmic integrals in Eqs. (14); e.g., for the $\ell = 2$ modes we have

$$U_{ij}^{\text{mem}}(t_r) = -\frac{2G}{7c^5} \int_{-\infty}^{t_r} d\tau I_{a(i)}^{(3)}(\tau) I_{j)a}^{(3)}(\tau). \quad (50)$$

There are two types of memory arising from these integrals: DC (or ‘‘direct current’’) memory and oscillatory memory. The DC memory is a slowly increasing, nonoscillatory contribution to the gravitational-wave amplitude, entering at Newtonian order. This leads to a difference in the amplitude between early and late times:

$$\Delta h_{\text{mem}} = \lim_{t \rightarrow +\infty} h(t) - \lim_{t \rightarrow -\infty} h(t). \quad (51)$$

The oscillatory memory, on the other hand, is a normal periodic contribution entering the gravitational-wave amplitude at higher PN order. In Refs. [61] and [51], the authors give expressions for both leading-order DC and oscillatory memory in the circular limit. The calculation of DC memory has been extended to 3PN order for circular binaries in Ref. [62] and to Newtonian order for eccentric binaries in Ref. [63]. In this paper, we will only briefly discuss the leading-order contributions to the DC and oscillatory memory for eccentric binaries, such that we can compare our results to the circular limit in Ref. [51]. The complete post-Newtonian corrections to the nonlinear memory are dealt with in a subsequent paper [50], completing the hereditary contributions to the gravitational-wave amplitudes for nonspinning eccentric binaries.

Following the same steps as in the previous section, we can calculate the derivatives of the source moments, and we find, e.g., for the 20-mode:

$$h_{\text{DC}}^{20} = \frac{256 Gm\nu}{7 c^2 R} \sqrt{\frac{\pi}{30}} \int_{-\infty}^{t_r} dt \left(1 + \frac{313}{48} e^2\right) x^5. \quad (52)$$

We find that all DC memory modes will consist of such integrals of the form

$$h_{\text{DC}}^{\ell 0} \propto \int_{-\infty}^{t_r} dt x^p(t) e^q(t). \quad (53)$$

One can rewrite this as an integral over the eccentricity

$$h_{\text{DC}}^{\ell 0} \propto \int_{e_i}^{e(t_r)} de \left(\frac{de}{dt}\right)^{-1} x^p(e) e^q, \quad (54)$$

where e_i is some initial eccentricity at early times. Solving the evolution equations (33) to leading order, we find

$$x(e) = x_0 \left(\frac{e_0}{e}\right)^{12/19}, \quad (55)$$

where $x(e_0) = x_0$. We can insert this into Eq. (54) together with the evolution equation de/dt and integrate

over e . We then find DC memory at leading Newtonian order in the 20-mode and 40-mode:

$$h_{\text{DC}}^{20} = \frac{8Gm\nu}{c^2 R} x \sqrt{\frac{\pi}{5}} \frac{-5}{14\sqrt{6}} \left\{1 - \left(\frac{e}{e_i}\right)^{12/19}\right\}, \quad (56a)$$

$$h_{\text{DC}}^{40} = \frac{8Gm\nu}{c^2 R} x \sqrt{\frac{\pi}{5}} \frac{-1}{504\sqrt{2}} \left\{1 - \left(\frac{e}{e_i}\right)^{12/19}\right\}. \quad (56b)$$

The time derivatives of the oscillatory modes are computed in the same way. We find that they consist of integrals of the form

$$h_{\text{osc}}^{\ell m} \propto \int_{-\infty}^{t_r} dt x^p(t) e^q(t) e^{i(s\lambda + r_l)}, \quad (57)$$

which can be integrated to give

$$h_{\text{osc}}^{\ell m} \propto -\frac{i}{n(r + (1+k)s)} x^p e^q e^{i(s\lambda + r_l)}. \quad (58)$$

Note that there are oscillatory memory contributions entering the waveform at 1.5, 2, 2.5 and 3PN order. We list here only the 2.5 and 3PN terms that have a circular limit, as to compare our results to Ref. [51]. We refer to our follow-up work [50] for a complete treatment of nonlinear memory. The modes have the following structure:

$$h_{\text{osc}}^{\ell m} = \frac{8Gm\nu}{c^2 R} x \sqrt{\frac{\pi}{5}} e^{-im\phi} H_{\text{osc}}^{\ell m}. \quad (59)$$

