Deterministic Constructions for Large Girth Protograph LDPC Codes

Asit Kumar Pradhan
Dept. of Electrical Engg.
IIT Madras, Chennai, India
Email: ee11m004@ee.iitm.ac.in

Arunkumar Subramanian
Link-A-Media Devices Corp.
Santa Clara, CA, USA

Andrew Thangaraj
Dept. of Electrical Engg.
IIT Madras, Chennai,India
Email: andrew@ee.iitm.ac.in

***Abstract*—For certain degree-distribution pairs with non-zero fraction of degree-two bit nodes, the bit-error threshold of the standard ensemble of Low Density Parity Check (LDPC) codes is known to be close to capacity. However, the degree-two bit nodes preclude the possibility of a block-error threshold. Interestingly, LDPC codes constructed using protographs allow the possibility of having both degree-two bit nodes and a block-error threshold. In this paper, we analyze density evolution for protograph LDPC codes over the binary erasure channel and show that their bit-error probability decreases double exponentially with the number of iterations when the erasure probability is below the bit-error threshold and long chain of degree-two variable nodes are avoided in the protograph. We present deterministic constructions of such protograph LDPC codes with girth logarithmic in blocklength, resulting in an exponential fall in bit-error probability below the threshold. We provide optimized protographs, whose block-error thresholds are better than that of the standard ensemble with minimum bit-node degree three. These protograph LDPC codes are theoretically of great interest, and have applications, for instance, in coding with strong secrecy over wiretap channels.**

I. Introduction

Constructing a sequence of codes with efficient encoders/decoders and a guarantee that block-error rate tends to zero with increasing block-length is one of the major goals of coding theory. At rates below capacity, such "good" sequences are known to exist, but many classical code sequences do not have this property. Modern code constructions, such as Low Density Parity Check (LDPC) codes, define a sequence of ensembles of codes with efficient decoders and probabilistic concentration results that come close to achieving the goal of constructing good code sequences [1]. Recently, polarization [2] and spatial coupling [3] have been used to construct good code sequences for binary symmetric channels.

In this work, we are primarily interested in deterministic constructions of sequences of good LDPC codes with block-error thresholds nearing capacity limits. We will stick to the binary erasure channel, though the work can be extended to other binary-input symmetric channels. Most of the prior work in this area provides probabilistic guarantees on ensembles of LDPC codes, and most of these guarantees are for bit-error probabilities. The block-error threshold problem for LDPC codes was first studied in [4], where standard ensembles with a minimum bit-node degree (denoted $l_{\min}$) of three was shown to have block-error thresholds. For the standard irregular ensemble with $l_{\min} = 2$, the block-error rate, surprisingly, tends to a constant as block-length increases. The main cause for this problem is the presence of long chains of degree-two nodes in the standard ensemble. However, (bit-error) capacity-approaching LDPC degree distributions have a significant fraction of degree-two bit nodes. For instance, the best threshold for rate-1/2 codes with minimum left degree three is only about 0.461 leaving a significant gap to the capacity threshold of 0.5. So, while degree-two nodes are needed to approach capacity, they preclude the possibility of a block-error threshold. One of the goals of this work is to construct LDPC codes with block-error thresholds that improve this gap to capacity.

A key idea in the construction of LDPC code ensembles with degree-two nodes and decaying block-error performance is the notion of multi-edge type (MET) ensembles [1], [5], of which the protograph LDPC code ensemble [6] has received considerable practical attention because of ease of implementation. In [5], the standard ensemble is restricted in a suitable fashion to limit the impact of degree-two nodes. In [6], density evolution and optimization for protograph LDPC code ensembles was described and carried out. In [7], protographs are optimized for thresholds nearing capacity, and linear growth of ensemble-averaged weight distribution is established for protograph LDPC code ensembles. There have been numerous other work in the construction of protographs in practical implementations.

