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Eindhoven University  
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## Department of Mathematics and Computing Sciences

Memorandum COSOR 99-14

### **Families of birth-death processes with similar time-dependent behaviour**

R.B. Lenin  
P.R. Parthasarathy  
W.R.W. Scheinhardt  
E.A. van Doorn

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The Netherlands

# Families of birth-death processes with similar time-dependent behaviour

R.B. Lenin\*, P.R. Parthasarathy†, W.R.W. Scheinhardt‡, E.A. van Doorn\*

\* Faculty of Mathematical Sciences  
University of Twente  
P.O. Box 217, 7500 AE Enschede, The Netherlands  
E-mail: {r.b.lenin, e.a.vandoorn}@math.utwente.nl

† Department of Mathematics  
Indian Institute of Technology, Madras  
Chennai 600 036, India  
E-mail: prp@pallava.iitm.ernet.in

‡ Department of Mathematics and Computing Science  
Eindhoven University of Technology  
P.O. Box 513, 5600 MB Eindhoven, The Netherlands  
E-mail: werner@win.tue.nl

**Abstract:** We consider birth-death processes taking values in  $\mathcal{N} \equiv \{0, 1, \dots\}$ , but allow the death rate in state 0 to be positive, so that escape from  $\mathcal{N}$  is possible. Two such processes with transition functions  $\{p_{ij}(t)\}$  and  $\{\tilde{p}_{ij}(t)\}$  are said to be *similar* if, for all  $i, j \in \mathcal{N}$ , there are constants  $c_{ij}$  such that  $\tilde{p}_{ij}(t) = c_{ij}p_{ij}(t)$  for all  $t \geq 0$ . We determine conditions on the birth and death rates of a birth-death process for the process to be a member of a family of similar processes, and we identify the members of such a family.

*Keywords and phrases:* chain sequence, transient behaviour, transition function

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# 1 Introduction

In this paper a *birth-death process*  $\mathcal{X} \equiv \{X(t), t \geq 0\}$ , say, will always be a process taking values in  $\mathcal{N} \equiv \{0, 1, \dots\}$  with birth rates  $\{\lambda_n, n \in \mathcal{N}\}$  and death rates  $\{\mu_n, n \in \mathcal{N}\}$ , all strictly positive except  $\mu_0$ , which might be equal to 0. When  $\mu_0 = 0$  the process is irreducible, but when  $\mu_0 > 0$  the process may escape from  $\mathcal{N}$ , via 0, to an absorbing state  $-1$ .

The  $q$ -matrix of transition rates of  $\mathcal{X}$ , restricted to the states in  $\mathcal{N}$ , will be denoted by  $Q \equiv (q_{ij}, i, j \in \mathcal{N})$ , so that

$$Q = \begin{pmatrix} -(\lambda_0 + \mu_0) & \lambda_0 & 0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (1)$$

We shall assume that the transition functions

$$p_{ij}(t) \equiv \Pr\{X(t) = j \mid X(0) = i\}, \quad i, j \in \mathcal{N}, t \geq 0,$$

constitute the unique  $Q$ -function (i.e., set of transition functions having  $Q$  as  $q$ -matrix) satisfying both the Kolmogorov backward equations

$$p'_{ij}(t) = \sum_{k \in \mathcal{N}} q_{ik} p_{kj}(t), \quad i, j \in \mathcal{N}, t \geq 0, \quad (2)$$

and forward equations

$$p'_{ij}(t) = \sum_{k \in \mathcal{N}} p_{ik}(t) q_{kj}, \quad i, j \in \mathcal{N}, t \geq 0. \quad (3)$$

Equivalently, see Anderson [2, pp. 262-263], we assume that the *potential coefficients*

$$\pi_0 \equiv 1 \quad \text{and} \quad \pi_n \equiv \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}, \quad n = 1, 2, \dots,$$

satisfy the condition

$$\sum_{n=0}^{\infty} (\pi_n + (\lambda_n \pi_n)^{-1}) = \infty. \quad (4)$$

The problem of solving the Kolmogorov equations explicitly for a specific set of birth and death rates has been approached in the literature in many different ways (of which the method of Karlin and McGregor [8] involving orthogonal polynomials has probably been the most successful). Our approach to the problem of finding solutions to the Kolmogorov equations will be to investigate whether a known solution to the equations for one birth-death process can help us identify the transition functions of other birth-death processes, by establishing whether the given process belongs to a class of birth-death processes whose transition functions behave – in a sense to be defined – “similarly”.

Thus consider, besides  $\mathcal{X}$ , another birth-death process  $\tilde{\mathcal{X}}$ , determined by birth rates  $\{\tilde{\lambda}_n, n \in \mathcal{N}\}$  and death rates  $\{\tilde{\mu}_n, n \in \mathcal{N}\}$ , with potential coefficients  $\{\tilde{\pi}_n\}$  and transition functions  $\{\tilde{p}_{ij}(t)\}$ .

