TORIC CO-HIGGS BUNDLES ON TORIC VARIETIES

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Abstract. Starting from the data of a nonsingular complex projective toric variety, we define an associated notion of *toric co-Higgs bundle*. We provide a Lie-theoretic classification of these objects by studying the interaction between Klyachko's fan filtration and the fiber of the co-Higgs bundle at a closed point in the open orbit of the torus action. This can be interpreted, under certain conditions, as the construction of a coarse moduli scheme of toric co-Higgs bundles of any rank and with any total equivariant Chern class.

1. Introduction

We begin with an algebraic or, equivalently, holomorphic vector bundle V over a nonsingular complex projective variety X with tangent bundle TX. Then, a *co-Higgs field* for V is a holomorphic section ϕ of the twisted endomorphism bundle $\mathrm{End}(V)\otimes TX$, subject to the integrability condition that the quadratic section $\phi\otimes\phi$ is symmetric — that is, that the section $\phi\wedge\phi$ of $\mathrm{End}(V)\otimes\Lambda^2 TX$ vanishes identically. A pair (V, ϕ) satisfying the above conditions is referred to as a *co-Higgs bundle*. Co-Higgs bundles were introduced simultaneously by Hitchin [11] and the fourth-named author [15] in the context of generalized complex geometry. The name *co-Higgs* speaks to a duality with Higgs bundles in the sense of Hitchin [9, 10] and Simpson [18], where the Higgs fields are T^*X-valued.

Co-Higgs bundles have been classified and/or constructed on $\mathbb{P}^1$ [16, 5], $\mathbb{P}^2$ [17], $\mathbb{P}^1\times\mathbb{P}^1$ [20], and logarithmic curves [2], for example. Over singular varieties, they have been used to some effect towards establishing inequalities related to vector-valued modular forms [7]. At the same time, there are "no-go" theorems for the existence of nontrivial co-Higgs bundles in some instances, such as over the moduli space of stable bundles on a nonsingular complex curve of genus at least 2 [4]. Recently, the first-named and fourth-named authors classified homogeneous co-Higgs bundles on Hermitian symmetric spaces [6]. The goal of the present work is to extend this to toric varieties. Accordingly, we define below a natural notion of *toric co-Higgs bundle*.

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We classify toric co-Higgs structures for a fixed toric bundle V using Klyachko's seminal work on classification of toric vector bundles [12]. We recall from [12, Theorem 2.2.1] that the category of toric bundles on X with equivariant homomorphisms is equivalent to the category of compatible Σ-filtered vector spaces, where Σ is the fan of X. Our strategy for classifying toric co-Higgs bundles (V, ϕ) is thus to reduce the data of (V, ϕ) to that of a tuple of commuting Σ-filtered endomorphisms of the fiber V_{x_0}, where x_0 is a closed point in the open orbit of T in X. This is the content of Theorem 3.2, which is the main theorem in this note.

Similar to the symmetric space case [4], the resulting classification is Lie-theoretic in nature and admits an interpretation as a moduli construction. Subject to certain conditions (namely, the freeness of a certain group action), we identify a scheme that parametrizes toric co-Higgs bundles of fixed rank and total equivariant Euler characteristic in the sense of [13]. This scheme fibers over an associated moduli scheme of toric bundles constructed in [14].

We also wish to point out similar work in [1], completed independently and at roughly the same time.

2. Set-up and examples

2.1. **Basic notions.** Throughout, X is a nonsingular complex projective variety. Assume that X admits an algebraic (equivalently, holomorphic) action of a complex torus $T \cong (\mathbb{C}^*)^n$ so that it is a *toric variety* in the sense of [8]. Furthermore, fix a holomorphic vector bundle V and suppose that it admits a lift of the action of T from X which is fiber-wise linear. In other words, V is equipped with the structure of a T-equivariant vector bundle. We will refer to V simply as a *toric bundle*.

There is subsequently an induced action of T on the vector space of global holomorphic sections of V:

$$(t \cdot s)(x) = ts(t^{-1}x) \tag{2.1}$$

for all $s \in H^0(X, V)$ and $t \in T$. A section $s \in H^0(X, V)$ is said to be *semi-invariant* if there exists a character χ of T such that

$$t \cdot s = \chi(t)s \tag{2.2}$$

for all $t \in T$. A semi-invariant section s is said to be *invariant* if the associated character $\chi(t)$ is trivial, meaning

$$t \cdot s = s \tag{2.3}$$

for all $t \in T$. Combining (2.1) and (2.3), we conclude that

$$ts(t^{-1}x) = s(x) \tag{2.4}$$

for any invariant section s.