The various contributions to $\mathcal{O}(e)$ are:

$$H_{\text{osc}}^{31} = \frac{-121 x^3 \nu \Delta}{45\sqrt{14}} \left(1 + e \left\{\frac{301}{242} e^{-il} + e^{il}\right\}\right), \quad (60a)$$

$$H_{\text{osc}}^{33} = \frac{11 x^3 \nu \Delta}{27\sqrt{210}} \left(1 + e \left\{\frac{9}{2} e^{-il} + \frac{3}{22} e^{il}\right\}\right), \quad (60b)$$

$$H_{\text{osc}}^{44} = \frac{i x^{5/2} \nu}{9\sqrt{35}} \left(1 + e \left\{\frac{7}{5} e^{-il} + 3e^{il}\right\}\right), \quad (60c)$$

$$H_{\text{osc}}^{51} = \frac{-13 x^3 \nu \Delta}{63\sqrt{385}} \left(1 + e \left\{\frac{251}{208} e^{-il} + e^{il}\right\}\right), \quad (60d)$$

$$H_{\text{osc}}^{53} = \frac{-x^3 \nu \Delta}{189\sqrt{330}} \left(1 + e \left\{\frac{201}{16} e^{il} - \frac{369}{32} e^{-il}\right\}\right), \quad (60e)$$

$$H_{\text{osc}}^{55} = \frac{9 x^3 \nu \Delta}{35\sqrt{66}} \left(1 + e \left\{\frac{2285}{1296} e^{-il} + \frac{985}{288} e^{il}\right\}\right). \quad (60f)$$

V. CONSTRUCTING THE FULL 3PN-ACCURATE WAVEFORM

We now want to construct the full 3PN-accurate waveform valid during the inspiral of a binary system. We begin by adding up the two contributions to the spherical harmonic modes:

$$h^{\ell m} = (h^{\ell m})_{\text{inst}} + (h^{\ell m})_{\text{hered}}. \quad (61)$$

Note that we are still missing some memory contributions. These will be computed in full in our follow-up work [50], and we will give expressions for the full waveform including memory there.

A. Instantaneous parts

The instantaneous parts $(h^{\ell m})_{\text{inst}}$ of the spherical harmonic modes for compact binaries in elliptical orbits have already been calculated to the third post-Newtonian order in Ref. [49], although the results do not include post-adiabatic corrections to the quasi-Keplerian parametrization. They are given in terms of the constants of motion x and $e = e_t$ and parametrized by the eccentric anomaly u . We will rewrite these in terms of the mean anomaly l by using the solution to the Kepler equation (25a). This

gives us expressions for the instantaneous contributions to the different modes in terms of the post-Newtonian parameter x and the time eccentricity e , parametrized by the angles ϕ and l . The modes again have the following structure:

$$h_{\text{inst}}^{\ell m} = \frac{8Gm\nu}{c^2 R} x \sqrt{\frac{\pi}{5}} e^{-im\phi} H_{\text{inst}}^{\ell m}. \quad (62)$$

The various contributions to, e.g., the H_{inst}^{22} mode are given to $\mathcal{O}(e)$ by

$$(H_{\text{inst}}^{22})_{\text{Newt}} = 1 + e \left\{ \frac{1}{4} e^{-il} + \frac{5}{4} e^{il} \right\}, \quad (63a)$$

$$(H_{\text{inst}}^{22})_{1\text{PN}} = x \left(-\frac{107}{42} + \frac{55\nu}{42} + e \left\{ e^{-il} \left[-\frac{257}{168} + \frac{169\nu}{168} \right] + e^{il} \left[-\frac{31}{24} + \frac{35\nu}{24} \right] \right\} \right), \quad (63b)$$

$$(H_{\text{inst}}^{22})_{2\text{PN}} = x^2 \left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512} + e \left\{ e^{il} \left[-\frac{2155}{252} - \frac{1655\nu}{672} + \frac{371\nu^2}{288} \right] + e^{-il} \left[-\frac{4271}{756} - \frac{35131\nu}{6048} + \frac{421\nu^2}{864} \right] \right\} \right), \quad (63c)$$

$$(H_{\text{inst}}^{22})_{2.5\text{PN}} = -x^{5/2} i\nu \left(\frac{56}{5} + e \left\{ \frac{7817}{420} e^{il} + \frac{2579}{84} e^{-il} \right\} \right), \quad (63d)$$

