EPJ manuscript No.
(will be inserted by the editor)

Theory of unitarity bounds and low energy form factors

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arXiv:1004.4257v2 [hep-ph] 24 Jun 2010

June 12, 2018

Abstract. We present a general formalism for deriving bounds on the shape parameters of the weak and electromagnetic form factors using as input correlators calculated from perturbative QCD, and exploiting analyticity and unitarity. The values resulting from the symmetries of QCD at low energies or from lattice calculations at special points inside the analyticity domain can be included in an exact way. We write down the general solution of the corresponding Meiman problem for an arbitrary number of interior constraints and the integral equations that allow one to include the phase of the form factor along a part of the unitarity cut. A formalism that includes the phase and some information on the modulus along a part of the cut is also given. For illustration we present constraints on the slope and curvature of the K_{l3} scalar form factor and discuss our findings in some detail. The techniques are useful for checking the consistency of various inputs and for controlling the parameterizations of the form factors entering precision predictions in flavor physics.

1 Introduction

Form factors are of central importance in strong interaction dynamics, providing information on the nature of the strong force and confinement. Phenomenologically, the weak form factors are of crucial importance for the determination of standard model parameters such as the elements of the Cabibbo-Kobayashi-Maskawa matrix.

Bounds on the form factors when a suitable integral of the modulus squared along the unitarity cut is known from an independent source were considered in the 1970's in the context of the hadronic contribution to muon anomaly and the kaon semileptonic decays [1,2,3,4,5,6,7,8,9,10, 11] (for a topical review of the results at that time, see [12]). Through complex analysis, this condition leads to constraints on the values at interior points or on the expansion parameters around $t = 0$, such as the slope and curvature, and belong to a class of problems referred to as the Meiman problem [13]. Mathematically, the problem belongs to the standard analytic interpolation theory for functions in the Hardy class H^2 [13,14,15,16]. The integral condition was provided either from an observable (like muon's $g-2$ in the case of the electromagnetic form factor of the pion), or from the dispersion relation satisfied by a suitable correlator, whose positive spectral function has, by unitarity, a lower bound involving the modulus squared of the relevant form factor. Therefore, the constraints derived in this framework are often referred to as "unitarity bounds".

An important step forward was achieved in [17], where it was noted that the correlator of interest for the K_{l3} form factors can be evaluated reliably in the deep Euclidean region by perturbative QCD. The modern approach clarified also the issue of the number of subtractions required in the dispersion relation for the correlator, which was not always treated correctly in the prior studies.

The first applications in the modern approach concerned mainly the form factors relevant for the $B \to D$ and $B \to D^*$ semileptonic decay, or the so-called Isgur-Wise function, where heavy quark symmetry provided strong additional constraints at interior points [18,19,20,21,22, 23,24,25]. More recent applications revisited the electromagnetic form factor of the pion [26,27,28,29], the strangeness changing $K\pi$ form factors [30,31,32,33], and the $B\pi$ vector form factor [34,35]. The results confirm that the approach represents a useful tool in the study of the form factors, complementary and free of additional assumptions inherent in standard dispersion relations.

The purpose of the present paper is to present in a systematic way the technique of unitarity bounds, including its most recent developments and offering explicit formulas that can be applied in future studies. There are several recent developments which increase the interest in these techniques and justify the present review. First, the correlators used as input are calculated now with greater precision in perturbative QCD, in many cases up to the order α_s^4. Also, calculations with greater precision in Chiral Perturbation Theory (ChPT), Heavy Quark Effective Theory (HQET), Soft Collinear Effective Theory (SCET), or on the lattice, provide improved values of the form factors at some specific points. The techniques presented here are the optimal frame of including inputs coming from sepa-

rate sources and testing their consistency. Moreover, improved information about the phase of the form factor is available by Watson's theorem [36] from the associated elastic scattering, and in some cases also the modulus is measured independently along a part of the cut.

We first present in section 2 our notation and formalism and in section 3 the general Meiman interpolation problem, for an arbitrary number of derivatives at the origin and an arbitrary number of values at points inside the analyticity domain. The solution is obtained either by Lagrange multipliers, or by the techniques of analytic interpolation theory [37,38], and is written in two equivalent ways, as a determinant of a suitable matrix and as a compact convex quadratic form.

