

# Examining the consistency relations describing the three-point functions involving tensors

V. Sreenath and L. Sriramkumar

Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

E-mail: [sreenath@physics.iitm.ac.in](mailto:sreenath@physics.iitm.ac.in), [sriram@physics.iitm.ac.in](mailto:sriram@physics.iitm.ac.in)

**Abstract.** It is well known that the non-Gaussianity parameter  $f_{\text{NL}}$  characterizing the scalar bi-spectrum can be expressed in terms of the scalar spectral index in the squeezed limit, a property that is referred to as the consistency relation. In contrast to the scalar bi-spectrum, the three-point cross-correlations involving scalars and tensors and the tensor bi-spectrum have not received adequate attention, which can be largely attributed to the fact that the tensors had remained undetected at the level of the power spectrum until very recently. The detection of the imprints of the primordial tensor perturbations by BICEP2 and its indication of a rather high tensor-to-scalar ratio, if confirmed, can open up a new window for understanding the tensor perturbations, not only at the level of the power spectrum, but also in the realm of non-Gaussianities. In this work, we consider the consistency relations associated with the three-point cross-correlations involving scalars and tensors as well as the tensor bi-spectrum in inflationary models driven by a single, canonical, scalar field. Characterizing the cross-correlations in terms of the dimensionless non-Gaussianity parameters  $C_{\text{NL}}^{\mathcal{R}}$  and  $C_{\text{NL}}^{\gamma}$  that we had introduced earlier, we express the consistency relations governing the cross-correlations as relations between these non-Gaussianity parameters and the scalar or tensor spectral indices, in a fashion similar to that of the purely scalar case. We also discuss the corresponding relation for the non-Gaussianity parameter  $h_{\text{NL}}$  used to describe the tensor bi-spectrum. We analytically establish these consistency relations explicitly in the following two situations: a simple example involving a specific case of power law inflation and a non-trivial scenario in the so-called Starobinsky model that is governed by a linear potential with a sharp change in its slope. We also numerically verify the consistency relations in three types of inflationary models that permit deviations from slow roll and lead to scalar power spectra with features which typically result in an improved fit to the data than the more conventional, nearly scale invariant, spectra. We close with a summary of the results we have obtained.

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## 1 Introduction

Our current understanding of the universe on large scales is based on the  $\Lambda$ CDM model, supplemented by the inflationary paradigm. While there exist certain limited alternatives to inflation, none of the alternatives seem to perform as effectively against the various cosmological data as inflation seems capable of. The inflationary paradigm was originally proposed to ensure that the initial conditions for the background cosmological model are more natural than what was possible within the hot big bang theory. But, it was soon recognized that the paradigm also provides a simple and efficient mechanism for the origin of the perturbations. Inflation can be achieved rather easily using scalar fields (usually referred to as the inflaton), and it is the quantum fluctuations associated with the scalar fields that sow the seeds of the primordial perturbations [1, 2]. These perturbations lead to the anisotropies in the Cosmic Microwave Background (CMB) and, eventually, to the formation of the large scale structure. Despite the measurements of the CMB anisotropies with ever increasing precision by missions such as WMAP and Planck, details concerning the dynamics of the scalar field driving inflation still remain to be satisfactorily understood [3–7]. Specifically, whereas the data seems to point to inflation driven by a slowly rolling scalar field, we are rather far from converging on the form of the potential governing the inflaton (in this context, see, for instance, Refs. [8, 9]).

Very often, inflationary models are compared with the cosmological data at the level of the scalar power spectrum. Over the last decade, it has been realized that non-Gaussianities and, in particular, the scalar bi-spectrum can provide a powerful handle to arrive at a much smaller class of viable inflationary models (for the earliest efforts in this direction, see Refs. [10]; for various theoretical efforts, see, for example, Refs. [11–13]; for work prior to Planck on arriving at constraints on non-Gaussianities from observations, see, for instance, Refs. [14, 15]). Such an expectation has been corroborated to a substantial extent by the strong constraints that have been arrived at from the Planck data on the three non-Gaussianity parameters, viz.  $(f_{\text{NL}}^{\text{loc}}, f_{\text{NL}}^{\text{eq}}, f_{\text{NL}}^{\text{ortho}})$ , that are commonly used to characterize the

scalar bi-spectrum [16]. While a considerable amount of effort has been dedicated to understanding the generation and imprints of the scalar bi-spectrum, a rather limited amount of attention has been paid to investigating the three-point functions involving the tensor perturbations [17–22]. This can be obviously ascribed to the fact that the tensors had remained undetected at the level of the power spectrum until very recently. Needless to add, the detection of the imprints of the primordial tensor perturbations on the CMB by BICEP2 [23], if it is also confirmed by, say, the forthcoming polarization data from Planck, would significantly alter the situation. Importantly, the high tensor-to-scalar ratio implied by the BICEP2 measurements provides hope that it may not be too far fetched to imagine that we can arrive at observational constraints on the three-point cross-correlations consisting of scalars and tensors and the tensor bi-spectrum in the foreseeable future<sup>1</sup>.

In a recent work, we had constructed a procedure (and developed a code) for numerically evaluating the three-point cross-correlations comprising of scalars and tensors as well as the tensor bi-spectrum for an arbitrary triangular configuration of the three wavenumbers involved [21]. We had also introduced dimensionless non-Gaussianity parameters, which we had denoted as  $C_{\text{NL}}^{\mathcal{R}}$  and  $C_{\text{NL}}^{\gamma}$ , to characterize the amplitude of the three-point scalar-tensor cross-correlations. It is well known that, in the squeezed limit wherein one of the wavenumbers is much smaller than the other two, the inflationary scalar bi-spectrum generated by a single scalar field can be expressed completely in terms of the scalar power spectrum, a result that is referred to as a consistency relation (for the original results, see, for instance, Refs. [11, 24]; for more recent discussions in this context, see Refs. [25]; for similar results involving higher order correlation functions, see, for example, Refs. [26]). Or, equivalently, the scalar non-Gaussianity parameter  $f_{\text{NL}}$  can be written purely in terms of the scalar spectral index. However, most of the work on the consistency relations have focussed on the scalar bi-spectrum, and it seems natural to expect that similar consistency relations will be satisfied by the three-point functions that involve the tensor perturbations as well (in this context, see Refs. [11, 18, 20, 22]). Our aim in this work is to investigate the validity of consistency relations involving the three-point scalar-tensor cross-correlations and the tensor bi-spectrum. We shall first express the corresponding consistency relations as relations between the non-Gaussianity parameters ( $C_{\text{NL}}^{\mathcal{R}}$  and  $C_{\text{NL}}^{\gamma}$  [21], and the quantity  $h_{\text{NL}}$  that is used to describe the purely tensor case [17]) and the scalar or the tensor spectral indices. We shall then analytically as well as numerically examine the validity of these consistency conditions in specific situations. As we shall illustrate, the consistency relations hold generically, and they prove to be valid even in scenarios involving substantial deviations from slow roll.

The structure of this paper is as follows. In the following section, we shall quickly introduce the quantities characterizing the scalar and tensor perturbations and the standard definitions of the power spectra in terms of these quantities. We shall also discuss the different three-point functions of interest and introduce the corresponding non-Gaussianity parameters. We shall further summarize the essential expressions regarding the evaluation of the three-point functions in the Maldacena formalism in single field inflationary models involving the canonical scalar field. In the subsequent section, we shall outline a proof of the consistency relations which describe the behavior of the three-point functions in the

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<sup>1</sup>We should hasten to add a clarifying remark here. A priori, the three-point functions are an independent measure of the primordial perturbations. So, a high tensor-to-scalar ratio does not necessarily imply significant amplitudes for the three-point functions involving tensors. But, evidently, detecting, say, the tensor bi-spectrum, seems less likely if the tensor-to-scalar ratio actually turns out to be considerably smaller than what BICEP2 has observed.

squeezed limit, wherein they can be expressed in terms of the scalar and the tensor power spectra. We shall also state the consistency relations as relations between the four non-Gaussianity parameters, viz.  $f_{\text{NL}}$ ,  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$ , and the scalar and the tensor spectral indices. In Sec. 4, we shall explicitly establish the consistency relations for the three non-Gaussianity parameters  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$  in two analytically tractable examples—firstly, in a simple situation involving a specific case of power law inflation and, secondly, in a non-trivial scenario in the so-called Starobinsky model that is described by a linear inflaton potential with a sudden change in its slope. In Sec. 5, we shall numerically investigate the validity of the consistency relations involving  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$  in models which permit deviations from slow roll. We shall consider three types of models which lead to features in scalar power spectrum and are known to result in an improved fit to the CMB data, and show numerically that the consistency relations hold in each of these cases. Finally, in Sec. 6, we conclude with a brief discussion of the results.

A few words on our notations and conventions are in order at this stage of our discussion. We shall work with units such that  $\hbar = c = 1$  and assume the Planck mass to be  $M_{\text{Pl}} = (8\pi G)^{-1/2}$ . We shall adopt the metric signature of  $(-, +, +, +)$ , and we shall make use of Latin indices to denote the spatial coordinates (barring  $k$ , which will represent the wavenumber). The quantities  $a$  and  $H$  shall represent the scale factor and the Hubble parameter of the spatially flat, Friedmann-Lemaître-Robertson-Walker (or, simply, Friedmann, hereafter) universe. We shall work in terms of either the cosmic time  $t$  or the conformal time  $\eta$ , and denote differentiation with respect to these quantities as an overdot and an overprime, respectively.

## 2 Three-point functions and the associated non-Gaussianity parameters

In this section, we shall quickly summarize a few expressions and results involving the scalar and tensor perturbations and the two and three-point correlation functions that will be essential for our discussion.