**Definition** *The birth-death processes  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  are said to be similar if the transition functions of  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  differ only by a constant factor, that is, if there are constants  $c_{ij}$ ,  $i, j \in \mathcal{N}$ , such that*

$$\tilde{p}_{ij}(t) = c_{ij}p_{ij}(t), \quad i, j \in \mathcal{N}, t \geq 0. \quad (5)$$

This concept of similarity is seemingly more general than that of Di Crescenzo [6], who calls the birth-death processes  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  similar if there are constants  $\nu_n$ ,  $n \in \mathcal{N}$ , such that

$$\tilde{p}_{ij}(t) = \frac{\nu_j}{\nu_i}p_{ij}(t), \quad i, j \in \mathcal{N}, t \geq 0. \quad (6)$$

However, the next theorem shows that the two definitions actually amount to the same thing, and that birth and death parameters and transition functions of two similar birth-death processes must be related in a very specific way. The proof of the theorem has been relegated to Section 5.

**Theorem 1** *If  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  are similar birth-death processes, then their birth and death parameters are related as*

$$\tilde{\lambda}_n + \tilde{\mu}_n = \lambda_n + \mu_n, \quad \tilde{\lambda}_n \tilde{\mu}_{n+1} = \lambda_n \mu_{n+1}, \quad n \in \mathcal{N}, \quad (7)$$

while their transition functions satisfy

$$\tilde{p}_{ij}(t) = \sqrt{\frac{\pi_i \tilde{\pi}_j}{\tilde{\pi}_i \pi_j}} p_{ij}(t), \quad i, j \in \mathcal{N}, t \geq 0. \quad (8)$$

One might suspect that, conversely, a birth-death process  $\tilde{\mathcal{X}}$  is similar to a given birth-death process  $\mathcal{X}$  if the birth and death rates of  $\tilde{\mathcal{X}}$  and  $\mathcal{X}$  are related as in (7), but it is doubtful whether this is true in general unless one imposes an additional condition, as in the next theorem.

**Theorem 2** *Let  $\mathcal{X}$  be a birth-death process satisfying*

$$\sum_{n=1}^{\infty} \pi_n \left( \sum_{k=0}^{n-1} (\lambda_k \pi_k)^{-1} \right)^2 = \infty \quad \text{if } \mu_0 > 0, \quad (9)$$

and let  $\tilde{\mathcal{X}}$  be a birth-death process the rates of which are related to those of  $\mathcal{X}$  as in (7). Then the processes  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  are similar.

Again, the proof of this theorem has been relegated to Section 5. In what follows we will assume the validity of condition (9), which is stronger than condition (4) when  $\mu_0 > 0$ .

The questions to be answered now are under which conditions the phenomenon of similarity occurs and, if it occurs, whether one can identify all the birth-death processes which are similar to a given process. These problems were partly resolved by Di Crescenzo [6] who restricted his analysis to the case  $\mu_0 > 0$  (see Di Crescenzo [7] for related results on *bilateral* birth-death processes). In the next section we obtain a complete solution, in which the crucial step is the application of a result of Chihara's [5] on *chain sequences*. Some examples are given in Section 3 and in Section 4 we briefly discuss similarity for birth-death processes on a finite state space. As announced, Section 5 contains the proofs of Theorems 1 and 2.

## 2 Families of birth-death processes

We will assume that the transition functions  $\{p_{ij}(t), i, j \in \mathcal{N}\}$  of the birth-death process  $\mathcal{X} \equiv \{X(t), t \geq 0\}$ , with birth rates  $\{\lambda_n, n \in \mathcal{N}\}$  and death rates  $\{\mu_n, n \in \mathcal{N}\}$  satisfying (4) and (9), are known. We let

$$\alpha_n \equiv \lambda_n + \mu_n, \quad \beta_{n+1} \equiv \lambda_n \mu_{n+1}, \quad n \in \mathcal{N}. \quad (10)$$

The question posed at the end of the previous section may now be phrased as follows. Can we identify, besides  $\{\lambda_n, n \in \mathcal{N}\}$  and  $\{\mu_n, n \in \mathcal{N}\}$ , all other sets of birth rates  $\{\tilde{\lambda}_n, n \in \mathcal{N}\}$  and death rates  $\{\tilde{\mu}_n, n \in \mathcal{N}\}$  such that

$$\tilde{\lambda}_n + \tilde{\mu}_n = \alpha_n, \quad \tilde{\lambda}_n \tilde{\mu}_{n+1} = \beta_{n+1}, \quad n \in \mathcal{N}. \quad (11)$$

This problem can be transformed into a problem involving *chain sequences*. Let us recall the definition and some basic results, see Chihara [3, Section III.5] and [5] for proofs and developments.