$$(H_{\text{inst}}^{22})_{3\text{PN}} = x^3 \left(\frac{761273}{13200} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^3}{99792} + \frac{856}{105} \ln \left(\frac{x}{x_0} \right) + e \left\{ e^{il} \left[\frac{6148781}{75600} + \left(-\frac{199855}{3024} + \frac{41\pi^2}{48} \right) \nu - \frac{9967\nu^2}{1008} + \frac{35579\nu^3}{36288} + \frac{3103}{210} \ln \left(\frac{x}{x_0} \right) \right] + e^{-il} \left[\frac{150345571}{831600} + \left(-\frac{121717}{20790} - \frac{41\pi^2}{192} \right) \nu - \frac{86531\nu^2}{8316} - \frac{33331\nu^3}{399168} + \frac{749}{30} \ln \left(\frac{x}{x_0} \right) \right] \right\} \right), \quad (63e)$$

where $x_0 = Gm/(c^3 \tau_0)$ is related to x'_0 by

$$\ln x'_0 = \frac{11}{18} - \frac{2}{3} \gamma_E - \frac{4}{3} \ln 2 + \frac{2}{3} \ln x_0. \quad (64)$$

B. Post-adiabatic corrections

We now move to include post-adiabatic corrections into the waveform. As already mentioned in Sec. IV, post-adiabatic corrections to the hereditary contributions will only enter at 4PN. We are thus left with computing the corrections to the instantaneous contributions as described in Sec. III. Schematically, the substitutions in Eq. (40) may be described as

$$\begin{aligned} & h^{\ell m}(x, e, l, \lambda) \\ & \Downarrow \\ & h^{\ell m}(\bar{x} + \tilde{x}, \bar{e} + \tilde{e}, \bar{l} + \tilde{l}, \bar{\lambda} + \tilde{\lambda}) \end{aligned}$$

$$\begin{aligned} & \Downarrow \\ & h^{\ell m}(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda}) + \left\{ \frac{\partial h^{\ell m}}{\partial x} \tilde{x} + \frac{\partial h^{\ell m}}{\partial e} \tilde{e} + \frac{\partial h^{\ell m}}{\partial l} \tilde{l} + \frac{\partial h^{\ell m}}{\partial \lambda} \tilde{\lambda} \right\} \\ & \Downarrow \\ & h^{\ell m}(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda}) + \frac{1}{c^5} h_{\text{post-ad}}^{\ell m}(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda}). \quad (65) \end{aligned}$$

In particular, we only need to make these substitutions at leading Newtonian and 0.5PN order. At higher orders, we simply replace the variables (x, e, l, λ) by their secular evolving parts $(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda})$ to get the secular evolving waveform.

The post-adiabatic contributions to the different modes in terms of the secular evolving parameters \bar{x} and \bar{e} , parametrized by the angles ϕ and \bar{l} , have the following form:

$$h_{\text{post-ad}}^{\ell m} = \frac{8Gm\nu}{c^2 R} \bar{x} \sqrt{\frac{\pi}{5}} e^{-im\phi} H_{\text{post-ad}}^{\ell m}. \quad (66)$$

For example, the $H_{\text{post-ad}}^{22}$ mode, that arises from including the post-adiabatic corrections in $(H_{\text{inst}}^{22})_{\text{Newt}}$, is given by

$$H_{\text{post-ad}}^{22} = \frac{192}{5} \bar{x}^{5/2} i\nu \left(1 + \bar{e} \left\{ \frac{401}{72} e^{-i\bar{l}} + \frac{293}{72} e^{i\bar{l}} \right\} \right). \quad (67)$$

We can combine these post-adiabatic contributions with the instantaneous ones to get the full secular evolving instantaneous waveform in terms of the variables $(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda})$. The result has again the following form:

$$h_{\text{inst}}^{\ell m} = \frac{8Gm\nu}{c^2 R} \bar{x} \sqrt{\frac{\pi}{5}} e^{-im\phi} H_{\text{inst}}^{\ell m}. \quad (68)$$

In e.g. the H_{inst}^{22} mode we find that the only term that is modified is the 2.5PN order:

$$(H_{\text{inst}}^{22})_{2.5\text{PN}} = -\bar{x}^{5/2} i\nu \left(24 + \bar{e} \left\{ \frac{43657}{420} e^{i\bar{l}} + \frac{1013}{140} e^{-i\bar{l}} \right\} \right). \quad (69)$$

All other orders are exactly as in Eqs. (63), but with (x, e, l, λ) replaced by $(\bar{x}, \bar{e}, \bar{l}, \bar{\lambda})$.