### 2.1 Primordial perturbations and power spectra

Upon taking into account the scalar perturbation described by the curvature perturbation  $\mathcal{R}$  and the tensor perturbation characterized by  $\gamma_{ij}$ , the spatially flat Friedmann metric can be expressed as [11]

$$ds^2 = -dt^2 + h_{ij}(t, \mathbf{x}) d\mathbf{x}^i d\mathbf{x}^j, \quad (2.1)$$

where the quantity  $h_{ij}$  is given by

$$h_{ij}(t, \mathbf{x}) = a^2(t) e^{2\mathcal{R}(t, \mathbf{x})} \left[ e^{\gamma(t, \mathbf{x})} \right]_{ij}. \quad (2.2)$$

Recall that, in the inflationary paradigm, the primordial perturbations are generated due to quantum fluctuations. On quantization, the curvature perturbation  $\hat{\mathcal{R}}$  and the tensor perturbation  $\hat{\gamma}_{ij}$  can be written in terms of the scalar and tensor Fourier modes, say,  $f_k$  and

$g_k$ , as follows:

$$\begin{aligned}\hat{\mathcal{R}}(\eta, \mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\mathcal{R}}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left( \hat{a}_{\mathbf{k}} f_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger f_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right),\end{aligned}\quad (2.3a)$$

$$\begin{aligned}\hat{\gamma}_{ij}(\eta, \mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left( \hat{b}_{\mathbf{k}}^s \varepsilon_{ij}^s(\mathbf{k}) g_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}_{\mathbf{k}}^{s\dagger} \varepsilon_{ij}^{s*}(\mathbf{k}) g_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right).\end{aligned}\quad (2.3b)$$

In these decompositions, the pairs of operators  $(\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger)$  and  $(\hat{b}_{\mathbf{k}}^s, \hat{b}_{\mathbf{k}}^{s\dagger})$  represent the annihilation and creation operators corresponding to the scalar and the tensor modes associated with the wavevector  $\mathbf{k}$ , and they satisfy the standard commutation relations. The quantity  $\varepsilon_{ij}^s(\mathbf{k})$  represents the polarization tensor of the gravitational waves with their helicity being denoted by the index  $s$ . The transverse and traceless nature of the gravitational waves leads to the conditions  $\varepsilon_{ii}^s(\mathbf{k}) = k_i \varepsilon_{ij}^s(\mathbf{k}) = 0$ . In this paper, we shall work with a normalization such that  $\varepsilon_{ij}^r(\mathbf{k}) \varepsilon_{ij}^{s*}(\mathbf{k}) = 2 \delta^{rs}$  [11].

In terms of the Mukhanov-Sasaki variables, viz.  $v_k = z f_k$  and  $u_k = M_{\text{pl}} a g_k / \sqrt{2}$ , where  $z = \sqrt{2\epsilon_1} M_{\text{pl}} a$ , with  $\epsilon_1 = -\dot{H}/H^2$  being the first slow roll parameter, the equations of motion governing the scalar and the tensor perturbations reduce to

$$v_k'' + \left( k^2 - \frac{z''}{z} \right) v_k = 0, \quad (2.4a)$$

$$u_k'' + \left( k^2 - \frac{a''}{a} \right) u_k = 0, \quad (2.4b)$$

respectively. The scalar and the tensor power spectra, viz.  $\mathcal{P}_s(k)$  and  $\mathcal{P}_T(k)$ , are defined as follows:

$$\langle \hat{\mathcal{R}}_{\mathbf{k}}(\eta) \hat{\mathcal{R}}_{\mathbf{k}'}(\eta) \rangle = \frac{(2\pi)^2}{2k^3} \mathcal{P}_s(k) \delta^{(3)}(\mathbf{k} + \mathbf{k}'), \quad (2.5a)$$

$$\langle \hat{\gamma}_{ij}^{\mathbf{k}}(\eta) \hat{\gamma}_{mn}^{\mathbf{k}'}(\eta) \rangle = \frac{(2\pi)^2}{2k^3} \frac{\Pi_{ij,mn}^{\mathbf{k}}}{4} \mathcal{P}_T(k) \delta^{(3)}(\mathbf{k} + \mathbf{k}'), \quad (2.5b)$$

where the expectation values on the left hand sides are to be evaluated in the specified initial quantum state of the perturbations, and the quantity  $\Pi_{ij,mn}^{\mathbf{k}}$  is given by [17, 19, 21]

$$\Pi_{ij,mn}^{\mathbf{k}} = \sum_s \varepsilon_{ij}^s(\mathbf{k}) \varepsilon_{mn}^{s*}(\mathbf{k}). \quad (2.6)$$

The vacuum state  $|0\rangle$  associated with the quantized perturbations is defined as the state that satisfies the conditions  $\hat{a}_{\mathbf{k}}|0\rangle = 0$  and  $\hat{b}_{\mathbf{k}}^s|0\rangle = 0$  for all  $\mathbf{k}$  and  $s$ . If one assumes the initial state of the perturbations to be the vacuum state  $|0\rangle$ , then, on making use of the decompositions (2.3) in the above definitions, the inflationary scalar and tensor power spectra  $\mathcal{P}_s(k)$  and  $\mathcal{P}_T(k)$  can be expressed as

$$\mathcal{P}_s(k) = \frac{k^3}{2\pi^2} |f_k|^2, \quad (2.7a)$$

$$\mathcal{P}_T(k) = 4 \frac{k^3}{2\pi^2} |g_k|^2. \quad (2.7b)$$

The amplitudes  $|f_k|$  and  $|g_k|$  on the right hand sides of the above expressions are to be evaluated when the modes are sufficiently outside the Hubble radius. It is useful to note here that the scalar and tensor spectral indices  $n_s$  and  $n_T$  are defined as

$$n_s = 1 + \frac{d \ln \mathcal{P}_s(k)}{d \ln k}, \quad (2.8a)$$

$$n_T = \frac{d \ln \mathcal{P}_T(k)}{d \ln k}. \quad (2.8b)$$

## 2.2 The non-Gaussianity parameters

The scalar bi-spectrum, the two scalar-tensor three-point cross-correlations and the tensor bi-spectrum in Fourier space, viz.  $\mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ ,  $\mathcal{B}_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ ,  $\mathcal{B}_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  and  $\mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ , evaluated towards the end of inflation at the conformal time, say,  $\eta_e$ , are defined as<sup>2</sup>

$$\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\eta_e) \hat{\mathcal{R}}_{\mathbf{k}_2}(\eta_e) \hat{\mathcal{R}}_{\mathbf{k}_3}(\eta_e) \rangle \equiv (2\pi)^3 \mathcal{B}_{\mathcal{R}\mathcal{R}\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (2.9a)$$

$$\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\eta_e) \hat{\mathcal{R}}_{\mathbf{k}_2}(\eta_e) \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3}(\eta_e) \rangle \equiv (2\pi)^3 \mathcal{B}_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (2.9b)$$

$$\langle \hat{\mathcal{R}}_{\mathbf{k}_1}(\eta_e) \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2}(\eta_e) \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3}(\eta_e) \rangle \equiv (2\pi)^3 \mathcal{B}_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (2.9c)$$

$$\langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1}(\eta_e) \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2}(\eta_e) \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3}(\eta_e) \rangle \equiv (2\pi)^3 \mathcal{B}_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3). \quad (2.9d)$$

For convenience, hereafter, we shall write these correlators as

$$\mathcal{B}_{\text{ABC}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^{-9/2} G_{\text{ABC}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \quad (2.10)$$

where A, B and C refer to either  $\mathcal{R}$  or  $\gamma$ .

As we have discussed before, in the purely scalar case, it proves to be convenient to express the level of non-Gaussianity in terms of the quantity  $f_{\text{NL}}$ , which is a suitable dimensionless ratio of the three and the two-point functions. In an analogous manner, one can consider similar dimensionless parameters to describe the scalar-scalar-tensor and the scalar-tensor-tensor correlators as well as the tensor bi-spectrum. Let us denote these non-Gaussianity parameters as  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$ , respectively. These can be introduced through the following expressions for the curvature and the tensor perturbations [21]:

$$\mathcal{R}(\eta, \mathbf{x}) = \mathcal{R}^{\text{G}}(\eta, \mathbf{x}) - \frac{3f_{\text{NL}}}{5} [\mathcal{R}^{\text{G}}(\eta, \mathbf{x})]^2 - C_{\text{NL}}^{\mathcal{R}} \mathcal{R}^{\text{G}}(\eta, \mathbf{x}) \gamma_{\bar{m}\bar{n}}^{\text{G}}(\eta, \mathbf{x}), \quad (2.11a)$$

$$\gamma_{ij}(\eta, \mathbf{x}) = \gamma_{ij}^{\text{G}}(\eta, \mathbf{x}) - h_{\text{NL}} \gamma_{ij}^{\text{G}}(\eta, \mathbf{x}) \gamma_{\bar{m}\bar{n}}^{\text{G}}(\eta, \mathbf{x}) - C_{\text{NL}}^{\gamma} \gamma_{ij}^{\text{G}}(\eta, \mathbf{x}) \mathcal{R}^{\text{G}}(\eta, \mathbf{x}), \quad (2.11b)$$

where  $\mathcal{R}^{\text{G}}$  and  $\gamma_{ij}^{\text{G}}$  denote the Gaussian quantities. In these relations, the overbars that appear on the indices of the Gaussian tensor perturbations imply that the indices should be summed over all allowed values (for further clarifications, see Ref. [21]). Upon using the above definitions and the Wick's theorem that is applicable to Gaussian random variables, we find that the four non-Gaussianity parameters, viz.  $f_{\text{NL}}$ ,  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$ , can be expressed in

<sup>2</sup>In this paper, we shall be primarily interested in the three-point functions involving the tensor perturbations. However, at a couple of locations, we shall also briefly touch upon the case of the scalar bi-spectrum and the corresponding non-Gaussianity parameter  $f_{\text{NL}}$  for the sake of completeness.

terms of the corresponding three-point functions and the scalar and the tensor power spectra as

$$f_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{5/6}{(2\pi^2)^2} [k_1^3 k_2^3 k_3^3 G_{\mathcal{R}\mathcal{R}\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)] \\ \times [k_1^3 \mathcal{P}_s(k_2) \mathcal{P}_s(k_3) + \text{two permutations}]^{-1}, \quad (2.12a)$$

$$C_{\text{NL}}^{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{4}{(2\pi^2)^2} [k_1^3 k_2^3 k_3^3 G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)] \\ \times \left( \Pi_{m_3 n_3, \bar{m} \bar{n}}^{\mathbf{k}_3} \right)^{-1} \left\{ [k_1^3 \mathcal{P}_s(k_2) + k_2^3 \mathcal{P}_s(k_1)] \mathcal{P}_T(k_3) \right\}^{-1}, \quad (2.12b)$$

$$C_{\text{NL}}^{\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{4}{(2\pi^2)^2} [k_1^3 k_2^3 k_3^3 G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)] \\ \times \left\{ \mathcal{P}_s(k_1) \left[ \Pi_{m_2 n_2, m_3 n_3}^{\mathbf{k}_2} k_3^3 \mathcal{P}_T(k_2) + \Pi_{m_3 n_3, m_2 n_2}^{\mathbf{k}_3} k_2^3 \mathcal{P}_T(k_3) \right] \right\}^{-1}, \quad (2.12c)$$

$$h_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{4^2}{(2\pi^2)^2} [k_1^3 k_2^3 k_3^3 G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)] \\ \times \left[ \Pi_{m_1 n_1, m_3 n_3}^{\mathbf{k}_1} \Pi_{m_2 n_2, \bar{m} \bar{n}}^{\mathbf{k}_2} k_3^3 \mathcal{P}_T(k_1) \mathcal{P}_T(k_2) + \text{five permutations} \right]^{-1}. \quad (2.12d)$$

### 2.3 The inflationary three-point functions in the Maldacena formalism

The most complete approach to evaluate the three-point functions generated during inflation is the formalism originally due to Maldacena [11]. The first step in the approach is to obtain the action describing the perturbations at the third order. Having obtained the third order action describing the perturbations, the different three-point correlation functions can be arrived at using the standard rules of perturbative quantum field theory [11, 17, 19, 21, 22].