The use of large-girth graphs in constructing LDPC codes started with Gallager's thesis [8], where regular LDPC codes with large girth were constructed. The Lubotzky-Phillips-Sarnak (LPS) constrction [9] of Ramanujan graphs has been used in the construction of regular and irregular LDPC codes in [10]. As shown in [11], large-girth LDPC codes with minimum left degree, $l_{\min} > 2$, achieve an exponential decay of bit error, i.e $\mathcal{O}(\exp(-c_1 n^{c_2}))$ for constants c_1, c_2, over a binary erasure channel BEC(ϵ), when ϵ is less than the density evolution threshold ϵ^*. So, large-girth LDPC codes have a block-error threshold equal to their bit-error thresholds, when $l_{\min} > 2$.

In this work, we provide deterministic constructions for a sequence of good LDPC codes by using large-girth graphs along with suitable protographs that contain degree-two nodes. This allows us to achieve BEC block-error thresholds as high as 0.486 with small (8×16) protographs. To do this, we begin by studying the density evolution for protograph ensembles, and show that bit-error decreases double exponentially in number of iterations at erasure probabilities smaller than the threshold even when $l_{\min} = 2$, if long chains of degree-two variable nodes are avoided through the protograph. To avoid chains of degree-two nodes, we allow at most one degree-two variable node to connect to a check node in the protograph.

We then provide a construction for large-girth protograph LDPC codes starting with a given large-girth regular graph and performing suitable node splitting operations. Using the LPS Ramanujan graphs, we provide deterministic constructions of large-girth protograph LDPC codes that achieve an exponential decay for bit-error probability with blocklength and still contain degree-two bit nodes. In comparison with prior work, we have analyzed the density evolution for protograph LDPC codes directly and showed the double-exponential decay with iterations even in the presence of degree-two nodes. Our node splitting construction is more general than the one in [10] and provides a deterministic construction with guaranteed block-error probability behavior.

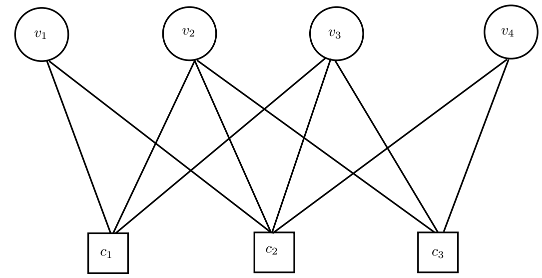
II. Protograph LDPC Codes

Following the notation in [6], a protograph $G = (V, C, E)$ is a bipartite graph with V and C being the sets of variable and check nodes, respectively, and E being the set of undirected edges that connect a vertex in V to a vertex in C. Multiple edges are allowed between a pair of nodes $(v, c) \in V \times C$.

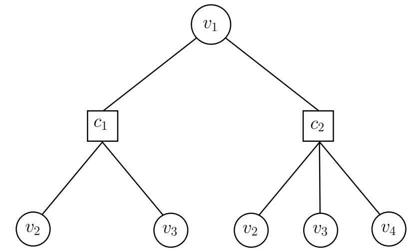
A protograph can be represented by a base matrix B, where $B(i, j)$ is the number of edges between the i-th check node (denoted c_i) and the j-th variable node (denoted v_j). For example, consider a base matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The protograph corresponding to the above base matrix is shown in Fig. 1(a).



(a) Example of a protograph.



(b) Computation graph $C_1(v_1)$.

Fig. 1. Protograph and computation graph.

A. Lifted Graphs

We can apply a copy and permute operation to a protograph to obtain expanded or lifted graphs of different sizes [6]. A given protograph G is copied, say T times, with the t-th copy having nodes (v, t) and (c, t), and edges (e, t). Then, for each edge e in the protograph, we assign a permutation π_e of the set $\{1, 2, \cdots, T\}$. In the permute operation, an edge (e, t) connecting (v, t) and (c, t) is permuted so as to connect variable node (v, t) to check node $(c, \pi_e(t))$. We will denote the lifted graph as $G' = (V', C', E')$. The lifted graph of a protograph can be thought of as a Tanner graph of an LDPC code, which is referred to as a protograph LDPC code. In general, $(T!)^{|E|}$ lifted graphs or protograph LDPC codes can be obtained from a protograph, each corresponding to a different permutation, where $|V|$ is the number of variable nodes in the protograph. The collection of these lifted graphs is called the protograph ensemble of LDPC codes. Protograph LDPC codes are a special class of MET-LDPC codes, with each edge in the protograph being of a different type. The degree distribution of check and variable nodes in the lifted graph is the same as that of the protograph. So, the (designed) rate of the protograph LDPC code is given by $1 - \frac{|C|}{|V|}$, where $|C|$ and $|V|$ denote the number of check and variable nodes in the protograph, respectively.