2.2. **T-equivariant structures and toric co-Higgs bundles.** Now, a T-equivariant structure on V induces a T-equivariant structure on $\mathrm{End}(V)$ in the following way. Given an element ψ in the fiber $(\mathrm{End}(V))_x = \mathrm{End}(V_x)$ over $x \in X$, we define $t\psi$ in $\mathrm{End}(V_{tx})$ by

$$(t\psi)(v) \;=\; t(\psi(t^{-1}v)) \tag{2.5}$$

for every $v \in V_{tx}$ and $t \in T$. Then, by (2.4), a holomorphic section ϕ of $\mathrm{End}(V)$ is invariant if and only if

$$t\phi(t^{-1}x) \;=\; \phi(x) \tag{2.6}$$

for all $x \in X$ and $t \in T$. By (2.5), equation (2.6) is equivalent to

$$t \circ \phi(t^{-1}x) = \phi(x) \circ t$$

for all $x \in X$ and $t \in T$. In other words, ϕ is an invariant section of $\mathrm{End}(V)$ if and only if ϕ is a T-equivariant endomorphism of V.

At the same time, the tangent bundle TX has a natural T-equivariant structure that is simply the linearization of the T-action on X. Together with the above T-action on $\mathrm{End}(V)$, we have an induced T-equivariant structure on the twisted bundle $\mathrm{End}(V) \otimes TX$. This allows us to formulate the following:

Definition 2.1. A *toric co-Higgs bundle* on a toric variety X is a pair $(V,\, \phi)$, where $V \longrightarrow X$ is a toric bundle and $\phi \in H^0(X,\, \mathrm{End}(V) \otimes TX)$ is an invariant co-Higgs field.

2.3. **Examples.** The tangent bundle TX of a nonsingular projective toric variety X always admits a nonzero invariant holomorphic section s. Let us take the tensor product of such a holomorphic vector field with the identity homomorphism 1 of *any* toric bundle V. Call this product ϕ. It follows immediately that

$$\phi \wedge \phi \;=\; (s \otimes 1) \wedge (s \otimes 1) \;=\; [s,\, s] \otimes (1 \wedge 1) \;=\; 0\,.$$

In other words, ϕ is a nontrivial invariant co-Higgs field on V, and hence every toric vector bundle on any toric variety is equipped with a family of invariant co-Higgs fields induced by the invariant holomorphic vector fields on X.

Another example, similar in spirit but for a specific bundle, is given by choosing $V = TX \oplus \mathcal{O}_X$ with its natural toric structure, where $\mathcal{O}_X$ is the structure sheaf of X. We can equip this V with the co-Higgs field

$$\phi \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where 1 is interpreted as the identity morphism on TX, which serves as the part of the co-Higgs structure that acts as $TX \longrightarrow \mathcal{O}_X \otimes TX$. Because ϕ is built from just the identity map, the co-Higgs field automatically has the required invariance. It satisfies the vanishing condition $\phi \wedge \phi \;=\; 0$ since, as a matrix, ϕ is nilpotent. More generally, this example is present on any complex variety — it is the so-called *canonical co-Higgs bundle* — and it is discussed in some detail from the point of view of slope stability and deformation theory in Chapter 6 of [15].

For contrast, we conclude with an example of a co-Higgs structure on a vector bundle that is not directly related to toric geometry. We briefly recall the definition of the jet bundle of a holomorphic vector bundle E on a projective variety Y. Let

$$p_j : Y \times Y \longrightarrow Y, \ j = 1,2$$

be the natural projection to the j-th factor. Let Δ be the diagonal of $Y \times Y$. For any non-negative integer k, consider the k-th jet bundle of E defined as

$$J^k(E) = p_{1*}(p_2^*(E)/(p_2^*E \otimes \mathcal{O}_{Y\times Y}(-(k+1)\Delta) \longrightarrow Y$$

One can check that the higher direct image of the above right hand side vanishes, which makes $J^k(E)$ a vector bundle. We have the following exact sequence

$$0 \longrightarrow E \otimes \Omega_Y \xrightarrow{f} J^1(E) \xrightarrow{g} J^0(E) \longrightarrow 0$$

The composition homomorphism $(g \otimes 1_{\Omega_Y}) \circ f$ defines a nonzero co-Higgs structure on $J^1(E)$. Even when Y itself is toric, this gives examples of non-toric co-Higgs bundles by taking E to be non-toric. For further examples of non-toric co-Higgs bundles on toric varieties, we refer the reader to [1].

