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A QUADTREE-BASED SCALED BOUNDARY FINITE ELEMENT METHOD FOR CRACK PROPAGATION MODELLING

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ABSTRACT

The quadtree is a hierarchical-type data structure where each parent is recursively divided into four children. This structure makes it particularly efficient for adaptive mesh refinement in regions with localised gradients. Compared with unstructured triangles, mesh generation is more efficient using quadtree decompositions. The finite number of patterns in the quadtree decomposition makes it efficient for data storage and retrieval. Motivated by these advantages, a crack propagation modelling approach using a quadtree-based scaled boundary finite element method (SBFEM) is developed. Starting from the formulation of an arbitrary *n*-sided polygon element, each quadrant in the quadtree mesh is treated as a polygon within the framework of the SBFEM. Special techniques to treat the hanging nodes are not necessary. Moreover, the SBFEM enables accurate calculation of the stress intensity factors directly from its solutions without local mesh refinement or asymptotic enrichment functions. When a crack propagates, it is only necessary to split each quadrant cut by the crack into two. These quadrants are polygons that can be directly modelled by the SBFEM. Changes to the mesh are minimal. The efficiency of this approach is demonstrated using numerical benchmarks.

KEYWORDS

Quadtree, scaled boundary finite element method, crack propagation, fracture.

INTRODUCTION

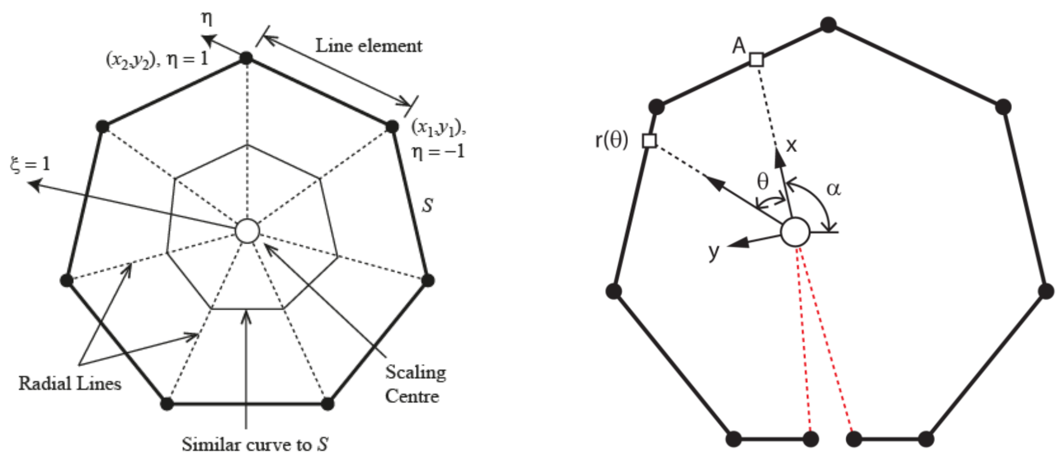
Quadtrees (Yerry and Shepard 1983) have unique properties that make them particularly suitable for computational modelling of problems that require high efficiency. Its hierarchical tree-like data structure makes it very efficient in adaptive meshing of complex geometries and regions experiencing localised gradients. Moreover, the finite number of patterns in a quadtree structure allows quick data storage and retrieval, making computations with quadtrees very efficient.

Despite of these advantages, adaptation of quadtree meshes into computational analyses within the framework of the finite element method (FEM) is not common in practice. The primary reason for this is the presence of hanging nodes between two adjacent elements of different sizes. The hanging nodes destroy the displacement compatibility between the elements and require special remedial techniques such as imposing constraints, subdivision into triangles and transitional elements (Gupta 1978). Sukumar and Tabarraei (2004) developed an approach to construct conforming shape functions for polygons with arbitrary number of sides and showed that this procedure can be directly employed for computations with quadtree meshes (Tabarraei and Sukumar 2005). It does not make any distinction between hanging nodes and corner nodes in a quadtree mesh.

The scaled boundary finite element method (SBFEM) is a semi-analytical technique that was developed by Song and Wolf (1997). It can be formulated on polygons with arbitrary number of sides (Ooi et al. 2012), allowing it for adaption with quadtree meshes. Each cell in the mesh, regardless of the presence of hanging nodes, is treated as a polygon. This enables the structure of the quadtree to be exploited for efficient computations. This polygon-based formulation also enables the SBFEM to better discretise curved boundaries. The cells that are trimmed by the boundaries are directly treated as polygons. When applied to fracture problems, stress singularities of any kind are analytically represented in the SBFEM's solution of stresses (Song and Wolf 1997) and therefore very accurate. Motivated by these advantages, an approach that combines a quadtree mesh with the SBFEM to model crack propagation is developed in this study.