Let v be a variable node in the lifted graph G'. The t-iteration computation graph associated with v, denoted $C_t(v)$, is defined as the subgraph of G' obtained by traversing from v up to the t-th iteration level along all edges [1]. The structure of the computation graph is completely determined by the protograph G for all lifted graphs G'. An example of a computation graph is shown in Fig. 1(b).

B. Density Evolution for Protograph Codes

Let us consider the standard message-passing decoder over a binary erasure channel (BEC(ϵ)) run on a lifted graph G' derived from a protograph $G = (V, C, E)$. Since the lifted graphs form an MET ensemble with $|E|$ edge types, density evolution proceeds with $|E|$ erasure probabilities, one for each edge in the protograph [12]. Let $E = \{e_1, e_2, \ldots, e_{|E|}\}$ with edge $e \in E$ connecting variable node v_e with check node c_e. Let $x_t(i)$ be the probability that an erasure is sent from variable node to check node along edge e_i in the t-th iteration. Similarly, let $y_t(j)$ be the probability that an erasure is sent from check node to variable node along edge e_j in the t-th iteration. The density evolution recursion [1] is given by

$$x_0(i) = \epsilon,$$

$$y_{t+1}(j) = 1 - \prod_{i \in E_c(e_j)} (1 - x_t(i)), \ \forall t \geq 1, \qquad (1)$$

$$x_{t+1}(i) = \epsilon \prod_{j \in E_v(e_i)} y_{t+1}(j), \ \forall t \geq 1, \quad (2)$$

where $E_c(e_j) = \{i \neq j : c_{e_j} = c_{e_i}\}$ is the set of all indices of edges adjacent to the same check node as the edge e_j, and $E_v(e_i) = \{j \neq i : v_{e_i} = v_{e_j}\}$ is the set of all indices of edges adjacent to the same variable node as the edge e_i. The density evolution threshold, denoted ϵ_{th}, for the protograph-based LDPC code ensemble is defined as the largest value of ϵ for which erasure probability on each edge of the protograph tends to zero, as $t \to \infty$. i.e. $\epsilon_{\text{th}} = \sup\{\epsilon : \max_i x_t(i) \to 0\}$. Clearly, this is also the threshold value below which the erasure probability of a variable node in the protograph goes to zero with increasing iterations (as long as $l_{\min} > 1$).

C. Asymptotic Behavior of Density Evolution

Theorem 1. *For $\epsilon < \epsilon_{th}$, $\max_i x_t(i)$ exhibits a double-exponential decay with t when not more than one degree-two variable node is connected to a check node in the protograph.*

Proof: We will repeatedly use the following inequality. For any $x \in [0, 1]$ and a positive integer d,

$$(d-1)x \geq 1 - (1-x)^{d-1}. \quad (3)$$

Let d be the maximum degree of a check node in the protograph, and let $\bar{x}_t = \max_i x_t(i)$. Since $\epsilon < \epsilon_{\text{th}}$, we have $\bar{x}_t \to 0$. We pick t large enough to have $0 \leq (d-1)\bar{x}_t < 1$.

We note that

$$1 - x_t(j) \geq 1 - \bar{x}_t, \quad \forall j$$
$$\Rightarrow \prod_{j \in E_c(e_i)} (1 - x_t(j)) \geq (1 - \bar{x}_t)^{(d-1)}, \quad \forall i$$

Using (1) and (3), we get

$$y_{t+1}(j) \leq (d-1)\bar{x}_t, \quad \forall j \quad (4)$$
$$\Rightarrow x_{t+1}(i) = \epsilon \prod_{e_j \in E_v(e_i)} y_{t+1}(j)$$
$$\leq \epsilon (d-1)^{(l_m - 1)} (\bar{x}_t)^{(l_m-1)}, \quad \forall i \quad (5)$$