A sequence  $\{a_n\}_{n=1}^{\infty}$  is a *chain sequence* if there exists a second sequence  $\{g_n\}_{n=0}^{\infty}$  such that

$$\begin{aligned} (i) \quad & 0 \leq g_0 < 1, \quad 0 < g_n < 1, \quad n = 1, 2, \dots, \\ (ii) \quad & a_n = (1 - g_{n-1})g_n, \quad n = 1, 2, \dots \end{aligned} \quad (12)$$

The sequence  $\{g_n\}$  is called a *parameter sequence* for  $\{a_n\}$ . If both  $\{g_n\}$  and  $\{h_n\}$  are parameter sequences for  $\{a_n\}$ , then

$$g_n < h_n, \quad n = 1, 2, \dots, \quad \text{if and only if} \quad g_0 < h_0. \quad (13)$$

Every chain sequence  $\{a_n\}$  has a *minimal* parameter sequence  $\{m_n\}$  uniquely determined by the condition  $m_0 = 0$ , and it has a *maximal* parameter sequence  $\{M_n\}$  characterized by the fact that  $M_0 > g_0$  for any other parameter sequence  $\{g_n\}$ . For every  $x$ ,  $0 \leq x \leq M_0$ , there is a unique parameter sequence  $\{g_n\}$  for  $\{a_n\}$  such that  $g_0 = x$ .

Returning to the context of the birth-death process  $\mathcal{X}$ , we let

$$\gamma_n \equiv \frac{\beta_n}{\alpha_{n-1}\alpha_n}, \quad n = 1, 2, \dots,$$

and observe that  $\{\gamma_n\}_{n=1}^{\infty}$  is a chain sequence, since we can write

$$\gamma_n = \left(1 - \frac{\mu_{n-1}}{\lambda_{n-1} + \mu_{n-1}}\right) \frac{\mu_n}{\lambda_n + \mu_n},$$

so that  $\{\mu_n/(\lambda_n + \mu_n)\}$  constitutes a parameter sequence for the chain sequence  $\{\gamma_n\}$ . Our task is now to find all parameter sequences for the chain sequence  $\{\gamma_n\}$ , since there is a one-to-one correspondence between parameter sequences for  $\{\gamma_n\}$  and sets of birth and death rates satisfying (11). Indeed, for every parameter sequence  $g \equiv \{g_n\}$  we can construct the corresponding birth rates  $\{\lambda_n^{(g)}\}$  and death rates  $\{\mu_n^{(g)}\}$  by letting

$$\lambda_n^{(g)} = \alpha_n(1 - g_n), \quad \mu_n^{(g)} = \alpha_n g_n, \quad n \in \mathcal{N}. \quad (14)$$

The problem of identifying all parameter sequences for a chain sequence for which one parameter sequence is known, has been solved completely by Chihara [5]. (Note that there is a slip in [5, Eq. (3.8)]). In our setting the solution may be formulated as follows.

**Case (i):**  $\mu_0 = 0$ . Let

$$S_{-1} \equiv 0, \quad S_n \equiv \lambda_0 \sum_{k=0}^n (\lambda_k \pi_k)^{-1}, \quad n \in \mathcal{N}, \quad \text{and} \quad S \equiv \lim_{n \rightarrow \infty} S_n \quad (15)$$

(possibly  $S = \infty$ ). Then all parameter sequences for  $\{\gamma_n\}$  are given by  $\{g_n(x)\}$ ,  $0 \leq x \leq 1/S$ , where

$$g_0(x) = x, \quad g_n(x) = \frac{\mu_n}{\lambda_n + \mu_n} \frac{1 - xS_{n-2}}{1 - xS_{n-1}}, \quad n \geq 1. \quad (16)$$

It follows in particular that  $\{\mu_n/(\lambda_n + \mu_n)\}$  is the only parameter sequence for  $\{\gamma_n\}$  if  $S = \infty$ .

**Case (ii):**  $\mu_0 > 0$ . Let

$$T_{-1} \equiv 0, \quad T_n \equiv \mu_0 \sum_{k=0}^n (\mu_k \pi_k)^{-1}, \quad n \in \mathcal{N}, \quad \text{and} \quad T \equiv \lim_{n \rightarrow \infty} T_n \quad (17)$$

(possibly  $T = \infty$ ). Then all parameter sequences for  $\{\gamma_n\}$  are given by  $\{g_n(x)\}$ ,  $-\infty \leq x \leq 1/T$ , where

$$g_n(x) = \frac{\mu_n}{\lambda_n + \mu_n} \frac{1 - xT_{n-1}}{1 - xT_n}, \quad n \in \mathcal{N}. \quad (18)$$

It is interesting to observe that the maximal parameter sequence is obtained for  $x = 1/T$ . So the sequence  $\{\mu_n/(\lambda_n + \mu_n)\}$  is the maximal parameter sequence for  $\{\gamma_n\}$  if  $T = \infty$ .

Translating these results in terms of birth and death rates we obtain the following theorem.