SCALED BOUNDARY FINITE ELEMENT METHOD

The SBFEM is briefly reviewed. A complete discussion on this subject has been reported in numerous instances in the literature e.g. Song and Wolf (1997). The SBFEM can be implemented on polygons with arbitrary number of sides (Ooi et al., 2012). The polygon geometry is required to satisfy the star convexity criterion i.e. there exists a point within the polygon where the entire boundary is directly visible from it. This point is called the scaling centre.



(a) normal polygon (b) crack polygon

Figure 1. SBFEM discretisation using polygons

The scaled boundary finite element coordinate system (ξ, η) of a generic polygon is shown in Figure 1a and a polygon modelling a crack in Figure 1b. ξ is the radial coordinate. ξ = 0 at the scaling centre and ξ = 1 at the polygon boundary. η is the local coordinate defined for each one-dimensional finite element that discretises the polygon boundary. The equilibrium condition in each polygon can be derived from the method of weighted residuals and results in the first order differential equation

$$\xi\left[\mathbf{u}(\xi) \quad \mathbf{q}(\xi)\right]^{\mathrm{T}}_{,\xi} = -\mathbf{Z}\left[\mathbf{u}(\xi) \quad \mathbf{q}(\xi)\right] \tag{1}$$

where $\mathbf{u}(\xi)$ are radial displacement functions, $\mathbf{q}(\xi)$ is the equivalent force vector and $\mathbf{Z}$ is the standard Hamiltonian matrix in the SBFEM (Song and Wolf 1997). $\mathbf{Z}$ can be decoupled into pairs of eigenvalues ($-\lambda_i$, λ_i) using an eigenvalue decomposition as

$$\mathbf{Z}\begin{bmatrix} \mathbf{\Phi}_n^{(u)} & \mathbf{\Phi}_p^{(u)} \\ \mathbf{\Phi}_n^{(q)} & \mathbf{\Phi}_p^{(q)} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_n^{(u)} & \mathbf{\Phi}_p^{(u)} \\ \mathbf{\Phi}_n^{(q)} & \mathbf{\Phi}_p^{(q)} \end{bmatrix}\begin{bmatrix} \mathbf{\Lambda}_n & 0 \\ 0 & \mathbf{\Lambda}_p \end{bmatrix} \tag{2}$$

where $\mathbf{\Phi}_n^{(u)}$, $\mathbf{\Phi}_n^{(q)}$, $\mathbf{\Phi}_p^{(u)}$ and $\mathbf{\Phi}_p^{(q)}$ are transformation matrices related to the deformation modes in a polygon. $\mathbf{\Lambda}_n$ and $\mathbf{\Lambda}_p$ are diagonal matrices containing the negative and positive eigenvalues -λ_i and λ_i respectively. The stiffness matrix of a polygon $\mathbf{K}_p$ can be evaluated as

$$\mathbf{K}_p = (\mathbf{\Phi}_n^{(q)})^{-1}\mathbf{\Phi}_n^{(u)} \tag{3}$$

The displacement field $\mathbf{u}(\xi, \eta)$ of a sector covered by a line element on a polygon boundary is

$$\mathbf{u}(\xi,\eta) = \mathbf{N}(\eta)\mathbf{\Phi}_n^{(u)}\xi^{-\mathbf{\Lambda}_n}\mathbf{c} \tag{4}$$

where $\mathbf{N}(\eta)$ is the shape functions of the finite elements and $\mathbf{c}$ are integration constants that can be determined from the boundary conditions. Correspondingly, the stress field $\boldsymbol{\sigma}(\xi, \eta)$ is

$$\boldsymbol{\sigma}(\xi,\eta) = \mathbf{\Psi}_\sigma(\eta)\xi^{-\mathbf{\Lambda}_n-\mathbf{I}}\mathbf{c} \tag{5}$$

where $\mathbf{\Psi}_\sigma(\eta) = \lfloor \mathbf{\Psi}_{\sigma_{xx}}(\eta) \quad \mathbf{\Psi}_{\sigma_{yy}}(\eta) \quad \mathbf{\Psi}_{\sigma_{zz}}(\eta) \rfloor^T$ is the stress mode (Song and Wolf 1997).