In this work, as we have discussed, we shall be interested in the three-point functions that involve the tensor perturbations. One can show that the scalar-scalar-tensor cross-correlation  $G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ , when evaluated in the perturbative vacuum, can be written as (see, for example, Ref. [21])

$$G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \sum_{C=1}^3 G_{\mathcal{R}\mathcal{R}\gamma(C)}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ = M_{\text{Pl}}^2 \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_{1i} \hat{n}_{2j} \sum_{C=1}^3 [f_{k_1}(\eta_e) f_{k_2}(\eta_e) g_{k_3}(\eta_e) \\ \times \mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \text{complex conjugate}], \quad (2.13)$$

where the quantities  $\mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  are described by the integrals

$$\mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -2i k_1 k_2 \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1 f_{k_1}^* f_{k_2}^* g_{k_3}^*, \quad (2.14a)$$

$$\mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{i}{2} \frac{k_3^2}{k_1 k_2} \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1^2 f_{k_1}^* f_{k_2}^* g_{k_3}^*, \quad (2.14b)$$

$$\mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{i}{2} \frac{1}{k_1 k_2} \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1^2 [k_1^2 f_{k_1}^* f_{k_2}^* + k_2^2 f_{k_1}^* f_{k_2}^*] g_{k_3}^*. \quad (2.14c)$$

The lower limit of the integrals, viz.  $\eta_i$ , denotes a sufficiently early time at which the initial conditions are imposed on the modes when they are well inside the Hubble radius. The upper limit  $\eta_e$  denotes a suitably late time which can, for instance, be conveniently chosen to be a time close to the end of inflation. Note that, for a given wavevector  $\mathbf{k}$ ,  $\hat{\mathbf{n}}$  denotes the unit vector  $\hat{\mathbf{n}} = \mathbf{k}/k$ . Hence, the quantities  $\hat{n}_{1i}$  and  $\hat{n}_{2i}$  represent the components of the unit vectors  $\hat{\mathbf{n}}_1 = \mathbf{k}_1/k_1$  and  $\hat{\mathbf{n}}_2 = \mathbf{k}_2/k_2$  along the  $i$ -spatial direction.

Similarly, the scalar-tensor-tensor cross-correlation  $G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ , evaluated in the perturbative vacuum, can be expressed as

$$\begin{aligned} G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \sum_{C=1}^3 G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3(C)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \\ &= M_{\text{Pl}}^2 \Pi_{m_2 n_2, ij}^{\mathbf{k}_2} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \sum_{C=1}^3 [f_{k_1}(\eta_e) g_{k_2}(\eta_e) g_{k_3}(\eta_e) \\ &\quad \times \mathcal{G}_{\mathcal{R}\gamma\gamma}^C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \text{complex conjugate}], \end{aligned} \quad (2.15)$$

with the quantities  $\mathcal{G}_{\mathcal{R}\gamma\gamma}^C(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  being given by

$$\mathcal{G}_{\mathcal{R}\gamma\gamma}^1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{i}{4} \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1 f_{k_1}^* g_{k_2}^* g_{k_3}^*, \quad (2.16a)$$

$$\mathcal{G}_{\mathcal{R}\gamma\gamma}^2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{i}{4} (\mathbf{k}_2 \cdot \mathbf{k}_3) \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1 f_{k_1}^* g_{k_2}^* g_{k_3}^*, \quad (2.16b)$$

$$\mathcal{G}_{\mathcal{R}\gamma\gamma}^3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{i}{4} \int_{\eta_i}^{\eta_e} d\eta a^2 \epsilon_1 f_{k_1}^* \left[ \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1^2} g_{k_2}^* g_{k_3}^* + \frac{\mathbf{k}_1 \cdot \mathbf{k}_3}{k_1^2} g_{k_2}^* g_{k_3}^* \right]. \quad (2.16c)$$

Lastly, the tensor bi-spectrum  $G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ , calculated in the perturbative vacuum, can be written as [11, 17, 19, 21, 22]

$$\begin{aligned} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= M_{\text{Pl}}^2 \left[ \left( \Pi_{m_1 n_1, ij}^{\mathbf{k}_1} \Pi_{m_2 n_2, im}^{\mathbf{k}_2} \Pi_{m_3 n_3, lj}^{\mathbf{k}_3} \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \Pi_{m_1 n_1, ij}^{\mathbf{k}_1} \Pi_{m_2 n_2, ml}^{\mathbf{k}_2} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \right) k_{1m} k_{1l} + \text{five permutations} \right] \\ &\quad \times [g_{k_1}(\eta_e) g_{k_2}(\eta_e) g_{k_3}(\eta_e) \\ &\quad \times \mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \text{complex conjugate}], \end{aligned} \quad (2.17)$$

where the quantity  $\mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  is described by the integral

$$\mathcal{G}_{\gamma\gamma\gamma}^1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{i}{4} \int_{\eta_i}^{\eta_e} d\eta a^2 g_{k_1}^* g_{k_2}^* g_{k_3}^*, \quad (2.18)$$

and we should emphasize that  $(k_{1i}, k_{2i}, k_{3i})$  denote the components of the three wavevectors  $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  along the  $i$ -spatial direction<sup>3</sup>.

<sup>3</sup>Such an emphasis seems essential to avoid confusion between  $k_1, k_2$  and  $k_3$  which denote the wavenumbers associated with the wavevectors  $\mathbf{k}_1, \mathbf{k}_2$  and  $\mathbf{k}_3$ , and the quantity  $k_i$  which represents the component of the wavevector  $\mathbf{k}$  along the  $i$ -spatial direction. We have made similar clarifications below wherever we are concerned some confusion in the notation may arise.



The delta functions that appear in the definitions (2.9) of the three-point functions imply that the wavevectors  $\mathbf{k}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{k}_3$  form a triangle. The squeezed limit of the three-point functions corresponds to the situation wherein one of the three wavenumbers, i.e.  $k_1$ ,  $k_2$  or  $k_3$ , vanishes. In the two cases of the scalar-tensor cross-correlations, the squeezed limit can evidently be arrived at by choosing the wavenumber of either the scalar or the tensor modes to be zero. However, the contributions to the scalar-scalar-tensor three-point function either explicitly involve the wavenumber of the scalar mode or its time derivative, both of which go to zero in the large scale limit. As a result, the scalar-scalar-tensor three-point function itself vanishes in the large scale limit of either of the scalar modes. For the same reasons, one finds that the scalar-tensor-tensor cross-correlation also vanishes in the limit wherein the wavenumber of any of the two tensor modes goes to zero. Therefore, in order to understand the behavior in the squeezed limit, we shall consider the large scale limits of the tensor and the scalar modes in the cases of the scalar-scalar-tensor and the scalar-tensor-tensor cross-correlations, respectively. In the squeezed limit, the expressions for the cross-correlations  $G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  and  $G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ , and the tensor bi-spectrum  $G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  can be written as

$$\begin{aligned} \lim_{k_3 \rightarrow 0} G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \lim_{k_3 \rightarrow 0} \sum_{C=1}^3 G_{\mathcal{R}\mathcal{R}\gamma(C)}^{m_3 n_3}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) \\ &= - \lim_{k_3 \rightarrow 0} M_{\text{Pl}}^2 \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_i \hat{n}_j \sum_{C=1}^3 [f_k(\eta_e) f_k(\eta_e) g_{k_3}(\eta_e) \\ &\quad \times \mathcal{G}_{\mathcal{R}\mathcal{R}\gamma}^C(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) + \text{complex conjugate}], \end{aligned} \quad (2.19a)$$

$$\begin{aligned} \lim_{k_1 \rightarrow 0} G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \lim_{k_1 \rightarrow 0} \sum_{C=1}^3 G_{\mathcal{R}\gamma\gamma(C)}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) \\ &= \lim_{k_1 \rightarrow 0} M_{\text{Pl}}^2 \Pi_{m_2 n_2, ij}^{\mathbf{k}} \Pi_{m_3 n_3, ij}^{-\mathbf{k}} \sum_{C=1}^3 [f_{k_1}(\eta_e) g_k(\eta_e) g_k(\eta_e) \\ &\quad \times \mathcal{G}_{\mathcal{R}\gamma\gamma}^C(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) + \text{complex conjugate}] \\ &= \lim_{k_1 \rightarrow 0} 2 M_{\text{Pl}}^2 \Pi_{m_2 n_2, m_3 n_3}^{\mathbf{k}} \sum_{C=1}^3 [f_{k_1}(\eta_e) g_k(\eta_e) g_k(\eta_e) \\ &\quad \times \mathcal{G}_{\mathcal{R}\gamma\gamma}^C(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) + \text{complex conjugate}], \end{aligned} \quad (2.19b)$$

$$\begin{aligned} \lim_{k_3 \rightarrow 0} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \lim_{k_3 \rightarrow 0} G_{\gamma\gamma\gamma}^{m_1 n_1 m_2 n_2 m_3 n_3}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) \\ &= - \lim_{k_3 \rightarrow 0} M_{\text{Pl}}^2 \Pi_{m_1 n_1, ij}^{\mathbf{k}} \Pi_{m_2 n_2, ij}^{-\mathbf{k}} \Pi_{m_3 n_3, ml}^{\mathbf{k}_3} k_m k_l \\ &\quad \times [g_k(\eta_e) g_k(\eta_e) g_{k_3}(\eta_e) \mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) \\ &\quad + \text{complex conjugate}] \\ &= - \lim_{k_3 \rightarrow 0} 2 M_{\text{Pl}}^2 \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} \Pi_{m_3 n_3, ml}^{\mathbf{k}_3} k_m k_l \\ &\quad \times [g_k(\eta_e) g_k(\eta_e) g_{k_3}(\eta_e) \mathcal{G}_{\gamma\gamma\gamma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) \\ &\quad + \text{complex conjugate}], \end{aligned} \quad (2.19c)$$

where, for simplicity, we have set  $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$  in the first and the third expressions, and  $\mathbf{k}_2 = -\mathbf{k}_3 = \mathbf{k}$  in the second. The overall minus sign in the above scalar-scalar-tensor corre-

lation arises due to the fact that, in the squeezed limit,  $\hat{n}_{1i} = -\hat{n}_{2i} = \hat{n}_i$ . The polarization factors in the tensor bi-spectrum simplify in the squeezed limit due to the transverse nature of the gravitational waves, i.e.  $k_i \varepsilon_{ij}^s(\mathbf{k}) = 0$ . Moreover, it is the normalization of the polarization tensor, viz.  $\varepsilon_{ij}^r(\mathbf{k}) \varepsilon_{ij}^{s*}(\mathbf{k}) = 2 \delta^{rs}$ , that leads to the overall factor of two in the cases of the scalar-tensor-tensor correlation and the tensor bi-spectrum. Note that the two three-point cross-correlations contain three independent terms. It is straightforward to argue that, in the squeezed limit of the tensor mode, it is only the first term that contributes in the case of the scalar-scalar-tensor correlation (as the other two contributions either depend explicitly on the wavenumber of the squeezed mode or its time derivative). Similarly, in the case of the scalar-tensor-tensor correlation, one finds that the third term does not contribute in the squeezed limit of the scalar mode.