Repeating the process, we get

$$y_{t+2}(j') \leq (d-1) x_{t+1}(i) \leq \epsilon (d-1)^{l_m} (\bar{x}_t)^{(l_m-1)}$$
$$x_{t+2}(i') \leq \epsilon^{l'_m} (d-1)^{l_m(l'_m-1)} (\bar{x}_t)^{(l_m-1)(l'_m-1)}$$
$$\Rightarrow \bar{x}_{t+2} \leq \epsilon^{l'_m} (d-1)^{l_m(l'_m-1)} (\bar{x}_t)^{(l_m-1)(l'_m-1)}$$

where l_m and l'_m are the minimum degrees of variable nodes in $E_v(e_i)$ and $E_v(e_{i'})$, respectively. Since a check node is connected to at most one degree-two variable node, we have that either $l_m \geq 3$ or $l'_m \geq 3$. So, $(l_m - 1)(l'_m - 1) \geq 2$. So, we have

$$\bar{x}_{t+2} \leq A(\bar{x}_t)^2, \quad (6)$$

where $A = \epsilon^{l'_m}(d-1)^{l_m(l'_m-1)}$ does not vary with t. Since $\epsilon < \epsilon_{\text{th}}$ and $\bar{x}_t \to 0$, there exists an R such that $A(\bar{x}_R)^2 < 1$ and $(d-1)\bar{x}_R < 1$. Following arguments similar to those in [11], we can show that

$$\bar{x}_{R+2i} \leq (A\bar{x}_R)^{2^i}, \quad (7)$$

which implies a double-exponential decay of $\bar{x}_t$ with t (for sufficiently large t). ■

For the standard ensemble of LDPC codes, density evolution analysis is approximate because of the following assumptions:

1. The computation graph is a tree.
2. The node degrees in the computation graph are independent.

Typically, probabilistic concentration results and asymptotic guarantees are used to support the practical validity of density evolution in the standard ensemble. In this work, we use protograph codes and MET density evolution — these make Assumption 2 above unnecessary. For Assumption 1 to be true, we consider large-girth graphs.

III. Large-Girth Protograph LDPC Codes

To achieve strong secrecy, Tanner graphs whose girth increases as $\log n$ with blocklength n have been used in [11]. In this work, we extend the construction in [11] to construct bipartite graphs from protographs with girth increasing as $\log n$. Once girth is $\Theta(\log n)$, message error rates for iterations up to $\Theta(\log n)$ will exactly follow the protograph density evolution of (1)- (2). Following the analysis in Section II-C, the message error rate falls double-exponentially in $\log n$, or exponentially in block-length n. This results in an inverse polynomial decay, $\mathcal{O}(1/n^k)$ for any k, for block-error rate.

A. LPS Graphs $X^{p,q}$

While the construction method can use any sequence of regular large-girth graphs, one explicit possibility is the LPS construction [9]. LPS graphs belong to the class of Cayley graphs. Given a group G and an inverse-closed subset S of G, i.e, $s^{-1} \in S, \forall s \in S$, the Cayley graph $(\Gamma(G, S))$ is the undirected simple graph defined as follows:

- The vertex set of $\Gamma(G, S)$ is G.
- For any $g \in G$ and $s \in S$, there is an edge between g and gs.

Let p and q be distinct, odd primes with $q > 2\sqrt{p}$. The LPS graph, denoted $X^{p,q}$ [9], is a connected, $(p+1)$-regular graph and has the following properties:

- If p is a quadratic residue $\mod q$, then $X^{p,q}$ is a non-bipartite graph with $q(q^2-1)/2$ vertices and girth $g(X^{p,q}) \geq 2\log_p q$.
- If p is a quadratic non-residue $\mod q$, then $X^{p,q}$ is a bipartite graph with $q(q^2-1)$ vertices and girth $g(X^{p,q}) \geq 4\log_p q - \log_p 4$.

When $X^{p,q}$ is non-bipartite, we can convert it to a bipartite graph using the following algorithm [11]:

- Given a graph G with vertices $V(G)$ and edges $E(G)$, construct a copy G' with a new vertex set $V(G')$ and a new edge set $E(G')$. Let $f : V(G) \to V(G')$ be the 1-1 mapping from a vertex in G to its copy in G'.