QUADTREE MESHES IN THE SCALED BOUNDARY FINITE ELEMENT METHOD

A cell in a quadtree mesh can be directly modelled by the SBFEM as a polygon having arbitrary number of sides, similar to the approach of Tabarraei and Sukumar (2005). No distinction is made between a corner node and a hanging node. All the cells in the quadtree mesh are directly modelled by a unified formulation within the framework of the SBFEM. From the point-of-view of finite element mesh generation algorithms, this is particularly appealing given that a quadtree decomposition is efficient and usually employed prior to triangularisation of the mesh. By employing quadtree meshes directly, the triangularisation process and reduces the computational resources for mesh generation.

In most applications, quadtree meshes are usually generated by constraining the size (lengths) between adjacent elements to 2:1, also known as the 2:1 rule. While the SBFEM does not have to adhere to this restriction, the 2:1 rule contains features that facilitate more efficient computations. When the 2:1 rule is enforced, the cells in the mesh are limited to only the patterns shown in Figure 2. Considering all orthogonal rotations of each of the cells, the total number of different cells that can be present in the quadtree mesh is 24. This can be reduced to 16 for isotropic homogeneous materials considering that rotation does not have effect on some of the cells. Only the stiffness matrices of these cells need to be computed in a linear elastic analysis. These can be pre-calculated and stored for quick data retrieval.

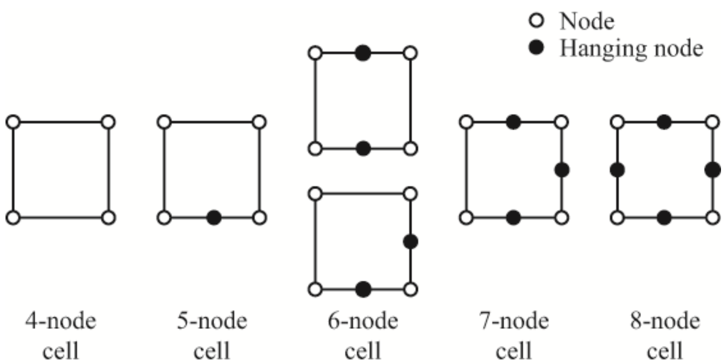


Figure 2. Regular cells in a quadtree mesh satisfying the 2:1 rule

FRACTURE ANALYSIS

Computation of Stress Intensity Factors

In the SBFEM, a crack is sufficiently modelled using a single polygon (Figure 1b). The crack surfaces are not discretised. The line elements discretising the polygon boundary do not form a closed loop. In a crack polygon, the matrix $\mathbf{\Lambda}_n$ contain two eigenvalues $\overline{\lambda}_i$, i = 1, 2 satisfying -1 < Re($\overline{\lambda}_i$) < 0. These eigenvalues lead to a singular stress field in Eq. (5) as $\xi \rightarrow 0$. This feature enables the SBFEM to model the stress singularity accurately. The singular stress field can be obtained from Eq. (5) as

$$\boldsymbol{\sigma}^{(s)}(\xi,\eta) = \mathbf{\Psi}_{\sigma}^{(s)}(\eta)\xi^{-\mathbf{\Lambda}_n^{(s)}-\mathbf{I}}\mathbf{c}^{(s)} \tag{6}$$

where $\mathbf{\Lambda}_n^{(s)}$ is a diagonal matrix containing the two eigenvalues $\overline{\lambda}_i$, $\mathbf{\Psi}_{\sigma}^{(s)}(\eta)$ is the singular stress mode and $\mathbf{c}^{(s)}$ are the corresponding integration constants (Song and Wolf 1997).

The stress intensity factors are directly evaluated from their definitions. For a crack that is aligned with the Cartesian coordinate system as shown in Figure 1b, the stress intensity factors are

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \lim_{r\to 0}\sqrt{2\pi r}\left[\sigma_{yy}\,|_{\theta=0} \quad \tau_{xy}\,|_{\theta=0}\right]^{\mathrm{T}} = \sqrt{2\pi L_A}\left[\mathbf{\Psi}_{\sigma_{yy}}^{(s)}(\eta\,|_{\theta=0})\mathbf{c}^{(s)} \quad \mathbf{\Psi}_{\tau_{xy}}^{(s)}(\eta\,|_{\theta=0})\mathbf{c}^{(s)}\right] \tag{7}$$

where L_A is the distance from the scaling centre to the boundary in the direction of the crack at θ = 0.