**Theorem 3** A birth-death process  $\mathcal{X}$  with birth rates  $\{\lambda_n, n \in \mathcal{N}\}$  and death rates  $\{\mu_n, n \in \mathcal{N}\}$  is not similar to any other birth-death process if and only if

$$\mu_0 = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} = \infty. \quad (19)$$

In the opposite case the process is similar to any member of an infinite, one-parameter family of birth-death processes. The sets of birth rates and death rates for the members of this family are given by  $\{\lambda_n(x), n \in \mathcal{N}\}$  and  $\{\mu_n(x), n \in \mathcal{N}\}$ , respectively, where  $\lambda_n(x) = \lambda_n + \mu_n - \mu_n(x)$ , and

$$\mu_0(x) \equiv \lambda_0 x, \quad \mu_n(x) \equiv \mu_n \frac{1 - x S_{n-2}}{1 - x S_{n-1}}, \quad n \geq 1, \quad 0 \leq x \leq 1/S, \quad (20)$$

if  $\mu_0 = 0$ , and

$$\mu_n(x) \equiv \mu_n \frac{1 - x T_{n-1}}{1 - x T_n}, \quad n \geq 0, \quad -\infty \leq x \leq 1/T, \quad (21)$$

if  $\mu_0 > 0$ . The quantities  $S_n$ ,  $S$ ,  $T_n$  and  $T$  are defined in (15) and (17).

**Remarks:** The condition (19) for nonsimilarity as well as the upper bound for  $\mu_0(x)$  in (20) were obtained previously by Karlin and McGregor [8, Lemma 1]. The condition (19) is actually equivalent to the birth-death process  $\mathcal{X}$  being recurrent, see, e.g., Karlin and McGregor [9]. Within the setting of similarity definition (6) the result for  $\mu_0 > 0$  was obtained earlier by Di Crescenzo in [6].

For any  $x$  in the intervals given in (20) or (21) the corresponding birth and death rates are the rates of a birth-death process  $\tilde{\mathcal{X}}$ , say, which is similar to  $\mathcal{X}$  and whose transition functions can be expressed in the transition functions of  $\mathcal{X}$  as indicated in (8). It is interesting to observe that the range of possible values for  $\tilde{\mu}_0 = \mu_0(x)$ , the death rate in state 0 of  $\tilde{\mathcal{X}}$ , can be represented as

$$0 \leq \tilde{\mu}_0 \leq \mu_0 + \left\{ \sum_{n=0}^{\infty} (\lambda_n \pi_n)^{-1} \right\}^{-1}, \quad (22)$$

both when  $\mu_0 = 0$  and when  $\mu_0 > 0$ . This can easily be verified from (20) and (21) by noting that  $\lambda_n \pi_n = \mu_{n+1} \pi_{n+1}$ . The minimal value  $\tilde{\mu}_0 = 0$  is obtained by letting  $x = 0$  when  $\mu_0 = 0$ , and  $x = -\infty$  when  $\mu_0 > 0$ , while the maximal value of  $\tilde{\mu}_0$  is obtained by choosing  $x = 1/S$  when  $\mu_0 = 0$ , and  $x = 1/T$  when  $\mu_0 > 0$ .

## 3 Examples

### 3.1 Example 1: Constant rates

We first want to make some additional remarks on the example already discussed by Di Crescenzo [6]. So let  $\mathcal{X} \equiv \{X(t), t \geq 0\}$  be the birth-death process with constant

birth rates  $\lambda_n \equiv \lambda$  and constant death rates  $\mu_n \equiv \mu$ ,  $n \in \mathcal{N}$ . Since  $\mu_0 > 0$  this process is transient. Ledermann and Reuter [11, formula (4.13)] were the first to show that its transition functions can be represented by

$$p_{ij}(t) = \left( \sqrt{\frac{\lambda}{\mu}} \right)^{j-i} e^{-(\lambda+\mu)t} \left\{ I_{j-i}(2t\sqrt{\lambda\mu}) - I_{j+i+2}(2t\sqrt{\lambda\mu}) \right\}, \quad t \geq 0, \quad (23)$$

where  $I_n(\cdot)$  is the  $n$ th modified Bessel function, see, e.g., Abramowitz and Stegun [1, Sect. 9.6]. It is readily seen that the quantities  $T_n$  of (17) are now given by

$$T_n = \frac{\lambda^{n+1} - \mu^{n+1}}{(\lambda - \mu)\lambda^n}, \quad n \in \mathcal{N}, \quad (24)$$

while

$$T = \begin{cases} \frac{\lambda}{\lambda - \mu} & \text{if } \lambda > \mu \\ \infty & \text{if } \lambda \leq \mu. \end{cases} \quad (25)$$

Hence, we conclude from Theorem 3 that for each  $x$  in the interval  $-\infty \leq x \leq 1/T$ , the process  $\mathcal{X}^{(x)}$  with rates

$$\lambda_n(x) = \lambda + \mu - \mu_n(x), \quad \mu_n(x) = \lambda\mu \frac{(\lambda - \mu)\lambda^{n-1} - x(\lambda^n - \mu^n)}{(\lambda - \mu)\lambda^n - x(\lambda^{n+1} - \mu^{n+1})}, \quad n \in \mathcal{N}, \quad (26)$$

is similar to  $\mathcal{X}$ , in accordance with Di Crescenzo [6]. In Figure 1 we show graphs of  $\mu_n(x)$  for  $n = 0, 1, \dots, 5$  in the case  $\lambda = 2$  and  $\mu = 1$ .