In the next section, we shall briefly outline a proof of the consistency relations obeyed by the different three-point functions and the corresponding non-Gaussianity parameters in the squeezed limit.

### 3 Consistency relations in the squeezed limit

A consistency relation basically links the three-point function to the two-point function in a particular limit of the wavenumbers involved<sup>4</sup>. It has been known for a while that the scalar bi-spectrum obeys a consistency relation in the squeezed limit [11, 24, 25]. In terms of the scalar non-Gaussianity parameter  $f_{\text{NL}}$ , it can be expressed as  $f_{\text{NL}} = 5(n_s - 1)/12$ , where  $n_s$  is the scalar spectral index [cf. Eq. (2.8)]. The scalar consistency relation is expected to be valid for any single field inflationary model, irrespective of the detailed dynamics of the field [24]. As we shall discuss in some detail below, this essentially occurs because of the fact that the amplitude of the long wavelength scalar modes freezes on super-Hubble scales. Due to this reason, their effects on the smaller wavelength modes can be treated as though they are evolving in a background with modified spatial coordinates. Since the tensor modes too behave in the same fashion as the scalar modes when they are sufficiently outside the Hubble radius (i.e. their amplitudes freeze as well), it seems natural to expect that there should exist similar consistency relations describing the three-point scalar-tensor cross-correlations and the tensor bi-spectrum [18, 20, 22]. In the remainder of this section, we shall arrive at the consistency relations governing all the four three-point functions in the squeezed limit.

As we have already pointed out, the amplitude of a long wavelength scalar or tensor mode would be a constant, since they would be well outside the Hubble radius (in this context, see, for instance, Refs. [27]). Due to the fact that the amplitude has frozen, they can be treated as a background as far as the smaller wavelength modes are concerned. Let us denote the constant amplitude (i.e. as far as their time dependence is concerned) of the long wavelength scalar and tensor modes as, say,  $\mathcal{R}^{\text{B}}$  and  $\gamma_{ij}^{\text{B}}$ , respectively. In the presence of such modes, the *background* metric will take the form

$$ds^2 = -dt^2 + a^2(t) e^{2\mathcal{R}^{\text{B}}} [e^{\gamma^{\text{B}}}]_{ij} dx^i dx^j, \quad (3.1)$$

i.e. the long wavelength modes lead to modified spatial coordinates. Such a modification is, in fact, completely equivalent to a spatial transformation of the form  $\mathbf{x}' = \Lambda \mathbf{x}$ , with

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<sup>4</sup>In fact, depending on the symmetries associated with the action governing the field(s) of interest, quantum field theory suggests such relations can exist between generic  $N$ -point and  $(N-1)$ -point correlation functions. Clearly, similar connections can be expected to arise from the various correlation functions generated during inflation as well (in this context, see, for instance, Refs. [26]).

the components of the matrix  $\Lambda$  being given by  $\Lambda_{ij} = e^{\mathcal{R}^B} [e^{\gamma^B/2}]_{ij}$ . Under such a spatial coordinate transformation, one can easily show that the Fourier modes of the small wavelength scalar and tensor perturbations transform as follows:  $\mathcal{R}_{\mathbf{k}} \rightarrow \det(\Lambda^{-1}) \mathcal{R}_{\Lambda^{-1}\mathbf{k}}$  and  $\hat{\gamma}_{ij}^{\mathbf{k}} \rightarrow \det(\Lambda^{-1}) \hat{\gamma}_{ij}^{\Lambda^{-1}\mathbf{k}}$ , where, evidently,  $\Lambda^{-1}$  represents the inverse of the original matrix  $\Lambda$ . Upon using the property that the determinant of the exponential of a matrix is the exponential of its trace and the fact that  $\gamma_{ij}$  is traceless, one arrives at the result  $\det(\Lambda^{-1}) = e^{-3\mathcal{R}^B}$ . At the leading order in  $\mathcal{R}^B$  and  $\gamma^B$ , one can also obtain that  $|\Lambda^{-1}\mathbf{k}| = [1 - \mathcal{R}^B - \gamma_{ij}^B k_i k_j / (2k^2)] k$ , where, as we have clarified earlier,  $k_i$  denotes the component of the wavevector  $\mathbf{k}$  along the  $i$ -spatial direction. Moreover, since  $\delta^{(3)}(\Lambda^{-1}\mathbf{k}_1 + \Lambda^{-1}\mathbf{k}_2) = \det(\Lambda) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2)$ , on combining the above results, one finds that the scalar and the tensor two-point functions in the presence of a long wavelength mode denoted by, say, the wavenumber  $k$ , can be written as

$$\begin{aligned} \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \rangle_k &= \frac{(2\pi)^2}{2k_1^3} \mathcal{P}_S(k_1) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &\times \left[ 1 - (n_S - 1) \mathcal{R}^B - \left( \frac{n_S - 4}{2} \right) \gamma_{ij}^B \hat{n}_{1i} \hat{n}_{1j} \right], \end{aligned} \quad (3.2a)$$

$$\begin{aligned} \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \rangle_k &= \frac{(2\pi)^2}{2k_1^3} \frac{\Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1}}{4} \mathcal{P}_T(k_1) \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \\ &\times \left[ 1 - n_T \mathcal{R}^B - \left( \frac{n_T - 3}{2} \right) \gamma_{ij}^B \hat{n}_{1i} \hat{n}_{1j} \right], \end{aligned} \quad (3.2b)$$

where, as we have mentioned,  $\hat{n}_{1i} = k_{1i}/k_1$ .

The above expressions for the two-point functions can then be utilized to arrive at the four three-point functions in the squeezed limit. We find that, in the presence of a long wavelength mode, the three-point functions can be obtained to be

$$\begin{aligned} \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle_{k_3} &\equiv \langle \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \rangle_{k_3} \hat{\mathcal{R}}_{\mathbf{k}_3} \rangle \\ &= -\frac{(2\pi)^{5/2}}{4k_1^3 k_3^3} (n_S - 1) \mathcal{P}_S(k_1) \mathcal{P}_S(k_3) \delta^3(\mathbf{k}_1 + \mathbf{k}_2), \end{aligned} \quad (3.3a)$$

$$\begin{aligned} \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle_{k_3} &\equiv \langle \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\mathcal{R}}_{\mathbf{k}_2} \rangle_{k_3} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle \\ &= -\frac{(2\pi)^{5/2}}{4k_1^3 k_3^3} \left( \frac{n_S - 4}{8} \right) \mathcal{P}_S(k_1) \mathcal{P}_T(k_3) \\ &\times \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_{1i} \hat{n}_{1j} \delta^3(\mathbf{k}_1 + \mathbf{k}_2), \end{aligned} \quad (3.3b)$$

$$\begin{aligned} \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle_{k_1} &\equiv \langle \hat{\mathcal{R}}_{\mathbf{k}_1} \langle \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle_{k_1} \rangle \\ &= -\frac{(2\pi)^{5/2}}{4k_1^3 k_2^3} \frac{n_T}{4} \mathcal{P}_S(k_1) \mathcal{P}_T(k_2) \Pi_{m_2 n_2, m_3 n_3}^{\mathbf{k}_2} \delta^3(\mathbf{k}_2 + \mathbf{k}_3), \end{aligned} \quad (3.3c)$$

$$\begin{aligned} \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle_{k_3} &\equiv \langle \langle \hat{\gamma}_{m_1 n_1}^{\mathbf{k}_1} \hat{\gamma}_{m_2 n_2}^{\mathbf{k}_2} \rangle_{k_3} \hat{\gamma}_{m_3 n_3}^{\mathbf{k}_3} \rangle \\ &= -\frac{(2\pi)^{5/2}}{4k_1^3 k_3^3} \left( \frac{n_T - 3}{32} \right) \mathcal{P}_T(k_1) \mathcal{P}_T(k_3) \\ &\times \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}_1} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_{1i} \hat{n}_{1j} \delta^3(\mathbf{k}_1 + \mathbf{k}_2), \end{aligned} \quad (3.3d)$$

where, in the cases of the scalar and the tensor bi-spectra and the scalar-scalar-tensor cross-correlation, we have considered  $\mathbf{k}_3$  to be the squeezed mode, while we have considered  $\mathbf{k}_1$  to

be the squeezed mode in the case of the scalar-tensor-tensor cross-correlation. Upon making use of the above expressions for the three-point functions in the definitions (2.12) for the non-Gaussianity parameters, we can express the consistency relations in the squeezed limit as follows:

$$\lim_{k_3 \rightarrow 0} f_{\text{NL}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) = \frac{5}{12} [n_{\text{S}}(k) - 1], \quad (3.4a)$$

$$\lim_{k_3 \rightarrow 0} C_{\text{NL}}^{\mathcal{R}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) = \left[ \frac{n_{\text{S}}(k) - 4}{4} \right] \left( \Pi_{m_3 n_3, \bar{m} \bar{n}}^{\mathbf{k}_3} \right)^{-1} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_i \hat{n}_j, \quad (3.4b)$$

$$\lim_{k_1 \rightarrow 0} C_{\text{NL}}^{\gamma}(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) = \frac{n_{\text{T}}(k)}{2} \left( \Pi_{m_2 n_2, m_3 n_3}^{\mathbf{k}} \right)^{-1} \Pi_{m_2 n_2, m_3 n_3}^{\mathbf{k}}, \quad (3.4c)$$

$$\begin{aligned} \lim_{k_3 \rightarrow 0} h_{\text{NL}}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) &= \left[ \frac{n_{\text{T}}(k) - 3}{2} \right] \left( 2 \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} \Pi_{m_3 n_3, \bar{m} \bar{n}}^{\mathbf{k}_3} + \Pi_{m_1 n_1, \bar{m} \bar{n}}^{\mathbf{k}} \Pi_{m_3 n_3, m_2 n_2}^{\mathbf{k}_3} \right. \\ &\quad \left. + \Pi_{\bar{m} \bar{n}, m_2 n_2}^{\mathbf{k}} \Pi_{m_3 n_3, m_1 n_1}^{\mathbf{k}_3} \right)^{-1} \\ &\quad \times \Pi_{m_1 n_1, m_2 n_2}^{\mathbf{k}} \Pi_{m_3 n_3, ij}^{\mathbf{k}_3} \hat{n}_i \hat{n}_j, \end{aligned} \quad (3.4d)$$

where we have explicitly illustrated the point that  $n_{\text{S}}$  and  $n_{\text{T}}$  are, in general, dependent on the wavenumber [a dependence which can be arrived at from the corresponding power spectra through the expressions (2.8)]. It is useful to note here that, during slow roll inflation, while the non-Gaussianity parameters  $f_{\text{NL}}$  and  $C_{\text{NL}}^{\gamma}$  are of the order of the slow parameters, the quantities  $C_{\text{NL}}^{\mathcal{R}}$  and  $h_{\text{NL}}$  prove to be of order unity. This does not imply that the parameters  $C_{\text{NL}}^{\mathcal{R}}$  and  $h_{\text{NL}}$  are ‘large’. They are of order of unity due to the manner in which they have been introduced.