C. Log cancellation

We observe that both instantaneous and tail terms still have some dependence on the arbitrary constant x'_0 (or x_0). We find that this dependence on x'_0 can be reabsorbed in a shift of the coordinate time t [61, 64] through a redefinition of the mean anomaly as

$$\xi = \bar{l} - \frac{3GM}{c^3} \bar{n} \ln \left(\frac{\bar{x}}{x'_0} \right), \quad (70)$$

where $M = m(1 - \nu\bar{x}/2)$ is the ADM mass. Note that there are no post-adiabatic corrections to n and x here, as phasing effects would only enter at 1.5 + 2.5PN order. This also means that both ξ and \bar{l} follow the same evolution, i.e., $d\xi/dt = d\bar{l}/dt = \bar{n}$, and they only differ by a constant factor. To simplify the final expressions, we also introduce a redefined phase ψ such that Eq. (41) gives the relation between ξ and ψ :

$$\psi = \bar{\lambda}_\xi + \bar{W}_\xi + \bar{\lambda}_\xi + (\tilde{v}_\xi - \tilde{l}_\xi). \quad (71)$$

Here

$$\bar{\lambda}_\xi = \bar{\lambda} - \frac{3GM}{c^3} (1 + \bar{k}) \bar{n} \ln \left(\frac{\bar{x}}{x'_0} \right), \quad (72)$$

is the phase $\bar{\lambda}$ evaluated at the shifted time defined by ξ , and \bar{W}_ξ , $\tilde{\lambda}_\xi$, \tilde{v}_ξ , and \tilde{l}_ξ are defined as in Eq. (41), but with \bar{l} replaced by ξ . From this, we can easily deduce that

$$\psi = \phi + \sum_{s=1}^{\infty} \frac{1}{s!} \left[(\xi - \bar{l})^s \left(\frac{d}{d\bar{l}} \right)^s + (\bar{\lambda}_\xi - \bar{\lambda})^s \left(\frac{d}{d\bar{\lambda}} \right)^s \right] \phi. \quad (73)$$

Note that the phase ψ does not have the same geometric interpretation as ϕ . Expanding these equations to $\mathcal{O}(\bar{x}^3)$ and $\mathcal{O}(\bar{e})$, we find

$$\bar{l} = \xi + 3 \left(\bar{x}^{3/2} - \bar{x}^{5/2} \left(3 + \frac{\nu}{2} \right) \right) \ln \left(\frac{\bar{x}}{x'_0} \right), \quad (74a)$$

$$\begin{aligned} \phi = \psi + & \left(\bar{x}^{3/2} (3 + 6\bar{e} \cos(\xi)) \right. \\ & \left. + \bar{x}^{5/2} \left(-\frac{3\nu}{2} + 6\bar{e}(2 - \nu) \cos(\xi) \right) \right) \ln \left(\frac{\bar{x}}{x'_0} \right) \\ & - 9\bar{x}^3 \bar{e} \sin(\xi) \ln^2 \left(\frac{\bar{x}}{x'_0} \right). \end{aligned} \quad (74b)$$

This redefinition of the time coordinate results in the cancellation of all log terms involving the arbitrary constant x'_0 .

D. Full waveform

The full waveform in terms of the redefined angles ξ and ψ – minus some memory contributions – has the following form:

$$h^{\ell m} = \frac{8Gm\nu}{c^2 R} \bar{x} \sqrt{\frac{\pi}{5}} e^{-im\psi} H^{\ell m}. \quad (75)$$

The various contributions to, e.g., the H^{22} mode are given to $\mathcal{O}(\bar{e})$ by

$$H_{\text{Newt}}^{22} = 1 + \bar{e} \left\{ \frac{1}{4} e^{-i\xi} + \frac{5}{4} e^{i\xi} \right\}, \quad (76a)$$

$$H_{\text{iPN}}^{22} = \bar{x} \left(-\frac{107}{42} + \frac{55\nu}{42} + \bar{e} \left\{ e^{-i\xi} \left[-\frac{257}{168} + \frac{169\nu}{168} \right] + e^{i\xi} \left[-\frac{31}{24} + \frac{35\nu}{24} \right] \right\} \right), \quad (76b)$$

$$H_{1.5\text{PN}}^{22} = \bar{x}^{3/2} \left(2\pi + \bar{e} \left\{ e^{-i\xi} \left[\frac{11\pi}{4} + \frac{27i}{2} \ln\left(\frac{3}{2}\right) \right] + e^{i\xi} \left[\frac{13\pi}{4} + \frac{3i}{2} \ln(2) \right] \right\} \right), \quad (76c)$$

$$H_{2\text{PN}}^{22} = \bar{x}^2 \left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512} + \bar{e} \left\{ e^{i\xi} \left[-\frac{2155}{252} - \frac{1655\nu}{672} + \frac{371\nu^2}{288} \right] + e^{-i\xi} \left[-\frac{4271}{756} - \frac{35131\nu}{6048} + \frac{421\nu^2}{864} \right] \right\} \right), \quad (76d)$$