- Create a bipartite graph H with vertex set $V(G) \cup V(G')$ and edge set $E(H) = \{(x, f(y)) : (x, y) \in E(G)\}$.

Following [13], it was shown in [11] that $g(H) \geq g(G)$.

For constructing a sequence of d-regular large-girth graphs for an arbitrary d using the LPS graphs, we use the following trick from [11]. There exists an infinite number of primes p such that $d|(p+1)$. For each such prime p and a suitable q, we construct $X^{p,q}$ and split each $(p+1)$-degree node into $(p+1)/d$ nodes of degree d. As shown in [11], the node splitting does not reduce girth and we have a large-girth graph of the required degree d.

B. Node Splitting for MET-LDPC Codes

The construction of a large-girth protograph LDPC code starts with a d-regular large-girth bipartite graph G with d being the number of edges in the protograph. The bipartition of G will contain $|V(G)|/2$ left and right vertices. We associate d sockets with each vertex of G, and associate each edge connected to a vertex with one of the sockets.

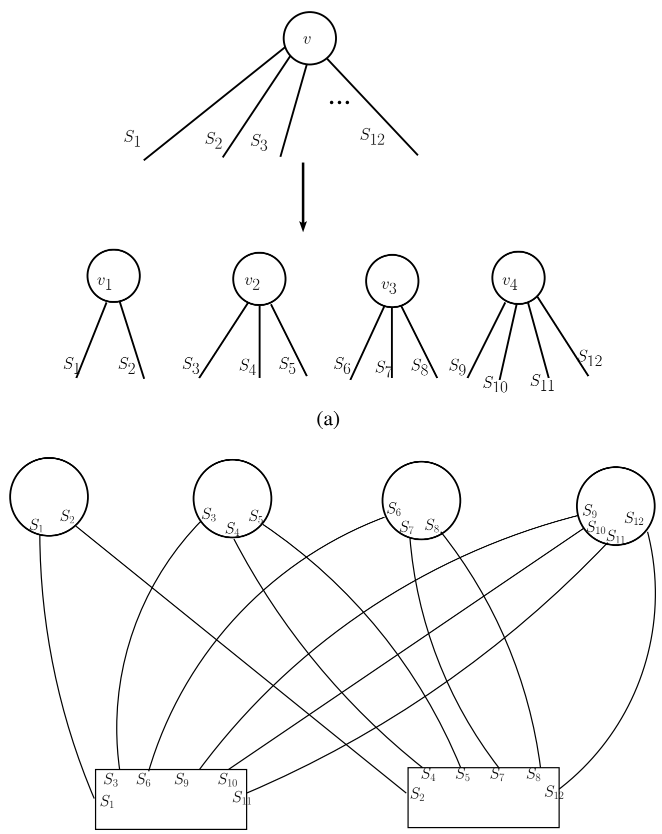
According to König's theorem, the edge chromatic number of a bipartite graph is equal to the maximum degree of its nodes. Therefore G has an edge coloring involving d colors. Based on this edge coloring, we define a coloring of the sockets in G by the colors $S = \{s_1, s_2, \ldots, s_d\}$. Let P and Q be two fixed partitions of S, with $P = \{P_1, P_2, \ldots, P_l\}$ and $Q = \{Q_1, Q_2, \ldots, Q_r\}$. If G is a Cayley graph, then the colors S can be associated with the generating set, along with a direction.

The main step in the construction is splitting the left and right vertices of G according to P and Q, respectively. A left vertex v is split into sub-vertices $v_1, v_2, \ldots, v_l$, such that for any i, the sockets of v in P_i get associated with v_i. A right vertex c is split into sub-vertices $c_1, c_2, \ldots, c_r$, such that for any j, the sockets of c in Q_j get associated with c_j. The resulting Tanner graph, denoted $T(G, P, Q)$, will have $l|V(G)|/2$ variable nodes and $r|V(G)|/2$ check nodes, and the associated MET-LDPC code will have design rate $1 - \frac{r}{l}$.

We note the following important properties of $T(G, P, Q)$.