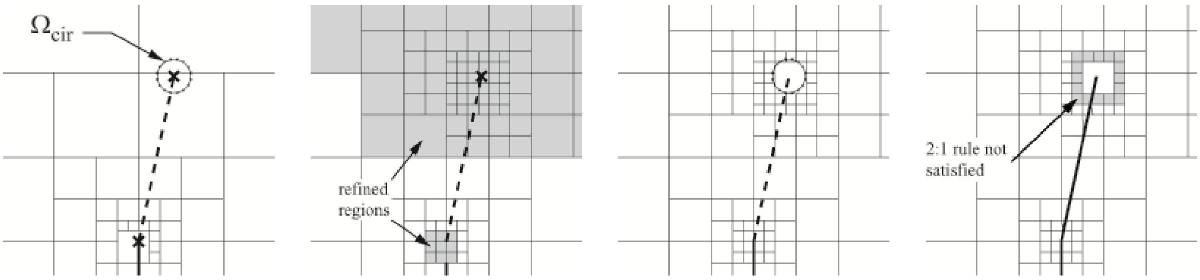
Crack Propagation Modelling

The crack propagation direction θ_p is determined from the stress intensity factors using the maximum circumferential stress criterion (Sukumar and Prevost 2003)

$$\theta_p = 2\tan^{-1}\left(\frac{-2K_{II}(\theta_c)/K_I(\theta_c)}{1+\sqrt{1+8(K_{II}(\theta_c)/K_I(\theta_c))^2}}\right) \tag{8}$$

where θ_c is the crack angle.

Automatic Remeshing Algorithm



(a) crack tip mesh (b) quadtree refinement (c) generate crack polygon (d) split cells cut

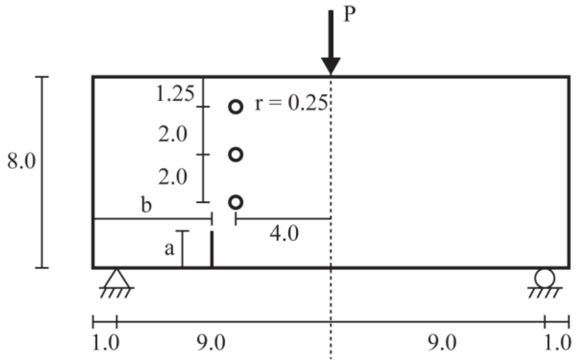
Figure 3. Automatic remeshing procedure

Figure 3a shows the mesh in the vicinity of a crack tip during a generic load step. The remeshing algorithm is depicted in Figure 3 and involves the following four steps:

1. Identify the smallest cell surrounding the crack polygon $h_{\min}$. Refine the mesh in the region near the crack tip to $h_{\min}$.
2. Identify the region around the new crack tip Ω_{cir}. Any cells that are in contact with this region is refined. The region affected by the remeshing can be larger than Ω_{cir} as the 2:1 rule is enforced during this process. The resulting mesh is depicted in Figure 3b.
3. Identify the cells that intersect or overlap Ω_{cir} and merge them into a single polygon as shown in Figure 3c.
4. Split the cells cut by the crack path into two, leading to the mesh in Figure 3d.

NUMERICAL EXAMPLE

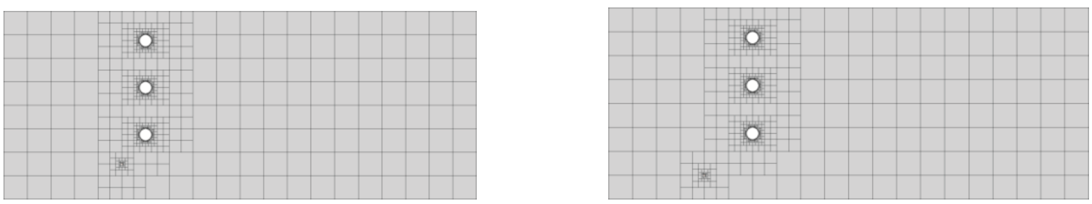
The cracked polymethylcrylate (PMMA) beam with three holes subjected to three-point bending shown in Figure 4 is considered. The load P = 4.45 N. This problem is a widely used numerical benchmark for crack propagation problems e.g. Bittencourt et al. (1996). The material properties of the beam are: Young's modulus E = 29 GPa and Poisson's ratio ν = 0.3. Plane stress conditions are assumed. Two cases, I and II, having different values of *a* and *b* as shown in Figure 4 are considered.