We will look more closely at the extremal cases  $x = 1/T$  and  $x = -\infty$ . First, if  $\lambda \leq \mu$ , then  $T = \infty$  and hence  $\mathcal{X}^{(1/T)} = \mathcal{X}^{(0)} = \mathcal{X}$ , the process we started with. But if  $\lambda > \mu$ , then we have

$$\lambda_n(1/T) = \mu, \quad \mu_n(1/T) = \lambda, \quad n \in \mathcal{N}. \quad (27)$$

So the process  $\mathcal{X}^{(1/T)}$  is the process we obtain by interchanging  $\lambda$  and  $\mu$ . Since the time-dependent factors of (23) are symmetric in  $\lambda$  and  $\mu$ , it is evident that  $\mathcal{X}^{(1/T)}$  is similar to  $\mathcal{X}$ .

Taking  $x = -\infty$  we readily obtain

$$\lambda_n(-\infty) = \frac{\lambda^{n+2} - \mu^{n+2}}{\lambda^{n+1} - \mu^{n+1}}, \quad \mu_n(-\infty) = \frac{\lambda\mu(\lambda^n - \mu^n)}{\lambda^{n+1} - \mu^{n+1}}, \quad n \in \mathcal{N}, \quad (28)$$

so that  $\mu_0(-\infty) = 0$  (as it should be). A little algebra subsequently reveals that the associated transition functions  $p_{ij}^{(-\infty)}(t)$  are given by

$$p_{ij}^{(-\infty)}(t) = \lambda^{i-j} \frac{\lambda^{j+1} - \mu^{j+1}}{\lambda^{i+1} - \mu^{i+1}} p_{ij}(t), \quad t \geq 0. \quad (29)$$

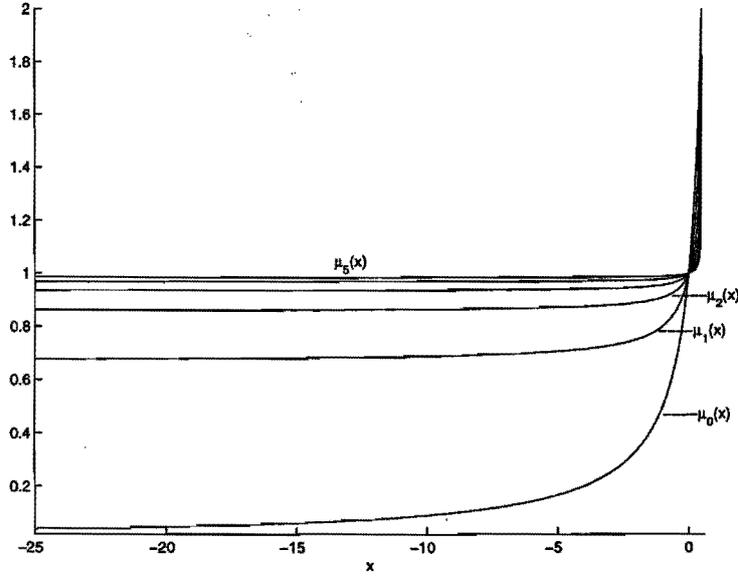


Figure 1:  $\mu_n(x)$ ,  $n = 0, 1, \dots, 5$ , for  $\lambda = 2$  and  $\mu = 1$

### 3.2 Example 2: Linear rates

We next look at the birth-death process  $\mathcal{X} \equiv \{X(t), t \geq 0\}$  with birth rates  $\lambda_n \equiv n + a$ , for some  $a > 0$ , and death rates  $\mu_n \equiv n$ ,  $n \in \mathcal{N}$ , so that  $\mu_0 = 0$ . The transition functions for  $\mathcal{X}$  can be represented by

$$p_{ij}(t) = \frac{i!}{\Gamma(a+i)} \int_0^\infty e^{-x(t+1)} L_i^{(a-1)}(x) L_j^{(a-1)}(x) x^{a-1} dx, \quad t \geq 0, \quad (30)$$

where  $\Gamma(\cdot)$  denotes the *gamma function*

$$\Gamma(z) \equiv \int_0^\infty e^{-x} x^{z-1} dx,$$

and  $L_n^{(\alpha)}(x)$  is the  $n$ th *Laguerre polynomial*, normalized such that  $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$ , see Karlin and McGregor [10] where  $\mathcal{X}$  is said to be of *type B*. The integrals (30) can easily be evaluated explicitly, yielding

$$p_{ij}(t) = \frac{t^i}{(1+t)^{a+i}} \sum_{k=0}^i \binom{i}{k} \left(\frac{1-t}{t}\right)^k \left(\frac{t}{1+t}\right)^{j-k} \frac{(a+i)_{j-k}}{(j-k)!}, \quad t \geq 0, \quad (31)$$

for  $i \leq j$ , see [10, p. 656], while it is obvious from (30) that

$$p_{ij}(t) = \frac{i!}{j!} \frac{(a)_j}{(a)_i} p_{ji}(t), \quad t \geq 0. \quad (32)$$

Here  $(x)_n \equiv \Gamma(x+n)/\Gamma(x)$ . It is not difficult to see that condition (19) for nonsimilarity is satisfied if and only if  $a \leq 1$ , so in what follows we will assume  $a > 1$ .

