

AN UPPER BOUND FOR THE REGULARITY OF BINOMIAL EDGE IDEALS OF TREES

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ABSTRACT. In this article we obtain an improved upper bound for the regularity of binomial edge ideals of trees.

Let G be a finite simple graph on $[n]$. The binomial edge ideal J_G is the ideal in $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ generated by the binomials $\{x_i y_j - x_j y_i \mid \{i, j\} \in E(G)\}$, where K is a field and $E(G)$ denotes the set of all edges of G . This notion was introduced by Herzog et al., [4] and independently by Ohtani [11]. Ever since then researchers have been trying to understand the interplay between the combinatorial invariants of the graph G and the algebraic invariants associated to the ideal J_G . In particular, there have been a lot of attempts on estimating the Castelnuovo-Mumford regularity of the binomial edge ideals using combinatorial invariants.

It is known that $\ell \leq \text{reg}(S/J_G) \leq n - 1$, where n is the number of vertices in G and ℓ denotes the length of a longest induced path in G , [10]. Further, in the same article, Matsuda and Murai conjectured that $\text{reg}(S/J_G) = n - 1$ if and only if G is a path. This conjecture was settled in the affirmative by Kiani and Saeedi Madani, [8].

A vertex v in G is said to be a *cut vertex* if $G \setminus \{v\}$ contains strictly more components than G . A block of a graph is a maximal induced subgraph without any cut vertex and a *block graph* is a graph in which every block is a complete graph. Saeedi Madani and Kiani proved that if $c(G)$ denotes the number of maximal cliques, then for a closed graph G , $\text{reg}(S/J_G) \leq c(G)$, [13]. For a block graph G , $c(G)$ is same as the number of blocks in G . Saeedi Madani and Kiani conjectured that the above inequality holds for all graphs. They proved the conjecture for the case of generalized block graphs, [7]. In [6], the authors obtained a lower bound for the regularity of the binomial edge ideal of trees and characterized the trees having minimal regularity. Recently, Herzog and Rinaldo computed one of the extremal Betti number of the binomial edge ideal of a block graph and classified block graphs admitting precisely one extremal Betti number, [5]. As a consequence, they generalized the lower bound obtained in [6] for block graphs and also characterized the block graphs attaining the lower bound. In [9], Mascia and Rinaldo computed the Krull dimension and regularity of block graphs.

Trees are an important subclass of block graphs. For a tree T on n vertices, $c(T) = n - 1$ thus making the bound $\text{reg}(S/J_T) \leq c(T)$ far from being sharp. Chaudhry et al. proved that a tree T is a caterpillar tree if and only if $\text{reg}(S/J_T) = \ell$, where ℓ is the length of a longest path in T , [2]. The authors of this article generalized this result to obtain an upper bound for the regularity of a class of trees known as lobster trees, [6]. In this article, we

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obtain an improved upper bound for $\text{reg}(S/J_T)$. The upper bound obtained is better than the presently known bound, $n - 1$, for most of the trees.

UPPER BOUND FOR REGULARITY OF TREES

Let G be a block graph. If two distinct blocks in G share a vertex, then it is a cut vertex. A block is said to be an *end-block* if it contains at most one cut vertex. We define the block degree $\text{bd}(v)$ of a cut vertex to be the number of blocks incident to v . A spine of a block graph G is defined to be a maximum length path P in G where every edge of P is a block in G . Note that it is possible that the spine is a single vertex. For $v \in V(G)$, let $\text{lbd}(v)$ denote the number of large blocks, i.e., blocks of size at least three, incident at v .

One of the terminology that we need is that of gluing of two graphs at a vertex. Let G be a graph. For a subset W of $V(G)$, let $G[W]$ denote the induced subgraph of G on the vertex set W . For a cut vertex v in G , let G_1, \dots, G_k denote the components of $G \setminus \{v\}$. Let $G'_i = G[V(G_i) \cup \{v\}]$. Then we say that G is obtained by *gluing* G_1, \dots, G_k at v , [12].