$$H_{2.5\text{PN}}^{22} = \bar{x}^{5/2} \left(-\frac{107\pi}{21} + \left(-24i + \frac{34\pi}{21} \right) \nu + \bar{e} \left\{ e^{i\xi} \left[-\frac{9i}{2} + \frac{229\pi}{168} + \left(-\frac{43657i}{420} + \frac{61\pi}{42} \right) \nu + \left(\frac{473i}{28} - \frac{3i\nu}{7} \right) \ln(2) \right] + e^{-i\xi} \left[-\frac{27i}{2} - \frac{1081\pi}{168} + \left(-\frac{1013i}{140} + \frac{137\pi}{42} \right) \nu + \left(\frac{27i}{4} + 9i\nu \right) \ln\left(\frac{3}{2}\right) \right] \right\} \right), \quad (76e)$$

$$H_{3\text{PN}}^{22} = \bar{x}^3 \left(\frac{27027409}{646800} + \frac{428i\pi}{105} + \frac{2\pi^2}{3} - \frac{856\gamma_E}{105} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^3}{99792} - \frac{1712 \ln(2)}{105} - \frac{428 \ln(\bar{x})}{105} + \bar{e} \left\{ e^{-i\xi} \left[\frac{219775769}{1663200} + \frac{749i\pi}{60} + \frac{49\pi^2}{24} - \frac{749\gamma_E}{30} + \left(-\frac{121717}{20790} - \frac{41\pi^2}{192} \right) \nu - \frac{86531\nu^2}{8316} - \frac{33331\nu^3}{399168} + \left(-\frac{2889}{70} + \frac{81i\pi}{2} \right) \ln\left(\frac{3}{2}\right) - \frac{81}{2} \ln^2\left(\frac{3}{2}\right) - \frac{749 \ln(2)}{15} - \frac{749 \ln(\bar{x})}{60} \right] + e^{i\xi} \left[\frac{55608313}{1058400} + \frac{3103i\pi}{420} + \frac{29\pi^2}{24} - \frac{3103\gamma_E}{210} + \left(-\frac{199855}{3024} + \frac{41\pi^2}{48} \right) \nu - \frac{9967\nu^2}{1008} + \frac{35579\nu^3}{36288} + \left(-\frac{6527}{210} + \frac{3i\pi}{2} \right) \ln(2) + \frac{3 \ln^2(2)}{2} - \frac{3103 \ln(\bar{x})}{420} \right] \right\} \right). \quad (76f)$$

For completeness all equations relating the different angles \bar{l} , $\bar{\lambda}$, ξ and ψ are listed in Appendix B.

E. Quasi-Circular limit

We now check our results against those in Ref. [51] in the quasi-circular limit. Note that the eccentricity is not a gauge-independent quantity and one thus has to be careful when talking about the circular limit. For a thorough discussion on different eccentricity parameters and discrepancies between them we refer to Refs. [65, 66].

Normally, one uses the orbital averaged description for the evolution of x and e , where one finds that the evolution equations (33) drive the eccentricity to zero during the inspiral. When introducing post-adiabatic corrections, this will not be true anymore, as the eccentricity is split into a orbital averaged part \bar{e} and a periodic oscillatory part \tilde{e} . The orbital averaged part \bar{e} will still follow the same evolution equations (33) and thus be driven to zero, but the periodic variations \tilde{e} will generally grow larger as the binary inspirals. As discussed in Ref. [66], the orbital averaged description also breaks down in the late inspiral, failing to capture a secular growth in the

eccentricity observed when directly integrating the two-body equations of motion.

In our case, it is reasonable to consider the circular limit as the limit where $\bar{x} \rightarrow x$ and $\bar{e} \rightarrow 0$, with x being the standard circular frequency parameter. Then, the evolution equations (33) reduce to the usual circular evolution equation

$$\dot{x} = \frac{64c^3\nu}{5Gm} x^5 + \mathcal{O}(x^6). \quad (77)$$

In this limit our redefined phase ψ reduces to

$$\psi|_{\bar{e}=0} = \phi - 3 \left(1 - \frac{\nu x}{2} \right) x^{3/2} \ln\left(\frac{x}{x'_0}\right), \quad (78)$$

which matches exactly the phase ψ used in Ref. [51]. We can thus directly compare our results to the circular limit by setting $\bar{e} = 0$ and $\bar{x}|_{\bar{e}=0} = x$. We find, e.g., for the h^{22} mode

$$h^{22} = \frac{8Gm\nu}{c^2R} x \sqrt{\frac{\pi}{5}} e^{-2i\psi} H^{22}, \quad (79)$$