1) It was shown in [11] that the above node-splitting procedure does not decrease girth. So, the girth of $T(G, P, Q)$ is not less than the girth of G for any P and Q.
2) The Tanner graph $T(G, P, Q)$ is, in fact, a lifted version of a protograph with l variable nodes indexed by P_i, $1 \leq i \leq l$, and r check nodes indexed by Q_j, $1 \leq j \leq r$. Variable node P_i in the protograph is connected by an edge to a check node Q_j, whenever $P_i \cap Q_j \neq \emptyset$. So, the number of edges in the protograph is $|S|$.
3) The protograph is copied $|G|/2$ times and the edge permutation is induced by the edge connections of the original graph G.

The procedure to generate a sequence of large-girth protograph LDPC codes can be summarized as follows. By fixing the degree d and the partitions P and Q, we fix a protograph. We then apply the above node-splitting procedure to a sequence of large-girth d-regular bipartite graphs. This results in a sequence of large-girth Tanner graphs that are liftings of the protograph defined by (d, P, Q). From the above construction, we have the following theorem.



(b)

Fig. 2. Illustration of (a) variable node splitting, and (b) protograph for III-C.

Theorem 2. *For a given protograph with threshold ϵ_{th}, there exists a deterministic sequence of large-girth liftings with increasing length n such that block error probability falls as ne^{-cn}, for a constant $c > 0$, over a BEC(ϵ) with $\epsilon < \epsilon_{th}$.*

C. An Example

Let $d = 12$, and let the socket colors $S = \{s_1, s_2, \cdots, s_{12}\}$ be split into two partitions P and Q given by

$$P_1 = \{s_1, s_2\} \qquad P_2 = \{s_3, s_4, s_5\}$$
$$P_3 = \{s_6, s_7, s_8\} \qquad P_4 = \{s_9, s_{10}, s_{11}, s_{12}\}$$

$$Q_1 = \{s_1, s_3, s_6, s_9, s_{10}, s_{11}\}$$
$$Q_2 = \{s_2, s_4, s_5, s_7, s_8, s_{12}\}$$

The variable nodes are split as shown in Fig. 2(a). The protograph generated by this choice of (d, P, Q) is shown in Fig. 2(b). The design rate of this protograph is $1/2$.

IV. Optimization of Protographs

We have optimized protographs using differential evolution [14] [15], where we use the threshold given by density evolution as the cost function. The salient steps of the differential evolution algorithm are described briefly in the following:

1. Initialization: For generation $G = 0$, we randomly choose N_P base matrices $B_{k,G}$, with $0 \leq k \leq N_P - 1$, of size $|C| \times |V|$, where $N_P = 10|C||V|$. Each entry of $B_{k,G}$ is binary, chosen independently and uniformly.
2. Mutation: Protographs of a particular generation are interpolated as follows.

$$M_{k,G} = [B_{r_1,G} + 0.5(B_{r_2,G} - B_{r_3,G})], \tag{8}$$

where r_1, r_2, r_3 are randomly-chosen distinct values in the range $[0, N_P - 1]$, and $[x]$ denotes the absolute value of x rounded to the nearest integer.
3. Crossover: A candidate protograph $B'_{k,G}$ is chosen as follows. The (i,j)-th entry of $B'_{k,G}$ is set as the (i,j)-th entry of $M_{k,G}$ with probability p_c, or as the (i,j)-th entry of $B_{k,G}$ with probability $1 - p_c$. We use $p_c = 0.88$ in our optimization runs. In $B'_{k,G}$, if any check node is connected to more than one degree-two variable node, edges are reassigned. So, each $B'_{k,G}$ avoids long chain of degree-two variable nodes.
4. Selection: For generation $G+1$, protographs are selected as follows. If the threshold of $B_{k,G}$ is greater than that of $B'_{k,G}$, set $B_{k,G+1} = B_{k,G}$; else, set $B_{k,G+1} = B'_{k,G}$.
5. Termination: Steps 2–4 are run for several generations (we run up to $G = 6000$) and the protograph that gives the best threshold is chosen as the optimized protograph.