In section 4 we present the complete treatment of the inclusion of the phase on a part of the unitarity cut, along with an arbitrary number of constraints of the Meiman type. No such treatment is found in the literature despite the long history of the problem [5,6,9,26,30]. We derive two equivalent sets of integral equations, using, as in section 3, either Lagrange multipliers or the analytic interpolation theory.

In section 5 we treat the situation when, in addition to the phase, some information on the modulus along a part of the cut is available from an independent source. This is a mathematically more complicated problem [8,12]. In this paper we shall present an approach proposed in [26], which uses the fact that the knowledge of the phase allows one to describe exactly the elastic cut of the form factor by means of the Omnès function. The problem is thus reduced to a standard Meiman problem on a larger analyticity domain. The method is very powerful and was recently employed to provide stringent constraints on the scalar $K\pi$ form factor at low energies [33]. However, while the method of treating the phase in section 4 automatically takes into account all the constraints of the original problem, this is not the case with the technique of this section: it provides necessary constraints for the input, which however may violate the original unitarity inequality. Therefore, the allowed domain for the input values is given by the intersection of the domains derived by the formulas of sections 4 and 5. We illustrate this fact in section 6, where we present constraints on the slope and curvature of the K_{l3} scalar form factor. This section extends the results previously reported in [33]. Finally, in the last section we summarize our conclusions and discuss possible applications.

This paper provides a comprehensive treatment of important mathematical and theoretical tools that are essential for improving precision studies of form factors that are of great importance to the standard model.

2 Notation and Formalism

Let $F(t)$ denote a generic form factor, which is real analytic in the complex t-plane cut along the positive real axis from the lowest unitarity branch point t_+ to ∞. The essential condition exploited in the present context is an inequality of the type:

$$\int_{t_+}^{\infty} dt\ \rho(t)|F(t)|^2 \leq I, \tag{1}$$

where $\rho(t) \geq 0$ is a positive semi-definite weight function and I is a known quantity. As mentioned in the Introduction, such inequalities can be obtained starting from a dispersion relation satisfied by a suitable correlator, evaluated in the deep Euclidean region by perturbative QCD, and whose spectral function is bounded from below by a term involving the modulus squared of the relevant form factor. An example will be presented in section 6.

In the analysis of the semileptonic decays one is interested in the parameters of the Taylor expansion at the origin, written as

$$F(t) = F(0)\left[1 + \lambda'\,\frac{t}{M^2} + \lambda''\,\frac{t^2}{2M^4} + \cdots\right], \tag{2}$$

where M is a suitable mass and λ' and λ'' denote the dimensionless slope and curvature, respectively. Also, from the symmetries of QCD at low energies and, more recently, from lattice calculations, one may know $F(t)$ at several special points inside the analyticity domain. The standard unitarity bounds exploit analyticity of the form factor and the inequality (1) in order to correlate in an optimal way these values and the expansion parameters in (2).

Additional information on the unitarity cut can be included in the formalism. According to Watson's theorem [36], below the inelastic threshold $t_{\rm in}$ the phase of $F(t)$ is equal (modulo π) to the phase $\delta(t)$ of the associated elastic scattering process. Thus,

$$F(t + i\epsilon) = |F(t)|e^{i\delta(t)}, \qquad t_+ < t < t_{\rm in}, \tag{3}$$

where $\delta(t)$ is known. Moreover, in certain cases also some information on the modulus $|F(t)|$, or a bound on it, is available on the same range $t_+ < t < t_{\rm in}$. In section 3 we consider the standard version of the unitarity bounds, with no information about the phase and the modulus, except the inequality (1). The inclusion of the phase and modulus will be discussed in sections 4 and 5.

For the subsequent treatment, the problem is brought to a canonical form by making the conformal transformation

$$z(t) = \frac{\sqrt{t_+} - \sqrt{t_+ - t}}{\sqrt{t_+} + \sqrt{t_+ - t}}\,, \tag{4}$$

that maps the cut t-plane onto the unit disc $|z| < 1$ in the $z \equiv z(t)$ plane, such that t_+ is mapped onto $z = 1$, the point at infinity to $z = -1$ and the origin to $z = 0$. After this mapping, the inequality (1) is written as

$$\frac{1}{2\pi}\int_0^{2\pi} {\rm d}\theta |g(e^{i\theta})|^2 \leq I, \tag{5}$$

where the analytic function $g(z)$ is defined as

$$g(z) = F(t(z))w(z). \tag{6}$$

Here $t(z)$ is the inverse of (4) and $w(z)$ is an *outer function*, *i.e.* a function analytic and without zeros in $|z| < 1$, such that its modulus on the boundary is related to $\rho(\theta)$ and the Jacobian of the transformation (4). In general, an outer function is obtained from its modulus on the boundary by the integral

$$w(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \frac{e^{i\theta}+z}{e^{i\theta}-z}\, \ln|w(e^{i\theta})|\right]. \tag{7}$$

In particular cases of physical interest, $w(z)$ has a simple analytic form (for an example see section 6).