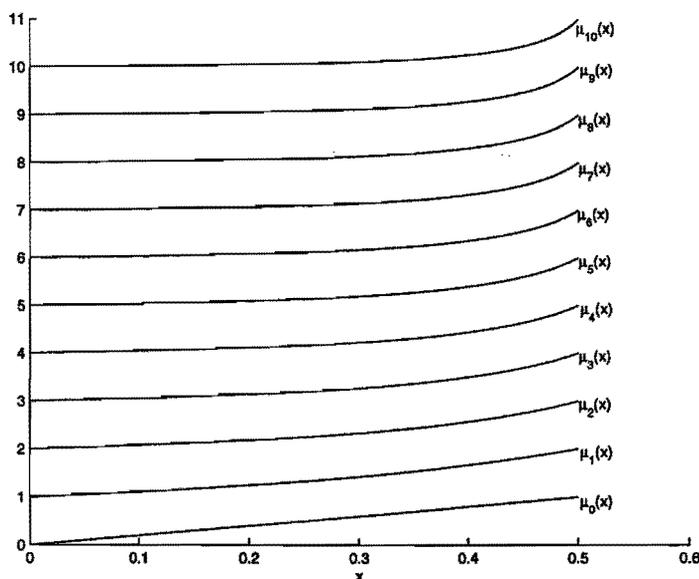


Figure 2:  $\mu_n(x)$ ,  $n = 0, 1, \dots, 10$ , for  $a = 2$

The quantities  $S_n$  of (15) are now given by

$$S_n = \sum_{k=0}^n \frac{k!}{(a+1)_k} = \frac{1}{a-1} \left( a - \frac{(n+1)!}{(a+1)_n} \right), \quad n \in \mathcal{N}, \quad (33)$$

as can easily be verified by induction. As a consequence,

$$S = \sum_{k=0}^{\infty} \frac{k!}{(a+1)_k} = \frac{a}{a-1}. \quad (34)$$

Hence, we conclude from Theorem 3 that for each  $x$  in the interval  $0 < x \leq 1 - 1/a$ , the process  $\mathcal{X}^{(x)}$  with rates

$$\mu_0(x) = ax, \quad \mu_n(x) = n(n+a-1) \frac{(a-1)_n - ax[(a)_{n-1} - (n-1)!]}{(a-1)_{n+1} - ax[(a)_n - n!]}, \quad n \geq 1, \quad (35)$$

and  $\lambda_n(x) = 2n + a - \mu_n(x)$ ,  $n \in \mathcal{N}$ , is similar to  $\mathcal{X} \equiv \mathcal{X}^{(0)}$ . In Figure 2 we show graphs of  $\mu_n(x)$  for  $n = 0, 1, 2, \dots, 10$  in the case  $a = 2$ .

Looking more closely at the extremal case  $x = 1 - 1/a$ , we have

$$\lambda_n(1 - 1/a) = n + 1, \quad \mu_n(1 - 1/a) = n + a - 1, \quad n \in \mathcal{N}. \quad (36)$$

Karlin and McGregor [10] have found explicit expressions for the transition functions of the process  $\mathcal{X}^{(1-1/a)}$  (which they call *of type A*), which are indeed similar to those of  $\mathcal{X} \equiv \mathcal{X}^{(0)}$  and given by

$$p_{ij}^{(1-1/a)}(t) = \frac{j!}{i!} \frac{(a)_i}{(a)_j} p_{ij}(t). \quad (37)$$

## 4 Birth-death processes with finite state space

Let us now consider a birth-death process  $\mathcal{X} \equiv \{X(t), t \geq 0\}$  taking values in the *finite* set  $\mathcal{N} \equiv \{0, 1, \dots, N\}$  with birth rates  $\{\lambda_n, n \in \mathcal{N}\}$  and death rates  $\{\mu_n, n \in \mathcal{N}\}$ , all strictly positive except  $\mu_0$  and  $\lambda_N$ , which may be equal to 0. When  $\mu_0 > 0$  the process may escape from  $\mathcal{N}$ , via 0, to an absorbing state  $-1$ , and when  $\lambda_N > 0$  the process may escape from  $\mathcal{N}$ , via  $N$ , to an absorbing state  $N + 1$ .

Again we can ask the question of whether there exist birth-death processes which are similar – in the sense of Section 1 – to  $\mathcal{X}$ . This problem may be analysed in a way which is similar to that of Section 2, but now involving *finite* chain sequences, that is, numerical sequences  $\{a_n\}_{n=1}^N$  for which there exists a sequence  $\{g_n\}_{n=0}^N$  – a *parameter sequence* for  $\{a_n\}$  – such that

$$\begin{aligned} (i) \quad & 0 \leq g_0 < 1, \quad 0 < g_n < 1, \quad n = 1, 2, \dots, N-1, \quad 0 < g_N \leq 1, \\ (ii) \quad & a_n = (1 - g_{n-1})g_n, \quad n = 1, 2, \dots, N. \end{aligned} \quad (38)$$

We will not give the details of the analysis leading to the following analogue of Theorem 3.