Finally, we would like to stress here the fact that we have arrived at the above consistency relations essentially assuming that the perturbations are initially in the Bunch-Davies vacuum and that the amplitude of the scalar and the tensor perturbations are frozen on super-Hubble scales. While we have focussed on single field models of inflation driven by the canonical scalar field, the amplitude of the perturbations are known to be conserved in any single field model. For this reason, one can expect the consistency relations to hold even in non-canonical models of inflation, provided the perturbations are in the Bunch-Davies vacuum [22].

## 4 Analytical examples

In this section, we shall explicitly confirm the validity of the above consistency relations involving the tensors in two analytically tractable examples. We shall first consider a particular case of power law inflation and then discuss the so-called Starobinsky model which permits a brief period of departure from slow roll.

### 4.1 A power law case

Power law inflation corresponds to the situation wherein the scale factor is given by

$$a(\eta) = a_1 \left( \frac{\eta}{\eta_1} \right)^{\gamma+1}, \quad (4.1)$$

where  $a_1$  and  $\eta_1$  are constants, while  $\gamma < -2$ . In such a situation, the first slow roll parameter proves to be a constant, and is given by  $\epsilon_1 = (\gamma+2)/(\gamma+1)$ . Further, since  $z \propto a$  in this case,

the scalar and the tensor modes are described by the same functions. One finds that, the solutions to the Mukhanov-Sasaki equations (2.4) can be expressed in terms of the Hankel functions of the first kind  $H_\nu^{(1)}(x)$  as (see, for instance, Refs. [28, 29])

$$v_k(\eta) = u_k(\eta) = \sqrt{\frac{-\pi\eta}{4}} e^{-i\pi(\nu-1/2)/2} H_{-\nu}^{(1)}(-k\eta), \quad (4.2)$$

where  $\nu = \gamma + 1/2$ . These specific solutions have been arrived at by demanding that they satisfy the Bunch-Davies initial conditions at very early times (i.e. as  $\eta \rightarrow -\infty$ ) [1, 2, 30]. The power spectra in power law inflation can be arrived at from the amplitudes of the Hankel functions, evaluated at late times (i.e. as  $\eta \rightarrow 0$ ). One can obtain the scalar and tensor power spectra to be

$$\mathcal{P}_s(k) = 16 \epsilon_1 \mathcal{P}_T(k) = \frac{1}{2\pi^3 M_{\text{Pl}}^2 \epsilon_1} \left( \frac{|\eta_1|^{\gamma+1}}{a_1} \right)^2 |\Gamma[-(\gamma + 1/2)]|^2 \left( \frac{k}{2} \right)^{2(\gamma+2)}, \quad (4.3)$$

where, recall that,  $\epsilon_1 = (\gamma + 2)/(\gamma + 1)$ , and  $\Gamma(x)$  represents the Gamma function. Note that the scalar and tensor spectral indices corresponding to these power spectra are constants, and are given by  $n_s - 1 = n_T = 2(\gamma + 2)$ . If the consistency relations (3.4) are indeed satisfied, then, upon setting each of the factors involving the polarization of the tensor perturbations to be unity, the above spectral indices would lead to the following values of non-Gaussianity parameters of our interest:

$$C_{\text{NL}}^{\mathcal{R}} = \frac{n_s - 4}{4} = \frac{2\gamma + 1}{4}, \quad (4.4a)$$

$$C_{\text{NL}}^\gamma = \frac{n_T}{2} = \gamma + 2, \quad (4.4b)$$

$$h_{\text{NL}} = \frac{n_T - 3}{8} = \frac{2\gamma + 1}{8}, \quad (4.4c)$$

which are constants independent of the wavenumber. Our task now would be to evaluate the three-point functions using the Maldacena formalism and examine if we indeed arrive at these values in the squeezed limit.

Ideally, it would have been desirable to arrive at analytic expressions describing the three-point functions in power law inflation for an arbitrary index  $\gamma$ . This clearly requires having to calculate the various integrals describing the correlations that we had summarized earlier in Subsec. 2.3. In fact, the spectral dependences of the three-point functions in power law inflation can be easily arrived at (in, say, the equilateral and the squeezed limits) without actually having to carry out the integrals involved [21, 31]. These results for the squeezed limit then immediately point to the fact that the non-Gaussianity parameters would be independent of scale. But, in order to be able to establish the consistency conditions (4.4) explicitly, apart from the spectral dependences, we shall require the amplitude of the integrals as well. But, care is required in handling the integrals in the extreme sub-Hubble limit (i.e. as  $\eta \rightarrow -\infty$ ) wherein the integrands oscillate with increasing frequency. This aspect seems to make it difficult to carry out the integrals and express them in a closed analytic form for a generic  $\gamma$ .

For the above reason, in order to establish the consistency relations, we shall focus on the specific case of  $\gamma = -3$  or, equivalently,  $\nu = -5/2$ . In this case, the scalar and tensor modes simplify to

$$f_k(\eta) = \frac{g_k(\eta)}{\sqrt{2}} = -\frac{1}{\sqrt{2} k^5 M_{\text{Pl}}} \frac{1}{a_1 \eta_1^2} (3 + 3 i k \eta - k^2 \eta^2) e^{-i k \eta}, \quad (4.5)$$

so that the corresponding derivatives are given by

$$f'_k(\eta) = \frac{g'_k(\eta)}{\sqrt{2}} = \frac{-i}{\sqrt{2} k^3 M_{\text{Pl}}} \frac{1}{a_1 \eta_1^2} (k^2 \eta^2 - i k \eta) e^{-i k \eta}. \quad (4.6)$$

As  $\eta \rightarrow 0$ , the scalar and tensor modes reduce to

$$\lim_{\eta \rightarrow 0} f_k(\eta) = \lim_{\eta \rightarrow 0} \frac{g_k(\eta)}{\sqrt{2}} = -\frac{3}{\sqrt{2} k^5 M_{\text{Pl}}} \frac{1}{a_1 \eta_1^2}. \quad (4.7)$$

We can arrive at the three-point functions of interest upon substituting the above scalar and tensor modes, their derivatives and their asymptotic behavior at late times, in the expressions (2.13), (2.15) and (2.17), and evaluating the various integrals involved. We find that, upon setting the factors containing the polarization tensor to be unity, in the squeezed limit, the three-point functions of interest are given by

$$\lim_{k_3 \rightarrow 0} k^3 k_3^3 G_{\mathcal{R}\mathcal{R}\gamma}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) = \frac{5}{4} \left( \frac{3}{M_{\text{Pl}} a_1 \eta_1^2} \right)^4 \frac{1}{k^2 k_3^2}, \quad (4.8a)$$

$$\lim_{k_1 \rightarrow 0} k_1^3 k^3 G_{\mathcal{R}\gamma\gamma}(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) = \left( \frac{3}{M_{\text{Pl}} a_1 \eta_1^2} \right)^4 \frac{1}{k_1^2 k^2}, \quad (4.8b)$$

$$\lim_{k_3 \rightarrow 0} k^3 k_3^3 G_{\gamma\gamma\gamma}(\mathbf{k}, \mathbf{k}, -\mathbf{k}_3) = \frac{5}{2} \left( \frac{3}{M_{\text{Pl}} a_1 \eta_1^2} \right)^4 \frac{1}{k^2 k_3^2}. \quad (4.8c)$$

The non-Gaussianity parameters corresponding to these three-point functions can be easily obtained to be  $C_{\text{NL}}^{\mathcal{R}} = -5/4$ ,  $C_{\text{NL}}^{\gamma} = -1$  and  $h_{\text{NL}} = -5/8$ . These values exactly match the results (4.4) with  $\gamma = -3$ , which implies that the consistency conditions are indeed satisfied in this case.

## 4.2 The case of the Starobinsky model

The second example that we shall consider is the Starobinsky model. In the Starobinsky model, the inflaton rolls down a linear potential which changes its slope suddenly at a particular value of the scalar field [32]. The governing potential is given by

$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{for } \phi > \phi_0, \\ V_0 + A_-(\phi - \phi_0) & \text{for } \phi < \phi_0, \end{cases} \quad (4.9)$$

where  $V_0$ ,  $A_+$ ,  $A_-$  and  $\phi_0$  are constants. As should be clear, there is a discontinuity in the slope of the potential at  $\phi_0$ . This discontinuity leads to a brief period of deviation from slow roll as the field traverses  $\phi_0$ , before slow roll is restored [32–35].

In order to calculate the three-point functions of our interest, evidently, we shall require the behavior of the scale factor, the first slow roll parameter  $\epsilon_1$ , and the scalar and the tensor modes [cf. Eqs. (2.13), (2.15) and (2.17)]. An important aspect of the Starobinsky model—which permits the background, the perturbations and the different correlation functions to be calculated analytically—is the assumption that the constant  $V_0$  is the dominant term in the potential that is driving the field as it crosses the discontinuity at  $\phi_0$ . In such a situation, we can consider the Hubble parameter to be a constant, say,  $H_0$ , determined by the relation  $H_0^2 \simeq V_0/(3 M_{\text{Pl}}^2)$ . Due to this reason, the scale factor essentially corresponds to that of de Sitter and, hence, the first slow parameter  $\epsilon_1$  remains small throughout the evolution.