In the next section, we classify toric co-Higgs structures on a fixed toric bundle.

3. Toric co-Higgs bundles and the Klyachko fan filtration

3.1. **Σ-filtrations.** Let X be a nonsingular complex projective toric variety equipped with an action of T (cf. [8]). Let Σ denote the fan of X, and let $\Sigma(d)$ be the set of d-dimensional cones in Σ. For any toric bundle V on X, Klyachko [12] constructed a compatible family of full filtrations of decreasing subspaces of the fiber $E = V_{x_0}$, where x_0 is a closed point in the open orbit of T in X. We will subsequently refer to x_0 simply as a "closed point for T". The family is indexed by the T-invariant divisors or equivalently by $|\Sigma(1)|$, the set of integral generators of the one-dimensional cones of Σ. In other words, we have a family of filtrations

$$\{E^\rho(i) \mid \rho \in |\Sigma(1)|, \ i \in \mathbb{Z}\}.$$

Note that "decreasing" means

$$E^\rho(i+1) \subseteq E^\rho(i)$$

for all i. For brevity, such a family of filtrations will be called a Σ-*filtration*. The compatibility condition mentioned above refers to the existence of cone-wise T-module structures on E giving rise to the Σ-filtration. We give below a short review of the construction of the Σ-filtration in the case of a projective toric variety. In this case, the description is slightly simpler as it suffices to consider only the cones of dimension n in the fan.

Let M and N denote the dual lattices of characters and co-characters of T, respectively. Let σ be a maximal cone of Σ and X_σ the corresponding affine toric variety. Denote by V_σ the restriction of V to X_σ. It was shown by Klyachko [12, Proposition 2.1.1] that there exists a framing (which is not unique) of V_σ by semi-invariant sections. Fix such a framing $(s_1, \ldots, s_r)$. Let S_σ be the T-submodule of $H^0(X_\sigma, V_\sigma)$ generated by the semi-invariant sections $s_1, \ldots, s_r$. Evaluation at x_0 gives an isomorphism of vector spaces $ev_{x_0} : S_\sigma \longrightarrow E$. This isomorphism induces a T–module structure on E, or equivalently, a decomposition

$$E = \bigoplus_{u \in M} E^\sigma_u . \tag{3.1}$$

The decompositions (3.1) may depend on the choice of the semi-invariant framing of V_σ. However, Klyachko showed that for each $\rho \in |\Sigma(1)|$, the subspaces

$$E^\rho(i) := \bigoplus_{u \in M, u(\rho) \geq i} E^\sigma_u , \quad \text{where } \sigma \in \Sigma(n) \text{ is such that } \rho \in |\Sigma(1)| \bigcap \sigma , \tag{3.2}$$

are independent of the choice of σ containing ρ and the choice of the framing.

A morphism of compatible Σ-filtered vector spaces $\{E^\rho(i)\}$ and $\{F^\rho(i)\}$ is a vector space map $\phi : E \longrightarrow F$ such that $\phi(E^\rho(i)) \subseteq F^\rho(i)$ for all ρ and i. We call such a morphism a *filtered linear map* of Σ-filtered vector spaces. Proposition 2.1.1(iii) of [12] proves that equivariant morphisms between two toric vector bundles over X correspond to filtered linear maps between their Σ-filtrations. Note that any linear map between the fibers at x_0 extend uniquely to a T-equivariant morphism of vector bundles over the open T-orbit. The regularity of such an extension over the boundary of the open orbit is naturally and intimately related to the weights of the T-action and the defining inequalities of the Σ-filtrations.

The discussion above can be summarized as following celebrated theorem of Klyachko.