A vertex v in a graph G is said to be a *free vertex* if it is part of exactly one maximal clique. Let G be a block graph and v be a vertex which is not a free vertex of a graph G . Let G' denote the graph obtained by adding edges between all the vertices of $N_G(v)$, G'' denote the graph $G \setminus \{v\}$ and H denote the graph $G' \setminus \{v\}$. Then there is an exact sequence, [3, 1]:

$$0 \longrightarrow \frac{S}{J_G} \longrightarrow \frac{S}{J_{G'}} \oplus \frac{S}{J_{G''}} \longrightarrow \frac{S}{J_H} \longrightarrow 0 \quad (1)$$

If G is obtained by identifying a vertex each of k cliques of size at least three, then by [6], $\text{reg}(S/J_G) = k$. Now, we consider block graphs having non-trivial spine.

Theorem 1. *Let G be a connected block graph in which every block of size at least three is an end-block. Let P be a spine of G of length $\ell(G) \geq 1$, $e_2(G) = |\{\{a, b\} \in E(G) \setminus E(P) \mid \text{bd}(a) \leq 2 \text{ and } \text{bd}(b) \leq 2\}|$, $C_G = \{v \in V(G) \setminus V(P) \mid \text{bd}_G(v) \geq 3\}$ and $b(G)$ be the number of large end-blocks that intersect the spine P . Then,*

$$\text{reg}(S/J_G) \leq e_2(G) + \ell(G) + b(G) + \sum_{v \in C_G} \max\{\text{lbd}(v), 2\}.$$

Proof. If there is no cut vertex in G , then G is an edge and hence the assertion holds, since $e_2(G) = 0$, $b(G) = 0$ and $C_G = \emptyset$.

Assume that G has at least one cut vertex. Let $d(x, P)$ denote the distance of the vertex x from the spine P and $d(G) = \sum_{x \text{ is a cut vertex in } G} d(x, P)$. We apply induction on $d(G)$. If $d(G) = 0$, then G is a graph with a spine P and some cliques attached to P . Therefore, the assertion follows from [6, Theorem 4.5].

Let $d(G) > 0$. Let v be a cut vertex in G such that $d(v, P)$ is maximum.

CASE I: If $\text{bd}(v) = 2$, then there exists a graph G_1 containing v as a free vertex and a clique C such that G is obtained by gluing G_1 and C at v . Then $e_2(G_1) = e_2(G) - 1$, $\ell(G) = \ell(G_1)$, $C_G = C_{G_1}$ and $b(G) = b(G_1)$. Moreover, $d(G_1) < d(G)$. By induction,

$$\text{reg}(S/J_{G_1}) \leq e_2(G_1) + \ell(G_1) + b(G_1) + \sum_{v \in C_{G_1}} \max\{\text{lbd}(v), 2\}.$$

By [6, Theorem 3.1], $\text{reg}(S/J_G) = \text{reg}(S/J_{G_1}) + 1$. Hence the assertion follows.

CASE II: Assume that $\text{bd}(v) \geq 3$. Then $v \in C_G$. Since v is not a free vertex, it follows from the exact sequence (1) that

$$\text{reg}\left(\frac{S}{J_G}\right) \leq \max\left\{\text{reg}\left(\frac{S}{J_{G'}} \oplus \frac{S}{J_{G''}}\right), \text{reg}\left(\frac{S}{J_H}\right) + 1\right\}.$$

Since H is an induced subgraph of G' , $\text{reg}(S/J_H) \leq \text{reg}(S/J_{G'})$. Therefore, we get

$$\text{reg}\left(\frac{S}{J_G}\right) \leq \max\left\{\text{reg}\left(\frac{S}{J_{G''}}\right), \text{reg}\left(\frac{S}{J_{G'}}\right) + 1\right\}.$$

We show that both the entries on the right hand side of the above inequality satisfies the bound given in the assertion.