In fact, under the assumption that  $V_0$  is suitably large compared to the other terms in the potential, the first slow roll parameter before and after the field crosses  $\phi_0$  can be obtained to be [32–35]

$$\epsilon_{1+}(\eta) \simeq \frac{A_+^2}{18 M_{\text{Pl}}^2 H_0^4}, \quad (4.10a)$$

$$\epsilon_{1-}(\eta) \simeq \frac{A_-^2}{18 M_{\text{Pl}}^2 H_0^4} \left[ 1 - \frac{\Delta A}{A_-} \left( \frac{\eta}{\eta_0} \right)^3 \right]^2, \quad (4.10b)$$

where  $\Delta A = A_- - A_+$ , and  $\eta_0$  denotes the conformal time when the transition takes place. Note that, in the above expressions and some that follow, a plus or minus sign in the subscript (or, when convenient, in the super-script) denote the quantities before and after the transition at  $\eta_0$ .

It should be clear from the above expressions for the first slow roll parameter that its value changes as the field traverses across  $\phi_0$ . Recall that, the second slow roll parameter is defined as  $\epsilon_2 = \dot{\epsilon}_1 / (H \epsilon_1)$ . Because of the change in the value of  $\epsilon_1$  at  $\eta_0$ , the magnitude of the second slow roll parameter  $\epsilon_2$  exhibits a sharp rise leading to brief period of departure from slow roll, before slow roll is restored again at a suitably later time. In order to calculate the non-vanishing contributions to the three-point functions in the squeezed limit of our interest, it suffices for us to know the scalar modes, and we do not require its time derivative which involves  $\epsilon_2$  [33, 35]. This, in turn, implies that we do not need the actual behavior of the the second slow roll parameter. Interestingly, due to the simple nature of the potential and the assumptions that one works under, one finds that the scalar modes are given by the standard Bunch-Davies modes in the de Sitter limit both before and after the transition. However, because of the transition that occurs at  $\eta_0$ , the modes post-transition are related to the modes prior to the transition by the standard Bogoliubov coefficients. One finds that the modes before and after the transition are given by [32–35]:

$$f_k^+(\eta) = \frac{i H_0}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1+}}} (1 + i k \eta) e^{-i k \eta}, \quad (4.11a)$$

$$f_k^-(\eta) = \frac{i H_0 \alpha_k}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} (1 + i k \eta) e^{-i k \eta} - \frac{i H_0 \beta_k}{2 M_{\text{Pl}} \sqrt{k^3 \epsilon_{1-}}} (1 - i k \eta) e^{i k \eta}, \quad (4.11b)$$

respectively. The quantities  $\alpha_k$  and  $\beta_k$  are the Bogoliubov coefficients, which can be determined by matching the modes at the transition. The Bogoliubov coefficients are found to be

$$\alpha_k = 1 + \frac{3 i \Delta A}{2 A_+} \frac{k_0}{k} \left( 1 + \frac{k_0^2}{k^2} \right), \quad (4.12a)$$

$$\beta_k = -\frac{3 i \Delta A}{2 A_+} \frac{k_0}{k} \left( 1 + \frac{i k_0}{k} \right)^2 e^{2 i k / k_0}, \quad (4.12b)$$

with  $k_0 = -1/\eta_0 = a_0 H_0$ , and  $a_0$  being the value of the scale factor at the transition.

In contrast to the scalar modes, the evolution of the tensor modes are determined only by the behaviour of the scale factor. Since the scale factor always remains that of de Sitter, the tensor modes are not affected by the transition at  $\phi_0$ , and are given by the standard Bunch-Davies solution, viz.

$$g_k(\eta) = \frac{i \sqrt{2} H_0}{M_{\text{Pl}} \sqrt{2 k^3}} (1 + i k \eta) e^{-i k \eta}, \quad (4.13)$$

over the complete domain in time that is of interest to us. The time derivative of the tensor mode, which we shall require, is given by

$$g'_k(\eta) = \frac{i\sqrt{2}H_0}{M_{\text{Pl}}\sqrt{2k^3}} k^2 \eta e^{-ik\eta}. \quad (4.14)$$

The power spectra can be easily arrived at from the above scalar and tensor modes. At late times, i.e. as  $\eta \rightarrow 0$ , the scalar power spectrum can be obtained to be

$$\begin{aligned} \mathcal{P}_s(k) &= \left(\frac{H_0}{2\pi}\right)^2 \left(\frac{3H_0^2}{A_-}\right)^2 |\alpha_k - \beta_k|^2 \\ &= \left(\frac{H_0}{2\pi}\right)^2 \left(\frac{3H_0^2}{A_-}\right)^2 \left[ \mathcal{I}(k) + \mathcal{I}_c(k) \cos\left(\frac{2k}{k_0}\right) + \mathcal{I}_s(k) \sin\left(\frac{2k}{k_0}\right) \right], \end{aligned} \quad (4.15)$$

where the quantities  $\mathcal{I}(k)$ ,  $\mathcal{I}_c(k)$  and  $\mathcal{I}_s(k)$  are given by

$$\mathcal{I}(k) = 1 + \frac{9}{2} \left(\frac{\Delta A}{A_+}\right)^2 \left(\frac{k_0}{k}\right)^2 + 9 \left(\frac{\Delta A}{A_+}\right)^2 \left(\frac{k_0}{k}\right)^4 + \frac{9}{2} \left(\frac{\Delta A}{A_+}\right)^2 \left(\frac{k_0}{k}\right)^6, \quad (4.16a)$$

$$\mathcal{I}_c(k) = \frac{3\Delta A}{2A_+} \left(\frac{k_0}{k}\right)^2 \left[ \left(\frac{3A_-}{A_+} - 7\right) - \frac{3\Delta A}{A_+} \left(\frac{k_0}{k}\right)^4 \right], \quad (4.16b)$$

$$\mathcal{I}_s(k) = -\frac{3\Delta A}{A_+} \frac{k_0}{k} \left[ 1 + \left(\frac{3A_-}{A_+} - 4\right) \left(\frac{k_0}{k}\right)^2 + \frac{3\Delta A}{A_+} \left(\frac{k_0}{k}\right)^4 \right]. \quad (4.16c)$$

The above scalar power spectrum exhibits a step-like feature with two nearly scale invariant rungs (that correspond to the two domains of slow roll) and a burst of oscillations (associated with the modes that leave the Hubble radius during the period of fast roll) connecting the two rungs (in this context, see, for instance, Fig. 3 of Ref. [33]). Since the tensor modes are given by the standard Bunch-Davies solution, the resulting tensor spectrum is strictly scale invariant, with the amplitude being given by

$$\mathcal{P}_T(k) = \frac{2H_0^2}{\pi^2 M_{\text{Pl}}^2}. \quad (4.17)$$

As in the power law case discussed in the previous sub-section, in order to establish the consistency relations, we first need to evaluate the scalar and tensor spectral indices from the above power spectra. We then need to obtain the non-Gaussianity parameters  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\mathcal{I}}$  and  $h_{\text{NL}}$  using the consistency conditions (3.4), and compare them with the non-Gaussianity parameters evaluated from the squeezed limit of the corresponding three-point functions.

The scalar spectral index  $n_s$  corresponding to the above scalar power spectrum can be obtained to be

$$\begin{aligned} n_s(k) &= \frac{1}{2} \left[ \mathcal{I}(k) + \mathcal{I}_c(k) \cos\left(\frac{2k}{k_0}\right) + \mathcal{I}_s(k) \sin\left(\frac{2k}{k_0}\right) \right]^{-1} \left\{ 8\mathcal{I}(k) - 3\mathcal{J}(k) \right. \\ &\quad \left. + [8\mathcal{I}_c(k) - 3\mathcal{J}_c(k)] \cos\left(\frac{2k}{k_0}\right) + [8\mathcal{I}_s(k) - 3\mathcal{J}_s(k)] \sin\left(\frac{2k}{k_0}\right) \right\}, \end{aligned} \quad (4.18)$$



where  $\mathcal{J}(k)$ ,  $\mathcal{J}_c(k)$  and  $\mathcal{J}_s(k)$  are given by

$$\mathcal{J}(k) = 2 + 15 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^2 + 42 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^4 + 27 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^6, \quad (4.19a)$$

$$\mathcal{J}_c(k) = \frac{\Delta A}{A_+} \left[ 4 + 3 \left( \frac{9A_-}{A_+} - 17 \right) \left( \frac{k_0}{k} \right)^2 + \frac{12\Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 - \frac{27\Delta A}{A_+} \left( \frac{k_0}{k} \right)^6 \right], \quad (4.19b)$$

$$\mathcal{J}_s(k) = \frac{2\Delta A}{A_+} \frac{k_0}{k} \left[ \left( \frac{3A_-}{A_+} - 11 \right) - 6 \left( \frac{3A_-}{A_+} - 4 \right) \left( \frac{k_0}{k} \right)^2 - \frac{27\Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 \right]. \quad (4.19c)$$

From this expression for  $n_s$ , upon suitably ignoring overall factors containing the polarization tensor, we can obtain the non-Gaussianity parameter  $C_{\text{NL}}^{\mathcal{R}}$  to be

$$C_{\text{NL}}^{\mathcal{R}}(k) = \frac{n_s(k) - 4}{4} = -\frac{3}{8} \frac{\mathcal{J}(k) + \mathcal{J}_c(k) \cos(2k/k_0) + \mathcal{J}_s(k) \sin(2k/k_0)}{\mathcal{I}(k) + \mathcal{I}_c(k) \cos(2k/k_0) + \mathcal{I}_s(k) \sin(2k/k_0)}. \quad (4.20)$$

We shall require the tensor spectral index  $n_T$  in order to evaluate the other two non-Gaussianity parameters  $C_{\text{NL}}^\gamma$  and  $h_{\text{NL}}$  using the consistency relations (3.4). However, since the tensor power spectrum (4.17) is strictly scale invariant at the level of approximation we are working with, the corresponding spectral index  $n_T$  vanishes identically. Moreover, note that since the tensor modes remain unaffected by the transition, the tensor bi-spectrum will be of the same form as in the de Sitter case, a situation wherein it is easy to establish analytically that  $h_{\text{NL}} = -3/8$  in the squeezed limit (see, for instance, Refs. [11, 17, 21]). In order to establish the consistency relation for the parameter  $C_{\text{NL}}^\gamma$ , we shall evaluate the tensor spectral index numerically, and compare the result with the analytical expressions that we shall obtain from the Maldacena formalism for the three-point functions in the squeezed limit.