Theorem 3.1 ([12, Theorem 2.2.1])**.** *The category of toric vector bundle over a toric variety X with fan Σ is equivalent to the category of complex vector spaces E with a family of decreasing filtrations*

$$\{E^\rho(i) \mid \rho \in |\Sigma(1)|, \ i \in \mathbb{Z}\} ,$$

which satisfy following compatibility condition :

For each $\sigma \in \Sigma(n)$, there exists a M-grading $E = \bigoplus_{u \in M} E^\sigma_u$ for which

$$E^\rho(i) := \bigoplus_{u \in M, u(\rho) \geq i} E^\sigma_u , \quad \text{where } \sigma \text{ is such that } \rho \in |\Sigma(1)| \bigcap \sigma.$$

3.2. **The main theorem.** Having built up necessary language around Σ-filtrations, we now pose the main theorem of this article.

Theorem 3.2. *Let X be a nonsingular complex projective toric variety equipped with an action of $T \cong (\mathbb{C}^*)^n$, let V by any toric bundle on X, and let $x_0 \in X$ be a closed point for T. Then, there is a $1:1$ correspondence between invariant co-Higgs fields ϕ and n-tuples of pairwise-commuting filtered linear maps of $E = V_{x_0}$ that respect the Klyachko Σ-filtration.*

Before proving Theorem 3.2, we need to understand the integrability condition $\phi \wedge \phi = 0$ locally. In [15, 11, 17], a local criterion for the vanishing of $\phi \wedge \phi$ is identified. Suppose that $\{z_1, \ldots, z_n\}$ is a holomorphic coordinate system on an affine chart U in a nonsingular variety X. We can write

$$\phi|_U = \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial z_i},$$

where each $\phi_i \in H^0(U, \operatorname{End}(V))$. Then

$$\phi \wedge \phi = 0 \text{ on } U \iff [\phi_i, \phi_j] = 0 \text{ on } U \; \forall\, 1 \leq i, j \leq n\,. \tag{3.3}$$

With this observation in hand, we can proceed with the proof of the main theorem.

Proof of Theorem 3.2. Let $(t_1, \ldots, t_n)$ be coordinates on T corresponding to an integral basis of $\operatorname{Lie}(T)$. We identify these with coordinates on the open dense T-orbit in X; this should not cause any confusion. To conform to (2.1), we consider the action of T on $\mathcal{O}_T$ (and $\mathcal{O}_X$) given by

$$(t \cdot f)(x) := f(t^{-1}x)\,. \tag{3.4}$$

Observe that the character t_i, and the derivation $\frac{\partial}{\partial t_i}$, have weights t_i^{-1} and t_i respectively under this action. The corresponding invariant vector fields $t_i \frac{\partial}{\partial t_i}$ are naturally T-invariant on the open orbit in X. By [3, Theorem 3.1], these vector fields admit T-invariant holomorphic extensions to the whole of X. In fact, by Lie theory, any T-invariant vector field on X is a complex linear combination of these fields.

Now, let $A_1, \ldots, A_n$ be pairwise-commuting linear endomorphisms of V_{x_0} that respect the Σ-filtration. Then by Klyachko's theorems [12, Proposition 2.1.1, Theorem 2.2.1], these define T-equivariant endomorphisms $\phi_1, \ldots, \phi_n$ of V such that $\phi_j(x_0) = A_j$. Therefore, each ϕ_j is an invariant section of $\operatorname{End}(V)$. Applying equation (2.4) to ϕ_j we see that

$$\{t\phi_j(t^{-1}x_0) = \phi_j(x_0)\} \implies \{\phi_j(t^{-1}x_0) = t^{-1}\phi_j(x_0)\}$$

for all $t \in T$. It then follows from (2.5) that the $\phi_j(t^{-1}x_0)$'s commute with each other for every $t \in T$. In other words, they commute mutually on the open dense

T-orbit in X. Therefore, by continuity, the ϕ_j's commute on entire X.