Note that v is not a cut vertex in G' and if $v \neq y \in V(G)$ is a cut vertex of G' , then it is a cut vertex of G as well. Therefore, $d(G') = d(G) - d(v, P) < d(G)$. We also have $C_{G'} = C_G \setminus \{v\}$. It can be seen that $e_2(G') \leq e_2(G)$ and $\ell(G) = \ell(G')$. By induction hypothesis,

$$\text{reg}(S/J_{G'}) \leq e_2(G') + \ell(G') + b(G') + \sum_{x \in C_{G'}} [\max\{\text{lbd}(x), 2\}].$$

If $d(v, P) = 1$, then $b(G') = b(G) + 1$ and for every $u \in C_{G'}$, $\text{lbd}_{G'}(u) = \text{lbd}_G(u)$. Therefore,

$$\sum_{x \in C_G} [\max\{\text{lbd}_G(x), 2\}] = \sum_{x \in C_{G'}} [\max\{\text{lbd}_{G'}(x), 2\}] + \max\{\text{lbd}_G(v), 2\}.$$

Hence

$$\begin{aligned} \text{reg}(S/J_{G'}) &\leq e_2(G) + \ell(G) + b(G) + 1 + \sum_{x \in C_{G'}} [\max\{\text{lbd}_G(x), 2\}] \\ &\leq e_2(G) + \ell(G) + b(G) + \sum_{x \in C_G} [\max\{\text{lbd}_G(x), 2\}] - 1. \end{aligned}$$

If $d(v, P) > 1$, then $b(G') = b(G)$. Further, there is a vertex $u_v \in C_{G'}$ which is the unique cut vertex neighbor of v . Moreover, we have $\text{lbd}_{G'}(u_v) = \text{lbd}_G(u_v) + 1$ and for every $u \in C_{G'} \setminus \{u_v\}$, $\text{lbd}_{G'}(u) = \text{lbd}_G(u)$. Therefore,

$$\begin{aligned}
\sum_{x \in C_{G'}} [\max\{\text{lbd}_G(x), 2\}] &= \sum_{x \in C_{G'} \setminus \{u_v\}} [\max\{\text{lbd}_{G'}(x), 2\}] + \max\{\text{lbd}_{G'}(u_v), 2\} \\
&= \sum_{x \in C_{G'} \setminus \{u_v\}} [\max\{\text{lbd}_G(x), 2\}] + \max\{\text{lbd}_G(u_v) + 1, 2\} \\
&\leq \sum_{x \in C_{G'} \setminus \{u_v\}} [\max\{\text{lbd}_G(x), 2\}] + \max\{\text{lbd}_G(u_v), 2\} + 1 \\
&\leq \sum_{x \in C_{G'} \setminus \{u_v\}} [\max\{\text{lbd}_G(x), 2\}] + \max\{\text{lbd}_G(u_v), 2\} \\
&\quad + \max\{\text{lbd}_G(v), 2\} - 1 \\
&= \sum_{x \in C_G} [\max\{\text{lbd}_G(x), 2\}] - 1.
\end{aligned}$$

Therefore,

$$\text{reg}(S/J_{G'}) \leq e_2(G) + \ell(G) + b(G) + \sum_{x \in C_G} [\max\{\text{lbd}_G(x), 2\}] - 1.$$

Now we consider the graph $G'' = G \setminus \{v\}$. Let $\text{lbd}_G(v) = r$. Then G'' is the disjoint union of G_1 which is the connected component of G'' containing P and C_1, \dots, C_r maximal cliques on at least 2 vertices and possibly some isolated vertices. Hence $\text{reg}(S/J_{G''}) = \text{reg}(S/J_{G_1}) + r$. For all $x \in V(G_1)$, $\text{lbd}_{G_1}(x) = \text{lbd}_G(x)$ and $d(G_1) = d(G) - d(v, P) < d(G)$. Therefore, by induction hypothesis

$$\text{reg}(S/J_{G_1}) \leq e_2(G'') + \ell(G_1) + b(G_1) + \sum_{x \in C_{G_1}} [\max\{\text{lbd}_{G_1}(x), 2\}].$$

Now, there are two possibilities, namely $e_2(G'') = e_2(G) + 1$ or $e_2(G'') = e_2(G)$.