The scalar-scalar-tensor cross-correlation in the Starobinsky model can be calculated analytically by dividing the integrals involved into two parts, corresponding to the epochs before and after the transition, and making use of the above expressions for the first slow roll parameter and the scalar and the tensor modes. In the squeezed limit of the tensor mode, on ignoring the polarization tensors, we find that the scalar-scalar-tensor three-point function can be written as

$$\lim_{k_3 \rightarrow 0} k^3 k_3^3 G_{\mathcal{R}\mathcal{R}\gamma}^{m_3 n_3}(\mathbf{k}, -\mathbf{k}, \mathbf{k}_3) = \frac{27 H_0^8}{8 M_{\text{Pl}}^2 A_-^2} \left[ \mathcal{J}(k) + \mathcal{J}_c(k) \cos\left(\frac{2k}{k_0}\right) + \mathcal{J}_s(k) \sin\left(\frac{2k}{k_0}\right) \right], \quad (4.21)$$

with  $\mathcal{J}(k)$ ,  $\mathcal{J}_c(k)$  and  $\mathcal{J}_s(k)$  being given by Eqs. (4.19). Upon making use of this expression and the power spectra (4.15) and (4.17) in the definition (2.12b) of the parameter  $C_{\text{NL}}^{\mathcal{R}}$  (and suitably ignoring the factors involving the polarization tensors), we find that one exactly arrives at the result (4.20), thereby establishing the consistency relation for this case. Similarly, in the squeezed limit of the scalar mode, scalar-tensor-tensor correlation can be obtained to be

$$\lim_{k_1 \rightarrow 0} k_1^3 k^3 G_{\mathcal{R}\gamma\gamma}^{m_2 n_2 m_3 n_3}(\mathbf{k}_1, \mathbf{k}, -\mathbf{k}) = \frac{H_0^4}{8 M_{\text{Pl}}^4} \left[ \mathcal{K}(k) + \mathcal{K}_c(k) \cos\left(\frac{2k}{k_0}\right) + \mathcal{K}_s(k) \sin\left(\frac{2k}{k_0}\right) \right], \quad (4.22)$$

where the quantities  $\mathcal{K}(k)$ ,  $\mathcal{K}_c(k)$  and  $\mathcal{K}_s(k)$  are given by

$$\mathcal{K}(k) = 4 \left( \frac{A_-}{A_+} \right)^2 + 9 \left( \frac{\Delta A}{A_+} \right)^2 \left( \frac{k_0}{k} \right)^6, \quad (4.23a)$$

$$\mathcal{K}_c(k) = \frac{3 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^2 \left[ 2 + 6 \frac{\Delta A}{A_+} \left( \frac{k_0}{k} \right)^2 - \frac{3 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^4 \right], \quad (4.23b)$$

$$\mathcal{K}_s(k) = \frac{3 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^3 \left[ \left( \frac{3 A_-}{A_+} - 4 \right) - \frac{6 \Delta A}{A_+} \left( \frac{k_0}{k} \right)^2 \right]. \quad (4.23c)$$

In Fig. 1, we have plotted the  $C_{\text{NL}}^\gamma$  that results from the above analytical expression for the scalar-tensor-tensor cross-correlation and the power spectra (4.15) and (4.17). In the same figure, we have also plotted the  $C_{\text{NL}}^\gamma$  that arises from the numerical determination of the tensor spectral index and the consistency condition (3.4c). It is clear from the figure that these two quantities match very well, indicating the fact that the consistency relation is valid in this case as well.

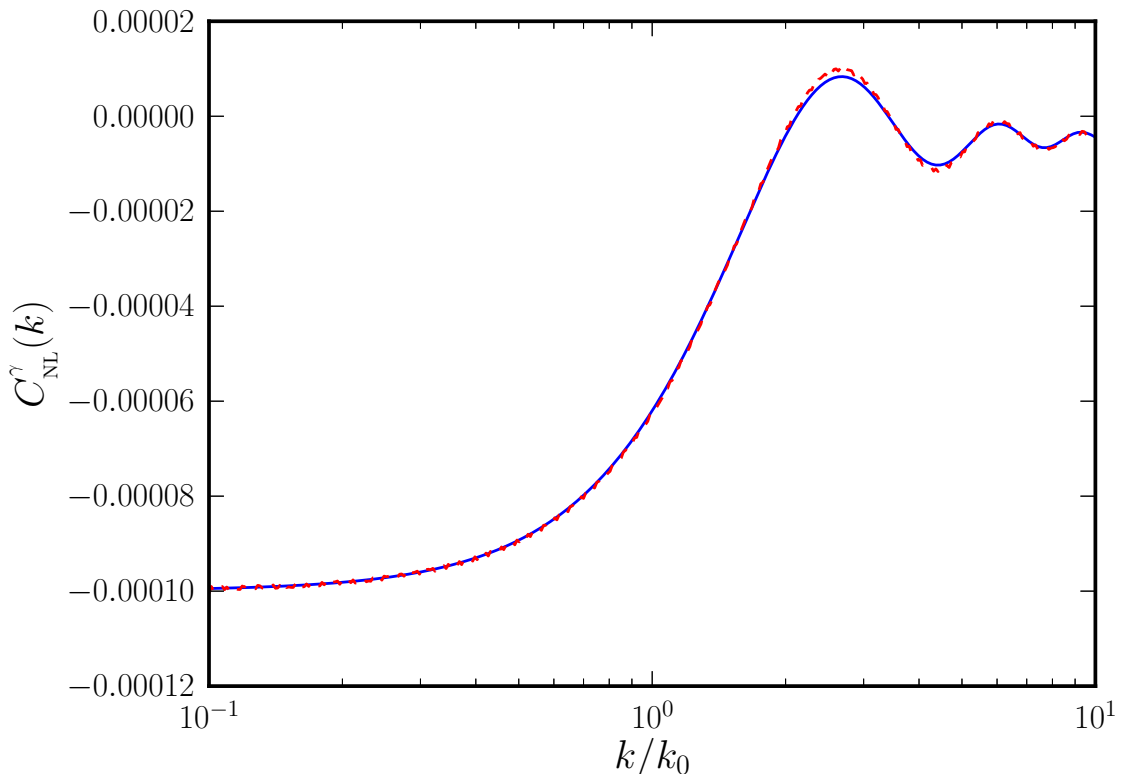
## 5 Numerical investigation of scenarios involving deviations from slow roll

In the last section, we had investigated the validity of the consistency relations comprising of the tensor perturbations in two specific situations that had proved to be analytically tractable. It is well known that the scalar consistency relation is valid in slow roll inflation [11, 24, 25], and its applicability in situations containing departures from slow roll has also been confirmed in a few instances (for analytical examples, see Refs. [36] and, for some numerical results, see Refs. [37]). Our general arguments in Sec. 3 as well as the analysis of the Starobinsky model in the previous section suggest that the consistency relations involving tensors too can be expected to be valid even in scenarios consisting of deviations from slow roll. It will be interesting to explicitly examine these relations in different models containing brief periods of fast roll.

While a nearly scale invariant inflationary scalar power spectrum is quite consistent with the CMB and other cosmological data [8, 9], it has been repeatedly noticed that certain features in the primordial scalar power spectrum fit the data better (for recent discussions, see, for instance, Refs. [38]). Typically, one finds that one or more of the following features lead to an improved fit to the data: (i) a sharp drop in power on large scales (see Refs. [39]; for some recent discussions in this context, see Refs. [40]), (ii) a short burst of oscillations over an intermediate range of scales [41–45], and (iii) small and persisting oscillations that extend over a wide range of scales [46–52]. In fact, it is exactly these three type of possibilities that have been considered by the Planck team in their analysis of probable features in the scalar power spectrum [7]. In this section, we shall consider three inflationary models that lead to features of the different types mentioned above and investigate numerically whether the consistency relations of our interest remain valid in these non-trivial situations as well.

The three inflationary models that we shall consider are as follows. The first example that we shall focus on is the inflationary model governed by the potential

$$V(\phi) = \frac{m^2}{2} \phi^2 - \frac{\sqrt{2 \lambda (n-1)} m}{n} \phi^n + \frac{\lambda}{4} \phi^{2(n-1)}, \quad (5.1)$$



**Figure 1.** The non-Gaussianity parameter  $C_{\text{NL}}^\gamma$  in the Starobinsky model, evaluated in the squeezed limit, has been plotted as a function of  $k/k_0$ . The solid blue curve represents the parameter arrived at from the analytical results for the scalar-tensor-tensor cross-correlation (obtained using the Maldacena formalism) and the scalar and the tensor power spectra. The dashed red curve corresponds to the non-Gaussianity parameter obtained from consistency condition (3.4c), with the tensor spectral index being determined numerically. Evidently, there is good agreement between the two results, indicating that the consistency relation holds even when departures from slow roll occur. Note that we have worked with the following values of the potential parameters in arriving at these results:  $\phi_0/M_{\text{Pl}} = 0.707$ ,  $V_0/M_{\text{Pl}}^4 = 2.37 \times 10^{-12}$ ,  $A_+/M_{\text{Pl}}^3 = 3.35 \times 10^{-14}$  and  $A_-/M_{\text{Pl}}^3 = 7.26 \times 10^{-15}$ . These values have been chosen so that the assumptions of the Starobinsky model, under which the analytical results have been arrived at, are valid (in this context, see, for instance, Ref. [33]).

which contains a point of inflection at

$$\phi_0 = \left[ \frac{2m^2}{(n-1)\lambda} \right]^{\frac{1}{2(n-2)}}. \quad (5.2)$$

For certain values of the parameters, this model permits two epochs of slow roll inflation, with a brief period of departure from inflation sandwiched in between, a scenario that has been dubbed punctuated inflation [39]. The sudden deviation from inflation and the rapid return to slow roll results in a steep drop in the scalar power on large scales, which can help fit the lower power seen at the large angular scales in the CMB. Notably, these models predict a rise in the tensor power over scales wherein the scalar power drop sharply. The second model that we shall consider is the quadratic potential containing a step that has

been introduced by hand. The complete potential is given by the expression [41–43]

$$V(\phi) = \frac{m^2}{2} \phi^2 \left[ 1 + \alpha \tanh \left( \frac{\phi - \phi_0}{\Delta\phi} \right) \right], \quad (5.3)$$

where, clearly,  $\alpha$  and  $\Delta\phi$  denote the height and the width of the step, while  $\phi_0$  represents its location. The step in the potential leads to a brief period of fast roll which, in turn, leads to a burst of oscillations in the power spectrum. These oscillations have been shown to improve the fit to the CMB data around the multipoles of  $\ell = 20$ –50. The last case that we shall consider is a potential motivated by string theory known as axion monodromy model. The model is described by the following potential [48, 50, 51]:

$$V(\phi) = \lambda \left[ \phi + \alpha \cos \left( \frac{\phi}{\beta} + \delta \right) \right]. \quad (5.4)$$

The continued oscillations in the potential lead to persistent oscillations in the inflationary scalar power spectrum. (For an illustration of the scalar power spectra that arise in these three models, we would refer the reader to Fig. 9 of Ref. [31].)