$$\begin{aligned}
H^{22} = & 1 + x \left(-\frac{107}{42} + \frac{55\nu}{42} \right) + 2\pi x^{3/2} + x^2 \left(-\frac{2173}{1512} - \frac{1069\nu}{216} + \frac{2047\nu^2}{1512} \right) + x^{5/2} \left(-\frac{107\pi}{21} + \left(-24i + \frac{34\pi}{21} \right) \nu \right) \\
& + x^3 \left(\frac{27027409}{646800} + \frac{428i\pi}{105} + \frac{2\pi^2}{3} - \frac{856\gamma_E}{105} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96} \right) \nu - \frac{20261\nu^2}{2772} + \frac{114635\nu^2}{99792} \right. \\
& \left. - \frac{1712}{105} \ln(2) - \frac{428}{105} \ln(x) \right). \tag{80}
\end{aligned}$$

This matches Eq. (9.4a) of Ref. [51]. Similarly, we can compare the other modes and find perfect agreement in all of them.

VI. CONCLUSION

In this work, we computed the tail contributions to the 3PN-accurate gravitational waveform from nonspinning compact binaries on eccentric orbits. This extends the work on instantaneous contributions in Ref. [49] and will be completed with the memory contributions in a follow-up paper [50]. We also include post-adiabatic corrections to the quasi-Keplerian parametrization when combining our tail results with the instantaneous ones, giving us the full waveform (neglecting memory) that can be compared to the circular one in the limit $e \rightarrow 0$. The tail contributions to the h^{22} mode are given at 3PN order and to $\mathcal{O}(e)$ in Eq. (48), the post-adiabatic corrections in Eq. (67). All other $h^{\ell m}$ modes up to $\ell = 5$ are listed in the supplemental *Mathematica* notebook [52]. To reiterate, all results are in MH coordinates, which differ from the SH coordinates at 3PN order.

Note that the instantaneous results in Ref. [49] can be applied to binary systems of arbitrary eccentricities, while the tail results presented here are calculated in a small eccentricity expansion. This is due to the complicated tail integrals over past history, which can only be analytically calculated when decomposing the integrand into harmonics of the orbital timescale using an eccentricity expansion. This means that our results are not applicable for large eccentricities $e \sim 1$, though they might give accurate results for moderate eccentricities $e \sim 0.4$ when combined with orbital evolution equations that are not expanded in eccentricity; see, e.g., Ref. [48].

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Appendix A: Integral relations

We calculate the average over one period as

$$\langle F \rangle = \frac{1}{T} \int_0^T F(t) dt = \frac{1}{2\pi} \int_0^{2\pi} F(l) dl = \frac{1}{2\pi} \int_0^{2\pi} F(u) \chi du, \tag{A1}$$

where $\chi = 1 - e \cos u$. Some helpful integration formulas are

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{\chi^{N+1}} = \frac{1}{(1-e^2)^{(N+1)/2}} P_N \left(\frac{1}{\sqrt{1-e^2}} \right), \tag{A2a}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \chi \ln(\chi) du = 1 - \sqrt{1-e^2} + \ln \left(\frac{1 + \sqrt{1-e^2}}{2} \right). \tag{A2b}$$

The zero-average primitive of a function $F(l)$ can be calculated as

$$\int F = \int_0^l F(l') dl' - \frac{1}{2\pi} \int_0^{2\pi} dl \int_0^l F(l') dl' = \int_0^l F(l') dl' - \frac{1}{2\pi} \int_0^{2\pi} (2\pi - l') F(l') dl'. \tag{A3}$$

Appendix B: Quasi-Keplerian relations

We list here again all equations relating the different angles \bar{l} , $\bar{\lambda}$, ξ , and ψ to the actual time coordinate t :

$$\xi = \bar{l} - \frac{3GM}{c^3} \bar{n} \ln\left(\frac{\bar{x}}{x'_0}\right), \quad (\text{B1a})$$

$$\bar{\lambda}_\xi = \bar{\lambda} - \frac{3GM}{c^3} (1 + \bar{k}) \bar{n} \ln\left(\frac{\bar{x}}{x'_0}\right) = \bar{\lambda} - 3 \left(1 - \frac{\nu \bar{x}}{2}\right) \bar{x}^{3/2} \ln\left(\frac{\bar{x}}{x'_0}\right), \quad (\text{B1b})$$