**Theorem 4** *A birth-death process  $\mathcal{X}$  with finite state space  $\mathcal{N} \equiv \{0, 1, \dots, N\}$ , birth rates  $\{\lambda_n, n \in \mathcal{N}\}$  and death rates  $\{\mu_n, n \in \mathcal{N}\}$  is not similar to any other birth-death process if and only if*

$$\mu_0 = \lambda_N = 0. \quad (39)$$

*In the opposite case the process is similar to an infinite, one-parameter family of birth-death processes. The sets of birth rates and death rates for the members of this family are given by  $\{\lambda_n(x), n \in \mathcal{N}\}$  and  $\{\mu_n(x), n \in \mathcal{N}\}$ , respectively, where  $\lambda_n(x) = \lambda_n + \mu_n - \mu_n(x)$ , and*

$$\mu_0(x) \equiv \lambda_0 x, \quad \mu_n(x) \equiv \mu_n \frac{1 - x S_{n-2}}{1 - x S_{n-1}}, \quad n = 1, 2, \dots, N, \quad 0 \leq x \leq (S_{N-1} + \lambda_0 D)^{-1}, \quad (40)$$

*if  $\mu_0 = 0$ , and*

$$\mu_n(x) \equiv \mu_n \frac{1 - x T_{n-1}}{1 - x T_n}, \quad n = 0, 1, \dots, N, \quad -\infty \leq x \leq (T_N + \mu_0 D)^{-1}, \quad (41)$$

*if  $\mu_0 > 0$ . Here  $D \equiv (\lambda_N \pi_N)^{-1}$ , which should be interpreted as  $\infty$  when  $\lambda_N = 0$ , and the quantities  $S_n$  and  $T_n$  are as in (15) and (17), respectively.*

It is easy to see that the ranges of possible values of  $\tilde{\mu}_0 = \mu_0(x)$  and  $\tilde{\lambda}_N = \lambda_N(x)$  are given by

$$0 \leq \tilde{\mu}_0 \leq \mu_0 + \lambda_N \pi_N \left\{ 1 + \lambda_N \pi_N \sum_{n=0}^{N-1} (\lambda_n \pi_n)^{-1} \right\}^{-1}, \quad (42)$$

and

$$0 \leq \tilde{\lambda}_N \leq \lambda_N + \mu_0 \pi_N^{-1} \left\{ 1 + \mu_0 \sum_{n=0}^{N-1} (\lambda_n \pi_n)^{-1} \right\}^{-1}, \quad (43)$$

for  $\mu_0 \geq 0$  and  $\lambda_N \geq 0$ . Moreover, when  $\tilde{\mu}_0$  attains its minimal value 0 then  $\tilde{\lambda}_N$  attains its maximal value and vice versa.

## 5 Proofs

**Proof of Theorem 1.** It will be convenient to let  $P(t) \equiv (p_{ij}(t), i, j \in \mathcal{N})$ , and write the Kolmogorov equations (2) and (3) in matrix notation as

$$P'(t) = QP(t) = P(t)Q, \quad t \geq 0. \quad (44)$$

With superscript  $(n)$  denoting (elementwise)  $n$ th derivative we then have  $P^{(n)}(0+) = Q^n$ , so if  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  are similar, that is, if (5) holds true, then

$$(\tilde{Q}^n)_{ij} = c_{ij}(Q^n)_{ij}, \quad i, j \in \mathcal{N}, \quad n = 0, 1, \dots \quad (45)$$

Taking  $n = |i - j|$  it now follows readily that

$$c_{ij} = \begin{cases} \frac{\tilde{\lambda}_i \tilde{\lambda}_{i+1} \dots \tilde{\lambda}_{j-1}}{\lambda_i \lambda_{i+1} \dots \lambda_{j-1}} & j > i \\ 1 & j = i \\ \frac{\tilde{\mu}_{j+1} \tilde{\mu}_{j+2} \dots \tilde{\mu}_i}{\mu_{j+1} \mu_{j+2} \dots \mu_i} & j < i. \end{cases} \quad (46)$$

Hence, choosing  $n = 1$  and  $j = i$  in (45) immediately gives us

$$\tilde{\lambda}_i + \tilde{\mu}_i = \lambda_i + \mu_i, \quad i \in \mathcal{N}. \quad (47)$$

The forward equations for  $\tilde{\mathcal{X}}$  imply

$$\tilde{p}'_{i+2, i+1}(t) = \tilde{\lambda}_i \tilde{p}_{i+2, i}(t) - (\tilde{\lambda}_{i+1} + \tilde{\mu}_{i+1}) \tilde{p}_{i+2, i+1}(t) + \tilde{\mu}_{i+2} \tilde{p}_{i+2, i+2}(t), \quad i \in \mathcal{N},$$

which, upon substitution of (5), (46) and (47), leads to

$$p'_{i+2, i+1}(t) = \tilde{\lambda}_i \frac{\tilde{\mu}_{i+1}}{\mu_{i+1}} p_{i+2, i}(t) - (\lambda_{i+1} + \mu_{i+1}) p_{i+2, i+1}(t) + \mu_{i+2} p_{i+2, i+2}(t), \quad i \in \mathcal{N}.$$

But the forward equations for  $\mathcal{X}$  tell us that the coefficient of  $p_{i+2, i}(t)$  should be  $\lambda_i$ , so we must have

$$\tilde{\lambda}_i \tilde{\mu}_{i+1} = \lambda_i \mu_{i+1}, \quad i \in \mathcal{N}, \quad (48)$$

as required. Moreover, as a consequence of (46) we have

$$c_{ij} = \frac{\pi_i \tilde{\pi}_j}{\tilde{\pi}_i \pi_j} c_{ji}, \quad (49)$$

while (46) and (48) imply  $c_{ij} = c_{ji}^{-1}$  for all  $i, j \in \mathcal{N}$ . Combining these results gives us (8), completing the proof of Theorem 1.