We shall now numerically examine the validity of the consistency relations involving the tensor perturbations in the above-mentioned models. As we had mentioned, in an earlier work, we had developed a numerical procedure and had constructed a Fortran code for evaluating the three-point scalar-tensor cross-correlations and the tensor bi-spectrum [21]. We shall make use of the code to compute the three-point functions in the squeezed limit as well as the scalar and the tensor spectral indices, in order to check the consistency conditions (3.4). We shall work with values of the parameters for the potentials that have been shown to lead to an improved fit to the CMB data (in this context, see Refs. [21, 31]). While evaluating the scalar and tensor power spectra, it has long been known that, in order to arrive at spectra with good accuracy, it is sufficient to numerically integrate the modes from a time when they are well inside the Hubble radius [say, when  $k/(aH) \simeq 10^2$ ] to a time when they are suitably outside [say, when  $k/(aH) \simeq 10^{-5}$ ] [53]. It has been shown that, while evaluating the three-point functions in the Maldacena formalism, it suffices to carry out the numerical calculations *roughly* over a similar time domain [21, 31, 54, 55]. However, there are three points that we need to emphasize in this regard. Firstly, to achieve higher levels of numerical accuracy, say, of the order of 1–3% or better, for the three-point functions, one may have to integrate from a time when the modes are deeper inside the Hubble radius than  $k/(aH) \simeq 10^2$ . In our calculations, we shall choose to integrate from  $k/(aH) \simeq 10^3$  for the punctuated inflation model (5.1) and the quadratic potential with a step (5.3). The oscillating nature of the potential (5.4) in the axion monodromy model leads to certain resonant behavior (see the first of the references in Ref. [36] and Refs. [21, 31]), and it typically requires one to integrate from further deep inside the Hubble radius, even in the case of the power spectrum. For this reason, we shall choose to integrate from  $k/(aH) \simeq 10^4$  in this case. Secondly, due to the rapid oscillations of the modes when they are inside the Hubble radius, a cut off in the integrands is required to regulate the integrals at early times<sup>5</sup>. We shall work with a cut off of the form  $\exp[-\kappa k/(aH)]$ , where  $\kappa$  is a parameter that has to be chosen according to the initial time from which the integrations are to be carried out. For instance, the earlier the initial time, the smaller the quantity  $\kappa$  has to be [21, 31, 54, 55]. Since, we shall integrate from  $k/(aH) \simeq 10^3$  in the cases of punctuated inflation and the quadratic

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<sup>5</sup>In theory, such a cut-off is mandatory to identify the correct perturbative vacuum [11, 12, 33].

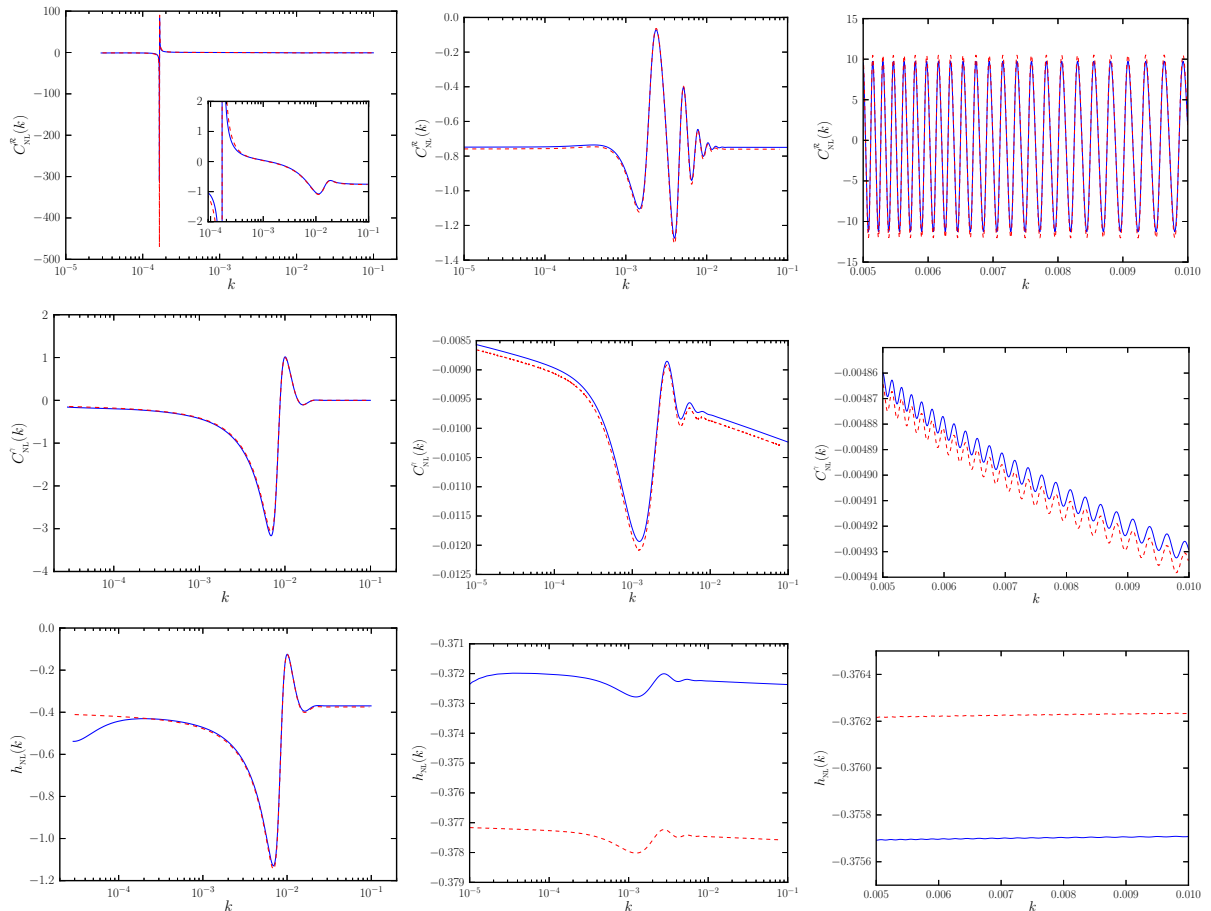
potential with a step, we shall work with  $\kappa = 1/50$  in these cases, which is known to lead to a good accuracy [21, 31]. However, as we integrate from deeper inside the Hubble radius in the axion monodromy model, we shall work with  $\kappa = 1/500$  in this case. The third and the last point concerns the implementation of the squeezed limit. To achieve this limit, we shall work with the smallest wavenumber (say, the largest scale mode that leaves the Hubble radius at the earliest possible time) that is numerically tenable. As a result, inherently, there will arise a weak wavenumber dependent effect when attempting to establish the consistency conditions numerically, with the small scale modes satisfying the condition better than the longer ones.

In Fig. 2, we have plotted the numerical results for the non-Gaussianity parameters  $C_{\text{NL}}^{\mathcal{R}}$ ,  $C_{\text{NL}}^{\gamma}$  and  $h_{\text{NL}}$  for the three models discussed above. We find that the results from the spectral indices match the numerical results for the non-Gaussianity parameters obtained using Maldacena formalism at the level of 3% or better in the cases of punctuated inflation and the quadratic potential with the step, with the largest differences arising for the smallest wavenumbers for the reasons discussed above. The match is slightly poorer in the axion monodromy model (with differences of about 7% for some wavenumbers), but the match improves if we carry out the integrals over a longer duration in time. (We should mention here that the seeming difference in the cases of the model with the step and the axion monodromy model in the last row of the figure is due to the fact that the non-Gaussianity parameter  $h_{\text{NL}}$  has been plotted over a rather small range in amplitude to highlight the mild variations.) Clearly, the consistency relations hold true even in inflationary models that contain deviations from slow roll inflation.

## 6 Discussion

Consistency relations link the three-point functions in the squeezed limit to the scalar and tensor power spectra. They essentially indicate that, in the squeezed limit, the three-point functions carry the same extent of information as the two-point functions do. Evidently, the consistency conditions can be conveniently expressed as relations between the non-Gaussianity parameters (two of which we had recently introduced [21]) and the scalar or the tensor spectral indices. The consistency relations arise essentially because of the fact that inflaton is the only clock in single field inflationary models. Due to this reason, the amplitude of the long wavelength modes freeze in such situations. As we had discussed in Sec. 3, this implies that, in the squeezed limit of interest, the long wavelength modes simply modify the background spatial coordinates. In this work, we have explicitly examined, both analytically and numerically, the validity of consistency relations involving the tensor perturbations in single field inflationary models. Corroborating the general arguments that have been outlined earlier, we find that the consistency conditions hold true even in non-trivial scenarios involving drastic deviations from slow roll, such as it occurs in the case of punctuated inflation.

It should be evident that the consistency conditions provide a remarkable tool to observationally confirm if inflation was driven by a single scalar field. Clearly, any observed deviation from the conditions would point to other scenarios such as inflation driven by multiple scalar fields or even exotic possibilities such as the ekpyrotic scenario. It would be worthwhile to repeat some of our analyses for non-canonical scalar fields, and then go to study the extent of deviations from the consistency conditions that arise in inflationary



**Figure 2.** The non-Gaussianity parameters,  $C_{NL}^R$  (the top row),  $C_{NL}^\gamma$  (the middle row) and  $h_{NL}$  (the bottom row), arrived at using the Maldacena formalism and from the scalar and tensor spectral indices through the consistency conditions, have been plotted as a function of the wavenumber for the three inflationary models of our interest, viz. punctuated inflation (the left column), the quadratic potential with a step (the middle column) and the axion monodromy model (the right column). While the solid blue lines correspond to the numerical results for the parameters obtained using the Maldacena formalism, the dashed red lines represent the values arrived at from the spectral indices and the consistency relations. The two results match at the level of 3% or better in the cases of punctuated inflation and the quadratic potential with the step, with the largest differences arising for the smallest wavenumbers due to the inherent limitation in implementing the squeezed limit numerically. Though the match is slightly poorer in the axion monodromy model (with differences of the order of 7% for certain wavenumbers), we find that the match can be improved by integrating for a longer duration. At a first look, the difference in the cases of the model with the step and the axion monodromy model in the last row may seem striking. But, we should clarify here that it is simply due to the fact that the non-Gaussianity parameter  $h_{NL}$  has been plotted in these cases over a rather small range in amplitude to highlight the mild variations. Needless to add, these figures confirm the validity of the consistency relations in single field inflationary models even in situations that allow strong departures from slow roll.

models comprising of more than one scalar field. We are currently investigating some of these issues.

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