$$\begin{aligned} \psi &= \bar{\lambda}_\xi + \bar{W}_\xi + \tilde{\lambda}_\xi + (\tilde{v}_\xi - \tilde{l}_\xi) \\ &= \bar{\lambda}_\xi + 2\bar{e} \sin(\xi) + \frac{5}{4} \bar{e}^2 \sin(2\xi) + \bar{x} \left((10 - \nu) \bar{e} \sin(\xi) + \left(\frac{31}{4} - \nu\right) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^2 \left(\frac{1}{12} (624 - 235\nu + \nu^2) \bar{e} \sin(\xi) + \frac{1}{24} (969 - 326\nu + 2\nu^2) \bar{e}^2 \sin(2\xi) \right) \\ &\quad - \bar{x}^{5/2} \nu \left(\frac{128}{5} + \frac{888}{5} \bar{e} \cos(\xi) + \frac{1}{45} \bar{e}^2 (10728 + 8935 \cos(2\xi)) \right) \\ &\quad + \bar{x}^3 \left(\left(292 + \left(-\frac{420131}{840} + \frac{287\pi^2}{32} \right) \nu + \frac{521\nu^2}{24} + \frac{\nu^3}{24} \right) \bar{e} \sin(\xi) \right. \\ &\quad \left. + \frac{1}{168} (35868 + (-55548 + 861\pi^2) \nu + 1925\nu^2 + 28\nu^3) \bar{e}^2 \sin(2\xi) \right), \end{aligned} \quad (\text{B1c})$$

where

$$\xi = \bar{u}_\xi - \bar{e} \sin \bar{u}_\xi + (\bar{g}_{4t} + \bar{g}_{6t}) (\bar{v}_\xi - \bar{u}_\xi) + (\bar{f}_{4t} + \bar{f}_{6t}) \sin \bar{v}_\xi + \bar{i}_{6t} \sin(2\bar{v}_\xi) + \bar{h}_{6t} \sin(3\bar{v}_\xi), \quad (\text{B2a})$$

$$\begin{aligned} \bar{u}_\xi &= \xi + \bar{e} \sin(\xi) + \frac{1}{2} \bar{e}^2 \sin(2\xi) + \bar{x}^2 \left(\frac{1}{8} (-60 + 9\nu + \nu^2) \bar{e} \sin(\xi) + \frac{3}{16} (-5 + \nu)(10 + \nu) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^3 \left(\left(-55 + \frac{104593\nu}{1680} + \frac{3\nu^2}{4} + \frac{\nu^3}{24} \right) \bar{e} \sin(\xi) \right. \\ &\quad \left. + \left(-\frac{315}{4} + \left(\frac{229219}{3360} + \frac{41\pi^2}{256} \right) \nu + \frac{53\nu^2}{8} - \frac{3\nu^3}{16} \right) \bar{e}^2 \sin(2\xi) \right), \end{aligned} \quad (\text{B2b})$$

$$\begin{aligned} \bar{v}_\xi &= 2 \arctan \left[\left(\frac{1 + \bar{e}_\phi}{1 - \bar{e}_\phi} \right)^{1/2} \tan \frac{\bar{u}_\xi}{2} \right] \\ &= \xi + 2\bar{e} \sin(\xi) + \frac{5}{4} \bar{e}^2 \sin(2\xi) + \bar{x} \left((4 - \nu) \bar{e} \sin(\xi) + (4 - \nu) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^2 \left(\frac{1}{12} (156 - 31\nu + \nu^2) \bar{e} \sin(\xi) + \frac{1}{24} (273 - 101\nu + 11\nu^2) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^3 \left(\left(64 + \left(-\frac{106181}{840} + \frac{41\pi^2}{32} \right) \nu + \frac{11\nu^2}{24} + \frac{\nu^3}{24} \right) \bar{e} \sin(\xi) \right. \\ &\quad \left. + \left(\frac{155}{4} + \left(-\frac{169649}{1680} + \frac{369\pi^2}{256} \right) \nu + \frac{49\nu^2}{6} - \frac{5\nu^3}{24} \right) \bar{e}^2 \sin(2\xi) \right), \end{aligned} \quad (\text{B2c})$$