Results from our optimization runs are given in Table I. We see that the optimized protographs give better thresholds than irregular standard ensemble codes with minimum degree 3. An optimized 4×8 protograph with threshold 0.479 is given by the following base matrix:

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 3 & 0 & 4 & 1 \\ 1 & 0 & 1 & 0 & 6 & 1 & 0 & 0 \end{bmatrix} \tag{9}$$

An optimized 8×16 protograph with threshold 0.486 is given by the following base matrix:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & 2 & 1 & 0 & 0 & 0 & 4 & 0 & 0 & 3 & 2 & 2 & 0 & 3 \\ 0 & 5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{10}$$

Note that the above protographs have block-error threshold same as the bit-error threshold, and the block-error rate falls inverse polynomially in block-length under the large girth construction as described in Sections II and III.

In future work, we hope to obtain better thresholds that are closer to capacity limits, by further increasing the size of the protograph.

Code type	Threshold
Standard ensemble ($l_{\min} = 3$)	0.461
4×8 protograph in (9)	0.479
8×16 protograph in (10)	0.486

TABLE I
OPTIMIZED PROTOGRAPHS AND THRESHOLDS (RATE 1/2).

V. CONCLUSION

In this work, we presented a deterministic construction for a sequence of codes for the binary erasure channel with block-error rate falling inverse polynomially with block-length at rates close to the capacity. The codes are protograph LDPC codes that avoid long degree-two variable node chains and are constructed from large-girth graphs. To the best of our knowledge, this is the first deterministic construction of LDPC. codes with guaranteed block-error thresholds nearing capacity limits.

REFERENCES

[1] T. Richardson and R. Urbanke, *Modern Coding Theory*. Cambridge University Press, 2008.

[2] E. Arıkan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3051 –3073, Jul. 2009.

[3] S. Kudekar, T. Richardson, and R. Urbanke, "Threshold saturation via spatial coupling: Why convolutional LDPC ensembles perform so well over the BEC," *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 803 –834, Feb. 2011.

[4] M. Lentmaier, D. Truhachev, K. Zigangirov, and D. Costello, "An analysis of the block error probability performance of iterative decoding," *IEEE Trans. Inf. Theory*, vol. 51, no. 11, pp. 3834 –3855, Nov. 2005.

[5] H. Jin and T. Richardson, "Block error iterative decoding capacity for LDPC codes," in *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on*, Sep. 2005, pp. 52 –56.

[6] J. Thrope, "Low-density parity-check (LDPC) codes constructed from protographs," *INP progress report*, vol. 53, no. 8, pp. 42 –154, Aug. 2005.

[7] D. Divsalar, S. Dolinar, C. Jones, and K. Andrews, "Capacity-approaching protograph codes," *IEEE J. Sel. Areas Commun.*, vol. 27, no. 6, pp. 876 –888, Aug. 2009.

[8] R. Gallager, "Low-density parity-check codes," Ph.D. dissertation, Massachusetts Institute of Technology, 1963.

[9] G. Davidoff, P. Sarnak, and A. Valette, *Elementary Number Theory, Group Theory and Ramanujan Graphs (London Mathematical Society Student Texts)*. Cambridge University Press, 2003.

[10] J. Rosenthal and P. Vontobel, "Constructions of regular and irregular LDPC codes using Ramanujan graphs and ideas from Margulis," in *Information Theory, 2001. Proceedings. 2001 IEEE International Symposium on*, 2001, p. 4.

[11] A. Subramanian, A. Thangaraj, M. Bloch, and S. McLaughlin, "Strong secrecy on the binary erasure wiretap channel using large-girth LDPC codes," *Information Forensics and Security, IEEE Transactions on*, vol. 6, no. 3, pp. 585–594, Sep. 2011.

[12] T. Richardson and R. Urbanke, "Multi-edge type LDPC codes," Apr. 2004, unpublished. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.106.7310

[13] B. Bollobás, *Extremal Graph Theory*. Dover Publications, 2004.

[14] A. Shokrollahi and R. Storn, "Design of efficient erasure codes with differential evolution," in *Information Theory, 2000. Proceedings. IEEE International Symposium on*, 2000, p. 5.

[15] R. Storn and K. Price, "Differential evolution—a simple and efficient heuristic for global optimization over continuous spaces," *Journal of Global Optimization*, vol. 11, pp. 341–359, 1997.