Next, we define $\phi \in H^0(X, \mathrm{End}(V) \otimes TX)$ by

$$\phi = \sum_j \phi_j \, t_j \, \frac{\partial}{\partial t_j} . \tag{3.5}$$

Consider any affine toric chart on X with coordinates $(z_1, \ldots, z_n)$. By [3, Lemma 3.1] we have

$$t_j \frac{\partial}{\partial t_j} = \sum_k c_{jk}(z_1, \ldots, z_n) \frac{\partial}{\partial z_k}, \tag{3.6}$$

where the c_{jk}'s are holomorphic functions. Substituting (3.6) in (3.5), we have the following representation of ϕ in the $(z_1, \ldots, z_n)$ coordinates:

$$\phi = \sum_k \psi_k \frac{\partial}{\partial z_k} ,$$

where $\psi_k = \sum_j c_{jk}(z_1, \ldots, z_n)\phi_j$. Since the ϕ_j's commute and the c_{jk}'s are scalars, the ψ_k's also mutually commute. Thus by (3.3), ϕ defines a co-Higgs structure on V. Hence, given a tuple $(A_1, \ldots, A_n)$ of commuting filtered endomorphisms of V, we obtain an equivariant co-Higgs structure on V.

In the other direction, given any equivariant co-Higgs structure ϕ on V, we may write ϕ, on the open orbit, as in (3.5). We use the fact any torus-equivariant vector bundle is trivial over the open orbit. As the vector fields $t_j \frac{\partial}{\partial t_j}$ are T-invariant, the ϕ_j's are also T-invariant. Moreover, as the open orbit is an affine toric chart, and ϕ is a co-Higgs field, the ϕ_j's commute mutually by (3.3). Then we define $A_j = \phi_j(x_0)$. As ϕ_j is a T-equivariant endomorphism of V, the endomorphisms A_j respect the Klyachko Σ-filtration.

It is now straightforward to check that the above association is a bijection. □

3.3. **Examples for the theorem.** As an example, we take V to be the tangent bundle TX. We will describe the toric co-Higgs fields on V for certain toric varieties X, and also some co-Higgs fields that are not invariant but only semi-invariant. Let $\rho_1, \ldots, \rho_m$ denote the elements of $|\Sigma(1)|$. We may assume without loss of generality that $\rho_1, \ldots, \rho_n$ form an integral basis of the co-character lattice of T. We choose the closed point x_0 in the open orbit to correspond to the identity element of T. Then the underlying vector space $E = V_{x_0}$ of the Klyachko Σ-filtration may be identified with

$$\mathbb{C}\left\langle \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n} \right\rangle \cong \mathbb{C}\langle \rho_1, \ldots, \rho_n \rangle .$$

Recall that under the action (3.4) the derivation $\frac{\partial}{\partial t_j}$ has weight t_j. It then follows from (3.2) that the Klyachko filtration on $T\mathbb{P}^n$ (cf. [12, p. 350]) is given by

$$E^{\rho_j}(i) = \begin{cases} 0 & \text{if } i > 1 \\ \mathbb{C}\langle \rho_j \rangle & \text{if } i = 1 \\ E & \text{if } i \leq 0\,. \end{cases}$$

Now, let $X = \mathbb{P}^n$. Then, we have $m = n+1$ and $\rho_{n+1} = -\sum_{j=1}^{n} \rho_j$. In this case, the only filtered endomorphisms of E are constant multiples of the identity map. Thus, any toric co-Higgs field on $T\mathbb{P}^n$ is of the form $s \otimes 1$ where $s = \sum a_j t_j \frac{\partial}{\partial t_j}$ for some $a_j \in \mathbb{C}$. However, $T\mathbb{P}^n$ admists others co-Higgs fields. For instance, each $\frac{\partial}{\partial t_j}$ is a global semi-invariant section of $T\mathbb{P}^n$ with weight t_j. Thus, $\frac{\partial}{\partial t_j} \otimes 1$ is a semi-invariant co-Higgs field on the bundle $T\mathbb{P}^n$, which is not toric according to our definition. Moreover, $\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) \otimes 1$ is a co-Higgs field on $T\mathbb{P}^n$, for $n \geq 2$, which is not even semi-invariant.