**Proof of Theorem 2.** We let

$$A \equiv \begin{pmatrix} -(\lambda_0 + \mu_0) & \sqrt{\lambda_0 \mu_1} & 0 & 0 & 0 & \dots \\ \sqrt{\lambda_0 \mu_1} & -(\lambda_1 + \mu_1) & \sqrt{\lambda_1 \mu_2} & 0 & 0 & \dots \\ 0 & \sqrt{\lambda_1 \mu_2} & -(\lambda_2 + \mu_2) & \sqrt{\lambda_2 \mu_3} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (50)$$

and note that  $A$  can be represented as

$$A = \Pi^{1/2} Q \Pi^{-1/2}, \quad (51)$$

where  $\Pi^{1/2}$  and  $\Pi^{-1/2}$  denote the diagonal matrices with entries  $\pi_n^{1/2}$ ,  $n \in \mathcal{N}$ , and  $\pi_n^{-1/2}$ ,  $n \in \mathcal{N}$ , respectively, on the diagonals. It now follows from (44) that

$$\Pi^{1/2} P'(t) \Pi^{-1/2} = A \Pi^{1/2} P(t) \Pi^{-1/2} = \Pi^{1/2} P(t) \Pi^{-1/2} A,$$

so that

$$R_P(t) \equiv \Pi^{1/2} P(t) \Pi^{-1/2}, \quad t \geq 0, \quad (52)$$

is a solution to the system

$$\begin{cases} R'(t) = AR(t) = R(t)A, & t \geq 0, \\ R(0) = I, \end{cases} \quad (53)$$

where  $I$  denotes the infinite identity matrix.

When the birth and death rates of  $\tilde{\mathcal{X}}$  are related to those of  $\mathcal{X}$  as in (7) we have

$$\tilde{\Pi}^{1/2} \tilde{Q} \tilde{\Pi}^{-1/2} = \Pi^{1/2} Q \Pi^{-1/2} = A. \quad (54)$$

Consequently, arguing as before, we find that  $R_{\tilde{P}}(t) \equiv \tilde{\Pi}^{1/2} \tilde{P}(t) \tilde{\Pi}^{-1/2}$  is also a solution to (53). If the solution to (53) is unique, we must therefore have

$$\tilde{\Pi}^{1/2} \tilde{P}(t) \tilde{\Pi}^{-1/2} = \Pi^{1/2} P(t) \Pi^{-1/2},$$

and hence

$$\tilde{P}(t) = \tilde{\Pi}^{-1/2} \Pi^{1/2} P(t) \Pi^{-1/2} \tilde{\Pi}^{1/2}, \quad (55)$$

so that  $\mathcal{X}$  and  $\tilde{\mathcal{X}}$  are similar. So it remains to be shown that the solution to (53) is unique.

Obviously, we are only interested in solutions  $R(t) \equiv (r_{ij}(t))$ ,  $i, j \in \mathcal{N}$ ,  $t \geq 0$ , to (53) which can be represented in terms of the rates and transition functions of some birth-death process, as in (52). Following the approach of Karlin and McGregor [8] it is not difficult to see that such a solution can also be written as

$$r_{ij}(t) = \int_0^\infty e^{-xt} q_i(x) q_j(x) \psi(dx), \quad i, j \in \mathcal{N}, \quad t \geq 0,$$

where  $\psi$  is a solution of the *Stieltjes moment problem* (Smp) associated with the matrix  $A$  of (51), and  $q_i(x)$  are the corresponding orthonormal polynomials. As a consequence,

there is a unique (relevant) solution to (53) if and only if the Smp associated with  $A$  is determined. From Karlin and McGregor [8, Ch. IV] or Chihara [4] we know that this is the case if and only if either  $\mu_0 = 0$  and (4) holds true, or,  $\mu_0 > 0$  and (9) holds true. This completes the proof of Theorem 2.

As an aside we note that when  $\mu_0 > 0$ , condition (9) must be equivalent to

$$\sum_{n=0}^{\infty} (\tilde{\pi}_n + (\tilde{\lambda}_n \tilde{\pi}_n)^{-1}) = \infty, \quad (56)$$

where  $\tilde{\lambda}_n$  and  $\tilde{\mu}_n$ ,  $n \in \mathcal{N}$ , constitute the (unique) solution of (11) satisfying  $\tilde{\mu}_0 = 0$ .

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