CASE	a	b
I	1.5	5.0
II	1.0	4.0

Figure 4. Cracked beam with three holes subjected to three-point bending

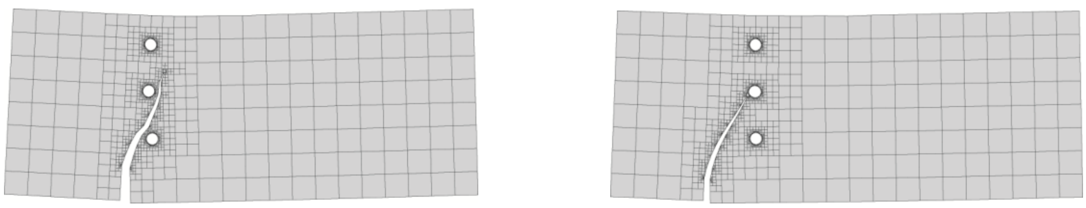
Figures 5a and 5b show the initial meshes of the beam for Cases I and II, respectively. The mesh for Case I has 473 cells of which 349 are regular square cells shown in Figure 1 and 124 are irregular cells. The total number of nodes for this mesh is 658. In Case II, the mesh has 476 cells of which 354 are regular square cells and 122 are irregular cells. The total number of nodes for this mesh is 661. In both Cases I and II, a crack incremental length Δ*a* = 0.5 mm is specified for the simulation.



(a) Case I (b) Case II

Figure 5. Initial meshes of the cracked beam

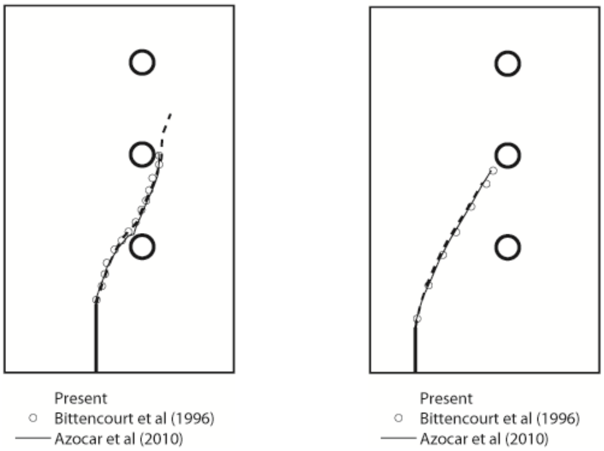
Figures 6a and 6b show the final meshes of the beam for Cases I and II, respectively. For Case I, the final mesh has 1075 cells of which, 264 are irregular. For Case II, the final mesh has 829 cells of which, 264 are irregular. The difference in the crack length and location, as outlined in Figure 2 for both the cases considered, is observed to affect the crack propagation path.



(a) Case I (b) Case II

Figure 6. Final meshes of the cracked beam

In Case I, the crack initially curves towards the bottom hole. As it approaches the bottom hole, the crack turns towards the the middle hole. The crack path once again deflects towards the loading point as it approaches the middle hole. In Case II, the crack propagation is a curved line evolving towards the middle hole. The crack trajectories in both cases obtained by the scaled boundary finite element method are reminiscence of the experimental results reported by Bittencourt et al. (1996) and the finite element simulation of Azocar et al. (2010) as shown in Figures 7a and 7b.



(a) Case I (b) Case II

Figure 7. Comparison of predicted crack paths with numerical results published in the literature

CONCLUSIONS

A crack propagation modelling technique has been developed by adapting quadtree meshes into the SBFEM. The use of quadtrees compliments the SBFEM in that the formulation makes no distinction between hanging nodes and corner nodes. This approach exploits the flexibility of the SBFEM in assuming polygons of arbitrary shapes, its accuracy in modelling stress singularities at the crack tip and the computational efficiency of the quadtree-polygon mesh. The evolution of the crack paths is conveniently modelled using an efficient local remeshing algorithm. The remeshing process is simple and involves only application of the quadtree decomposition in the region near the crack tips and splitting the cells cut by the crack into two. The developed technique was successfully validated using a standard numerical benchmark. The developed technique therefore, offers an interesting and viable alternative to the standard approaches reported in the literature.

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