More generally, we obtain a larger class of filtered endomorphisms of E when X is the product of projective spaces. For instance, let $X = \mathbb{P}^1 \times \mathbb{P}^1$. In this case, $|\Sigma(1)| = \{\rho_1, \ldots, \rho_4\}$ where $\rho_1 = (1, 0) = -\rho_3$ and $\rho_2 = (0, 1) = -\rho_4$. Thus, $E \cong \mathbb{C}\langle \rho_1 \rangle \oplus \mathbb{C}\langle \rho_2 \rangle$ and filtered endomorphisms of E are given by diagonal matrices with respect to this decomposition. Since any two diagonal matrices commute, a toric co-Higgs field on TX is of the form $\phi_1 t_1 \frac{\partial}{\partial t_1} + \phi_2 t_2 \frac{\partial}{\partial t_2}$ where ϕ_1 and ϕ_2 are arbitrary filtered endomorphisms of E.

4. Existence of a moduli scheme

We fix a toric bundle V on a non-singular toric variety X of dimension n with fan Σ. Let x_0 be a closed point of T and put $E = V_{x_0}$ as in the preceding section. We use H^ρ to refer to the parabolic subgroup of $\mathrm{GL}(E)$ that preserves the filtration E^ρ on E. Then, the group of endomorphisms of the Σ-filtration $\{E^\rho(i)\}$ coincides with the group $H_V := \bigcap_\rho H^\rho$. Notice that the group H_V contains the center of $\mathrm{GL}(E)$. Denote by $H_V[n]$ the set of n-tuples of pairwise-commuting elements of the group H_V. Now, Theorem 3.2 can be recast as:

Corollary 4.1. *If X is a nonsingular complex projective toric variety and V is any toric bundle on X, then invariant co-Higgs fields ϕ for V are in $1:1$ correspondence with elements of $H_V[n]$.*

Now, we wish to consider all isomorphism classes $[V]$ of toric bundles on X having fixed rank r and fixed T-equivariant Chern classes. These classes can be defined explicitly within the equivariant Chow cohomology ring of X in terms of Klyachko's filtration as per [13]. As per [14], let $\mathcal{V}_X^{fr}(r, \psi)$ be the fine moduli space of rank-r toric vector bundles framed at x_0 and with total equivariant Chern class ψ. It is then a result of Payne [14, Corollary 3.11] that if $\mathrm{PGL}(r)$ acts freely on $\mathcal{V}_X^{fr}(r,\,\psi)$, then there exists an associated coarse moduli scheme $\mathcal{V}_X(r,\psi)$ of toric bundles on X with that total equivariant Chern class. In light of this, our result gives rise to:

Corollary 4.2. *When the group* $\mathrm{PGL}(r)$ *acts freely on* $\mathcal{V}_X^{fr}(r,\,\psi)$*, there exists a scheme*

$$\mathcal{C}_X(r,\psi) \xrightarrow{\pi} \mathcal{V}_X(r,\psi),$$

with fibers $\pi^{-1}([V]) \cong H_V[n]$*, that can be identified with a quasiprojective coarse moduli scheme of toric co-Higgs bundles on* X *of fixed rank* r *and total equivariant Chern class* ψ*.*

We take the opportunity now to mention a few natural questions for further study. First, assuming the fibration π exists, when is it flat? Recall that flat morphisms, in reasonable circumstances, have strong topological properties. For instance, the fibers of a surjective, faithfully flat morphism of irreducible varieties will have the expected dimension, which is the difference in the dimensions of the ambient schemes.

Second, does $\mathcal{C}_X(r,\,\psi)$ inherit arbitrarily bad singularities from $\mathcal{V}_X(r,\psi)$, as per the "Murphy's Law" for toric bundles in [14, Section 4], and under which condition(s) does it becomes smooth? It is known that in the rank-2 case that the framed moduli $\mathcal{V}_X^{fr}(2,\psi)$ is smooth. It is natural to ask whether this holds for co-Higgs case.

Moreover, it would be desirable to understand the relationship of this construction to either Mumford-Takemoto or Gieseker stability for co-Higgs bundles in general. In particular, Simpson's moduli space of Λ-modules [19], where Λ is a coherent sheaf of $\mathcal{O}_X$-modules, produces a moduli space of Gieseker-stable coherent co-Higgs sheaves on X when $\Lambda = \mathrm{Sym}^\bullet(T^*X)$ (cf. [17, Section 2] for further details on this correspondence in the co-Higgs setting). Interpreting the variation of toric structures on V with regards to the moduli problem for Λ-modules is an interesting direction for further exploration. Finally, classifying toric co-Higgs bundles over singular toric varieties is a natural problem to explore.

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