$$\begin{aligned} \bar{W}_\xi &= (1 + \bar{k}) (\bar{v}_\xi - \xi) + (\bar{f}_{4\phi} + \bar{f}_{6\phi}) \sin(2\bar{v}_\xi) + (\bar{g}_{4\phi} + \bar{g}_{6\phi}) \sin(3\bar{v}_\xi) + \bar{i}_{6\phi} \sin(4\bar{v}_\xi) + \bar{h}_{6\phi} \sin(5\bar{v}_\xi) \\ &= 2\bar{e} \sin(\xi) + \frac{5}{4} \bar{e}^2 \sin(2\xi) + \bar{x} \left((10 - \nu) \bar{e} \sin(\xi) + \left(\frac{31}{4} - \nu\right) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^2 \left(\frac{1}{12} (624 - 235\nu + \nu^2) \bar{e} \sin(\xi) + \frac{1}{24} (969 - 326\nu + 2\nu^2) \bar{e}^2 \sin(2\xi) \right) \\ &\quad + \bar{x}^3 \left(\left(292 + \left(-\frac{420131}{840} + \frac{287\pi^2}{32} \right) \nu + \frac{521\nu^2}{24} + \frac{\nu^3}{24} \right) \bar{e} \sin(\xi) \right. \\ &\quad \left. + \frac{1}{168} (35868 + (-55548 + 861\pi^2) \nu + 1925\nu^2 + 28\nu^3) \bar{e}^2 \sin(2\xi) \right), \end{aligned} \quad (\text{B2d})$$

$$\tilde{\lambda}_\xi = -\nu \bar{x}^{5/2} \left[\frac{296}{3} \bar{e} \cos(\xi) + \frac{199}{5} \bar{e}^2 \cos(2\xi) \right] + \mathcal{O}(\bar{x}^{7/2}), \quad (\text{B2e})$$

$$\begin{aligned} \tilde{l}_\xi = & -\nu\bar{x}^{5/2} \left[\frac{64}{5\bar{e}} \cos(\xi) + \frac{352}{15} \cos(2\xi) + \bar{e} \left(\frac{1654}{15} \cos(\xi) + \frac{358}{9} \cos(3\xi) \right) \right. \\ & \left. + \bar{e}^2 \left(\frac{694}{15} \cos(2\xi) + \frac{1289}{20} \cos(4\xi) \right) \right] + \mathcal{O}(\bar{x}^{7/2}), \end{aligned} \quad (\text{B2f})$$

$$\begin{aligned} \tilde{u}_\xi = & -\nu\bar{x}^{5/2} \left[\frac{64}{5} + \frac{64}{5\bar{e}} \cos(\xi) + \frac{352}{15} \cos(2\xi) + \bar{e} \left(\frac{2198}{15} \cos(\xi) + \frac{358}{9} \cos(3\xi) \right) \right. \\ & \left. + \bar{e}^2 \left(\frac{340}{3} + \frac{5366}{45} \cos(2\xi) + \frac{1289}{20} \cos(4\xi) \right) \right] + \mathcal{O}(\bar{x}^{7/2}), \end{aligned} \quad (\text{B2g})$$

$$\tilde{e}_\xi = -\nu\bar{x}^{5/2} \left[\frac{64}{5} \sin(\xi) + \frac{352}{15} \bar{e} \sin(2\xi) + \bar{e}^2 \left(\frac{1138}{15} \sin(\xi) + \frac{358}{9} \sin(3\xi) \right) \right] + \mathcal{O}(\bar{x}^{7/2}), \quad (\text{B2h})$$

$$\begin{aligned} \tilde{v}_\xi = & \frac{\sqrt{1-\bar{e}^2}}{1-\bar{e}\cos\bar{u}_\xi} \tilde{u}_\xi + \frac{\sin\bar{u}_\xi}{\sqrt{1-\bar{e}^2}(1-\bar{e}\cos\bar{u}_\xi)} \tilde{e} \\ = & -\nu\bar{x}^{5/2} \left[\frac{128}{5} + \frac{64}{5\bar{e}} \cos(\xi) + \frac{352}{15} \cos(2\xi) + \bar{e} \left(\frac{946}{5} \cos(\xi) + \frac{358}{9} \cos(3\xi) \right) \right. \\ & \left. + \bar{e}^2 \left(\frac{1192}{5} + \frac{9226}{45} \cos(2\xi) + \frac{1289}{20} \cos(4\xi) \right) \right] + \mathcal{O}(\bar{x}^{7/2}), \end{aligned} \quad (\text{B2i})$$

where \bar{l} and $\bar{\lambda}$ are given by Eqs. (36) and where $\bar{f}_{4\phi}$, $\bar{f}_{6\phi}$, $\bar{g}_{4\phi}$, $\bar{g}_{6\phi}$, $\bar{i}_{6\phi}$, $\bar{h}_{6\phi}$, \bar{g}_{4t} , \bar{g}_{6t} , \bar{f}_{4t} , \bar{f}_{6t} , \bar{i}_{6t} , and \bar{h}_{6t} are the slowly evolving orbital functions given in Ref. [31], with x and e being replaced by \bar{x} and \bar{e} .

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