If $e_2(G'') = e_2(G) + 1$, then the unique cut vertex neighbor u_v of v has block degree 2 in G_1 . Therefore $C_{G_1} = C_G \setminus \{v, u_v\}$ so that

$$\begin{aligned}
\text{reg}(S/J_{G_1}) &\leq e_2(G) + 1 + \ell(G_1) + b(G_1) + \sum_{x \in C_{G_1}} \max\{\text{lbd}_{G_1}(x), 2\} \\
&\leq e_2(G) + \ell(G_1) + b(G_1) + \sum_{x \in C_{G_1}} [\max\{\text{lbd}_{G_1}(x), 2\}] + \max\{\text{lbd}_{G_1}(u_v), 2\} \\
&\leq e_2(G) + \ell(G_1) + b(G_1) + \sum_{x \in C_G \setminus \{v\}} [\max\{\text{lbd}_G(x), 2\}].
\end{aligned}$$

Note also that $r = \text{lbd}_G(v)$. Hence

$$\text{reg}(S/J_{G''}) = \text{reg}(S/J_{G_1}) + r \leq e_2(G) + \ell(G) + b(G) + \sum_{x \in C_G} \max\{\text{lbd}_G(x), 2\}.$$

For the case when $e_2(G'') = e_2(G)$, we have $C_{G_1} = C_G \setminus \{v\}$. Now as argued in the previous case, one can conclude that

$$\text{reg}(S/J_{G''}) \leq e_2(G) + \ell(G) + b(G) + \sum_{x \in C_G} \max\{\text{lbd}_G(x), 2\}.$$

□

As an immediate consequence, we generalize [6, Corollary 4.8] to get an upper bound for the regularity of all trees.

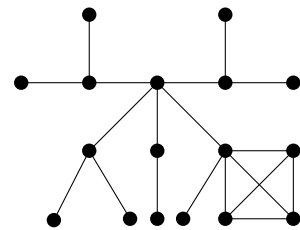
Corollary 2. *Let T be a tree on $[n]$ with spine P of length ℓ . Let e_2 denote the number of edges that are not in P and with both end points having degree at most 2 and d_3 denote the number of vertices, not in P , and having degree at least 3. Then*

$$\text{reg}(S/J_T) \leq e_2 + \ell + 2d_3.$$

Proof. Following the notation of Theorem 1, $b(T) = 0$ and $\text{lbd}_T(x) = 0$ for each $x \in V(T)$ so that $\max\{\text{lbd}_T(x), 2\} = 2$. Now the assertion follows directly from Theorem 1. □

Example 3. *Here we illustrate by an example a block graph considered in Theorem 1.*

Let G be the graph given on the right side. Following the notation in Theorem 1, we can see that $e_2(G) = 1, \ell(G) = 4, b(G) = 0$ and $|C_G| = 2$. Therefore, we get $\text{reg}(S/J_G) \leq 9$. We have computed the regularity of this graph using Macaulay 2 and have found that the graph attains the regularity upper bound.



We also note that the upper bound we obtained in Theorem 1 coincides with the lower bound for the regularity of *Flower graph* $F_{h,k}(v)$ proved in Corollary 3.5 of [9].

Corollary 4. *Let $F_{h,k}(v)$ denote the graph obtained by identifying a free vertex each of h copies of C_3 and $k \geq 1$ copies of $K_{1,3}$ at a common vertex v . Then $\text{reg}(S/J_{F_{h,k}(v)}) = 2k + h$.*

Proof. Let $G = F_{h,k}(v)$. Following the notation in Theorem 1, we get $e_2(G) = 0$ and $b(G) = h$. If $k \leq 2$, then $C(G) = \emptyset$ and if $k > 2$, then $C(G)$ consists of all the center vertices of $k - 2$ copies of $K_{1,3}$ outside a fixed spine. Therefore, it follows from Theorem 1 that $\text{reg}(S/J_G) \leq 2k + h$. Following the notation in the article [9], it can be seen that $i(F(v)) = k + 1$ and $\text{cdeg}(v) = h + k$. This proves the assertion. □

It may also be noted that the upper bound is not attained by all block graphs. For example, in the case of the graph considered in [9, Example 3.8], our bound gives the value 6 while the actual regularity is 5.

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