The function $g(z)$ is analytic within the unit disc and can be expanded as:

$$g(z) = g_0 + g_1 z + g_2 z^2 + \cdots, \tag{8}$$

and (1) implies

$$\sum_{k=0}^{\infty} g_k^2 \le I. \tag{9}$$

Using (6), the real numbers g_k are expressed in a straightforward way in terms of the coefficients of the Taylor expansion (2). The inequality (9) represents the simplest "unitarity bound" for the shape parameters defined in (2). In what follows we shall improve it by including additional information on the form factor.

3 Meiman problem

We consider the general case when the first K derivatives of $g(z)$ at $z = 0$ and the values at N interior points are assumed to be known:

$$\left[\frac{1}{k!}\frac{d^k g(z)}{dz^k}\right]_{z=0} = g_k, \quad 0 \le k \le K-1;$$
$$g(z_n) = \xi_n, \quad 1 \le n \le N, \tag{10}$$

where g_k and ξ_n are given numbers. They are related, by (6), to the derivatives $F^{(j)}(0)$, $j \le k$ of $F(t)$ at $t = 0$, and the values $F(t(z_n))$, respectively. For simplicity and in view of phenomenological inputs that we will use, we assume the points z_n to be real, so ξ_n are also real.

Meiman problem [13] requires us to find the optimal constraints satisfied by the numbers defined in (10) if (5) holds. This mathematical problem is also known as a general Schur-Carathéodory-Pick-Nevanlinna interpolation [14,15,16].

One can prove [37,38] that the most general constraint satisfied by the input values appearing in (10) is given by the inequality:

$$\mu_0^2 \le I, \tag{11}$$

where μ_0^2 is the solution of the minimization problem:

$$\mu_0^2 = \min_{g\in\mathcal{G}} ||g||^2_{L^2}. \tag{12}$$

Here $||g||^2_{L^2}$ denotes the L^2 norm, *i.e.* the quantity appearing in the l.h.s. of (5) or (9), and the minimum is taken over the class $\mathcal{G}$ of analytic functions which satisfy the conditions (10). In the next subsections we shall solve the minimization problem (12) by two different methods.

3.1 Lagrange multipliers

One may set up a Lagrangian for the minimization problem (12) with the constraints (10):

$$\mathcal{L} = \frac{1}{2}\sum_{k=0}^{\infty} g_k^2 + \sum_{n=1}^{N} \alpha_n \bar{\xi}_n \tag{13}$$

where α_n are Lagrange multipliers, and $\bar{\xi}_n$ are known numbers defined as

$$\bar{\xi}_n = \xi_n - \sum_{k=0}^{K-1} g_k z_n^k. \tag{14}$$

Solving the Lagrange equations obtained by varying with respect to g_k for all $k \ge K$, and eliminating the Lagrange multipliers yields the solution of the minimization problem (12). For purposes of illustration, when $N = 2$, the Lagrange equations yield

$$g_k = \alpha_1 z_1^k + \alpha_2 z_2^k, \qquad k \ge K, \tag{15}$$

and the solution of the minimization (12) takes the form:

$$\mu_0^2 = \sum_{k=0}^{K-1} g_k^2 + \alpha_1 \sum_{k=K}^{\infty} g_k z_1^k + \alpha_2 \sum_{k=K}^{\infty} g_k z_2^k. \tag{16}$$

Then inequality (11) can be expressed in terms of the two Lagrange multipliers as:

$$\alpha_1 \bar{\xi}_1 + \alpha_2 \bar{\xi}_2 \le \bar{I}, \tag{17}$$

where

$$\bar{I} = I - \sum_{k=0}^{K-1} g_k^2 \tag{18}$$

and the constraint conditions themselves are

$$\alpha_1 \frac{z_1^{2K}}{1-z_1^2} + \alpha_2 \frac{(z_1 z_2)^K}{1-z_1 z_2} = \bar{\xi}_1,$$
$$\alpha_1 \frac{(z_1 z_2)^K}{1-z_1 z_2} + \alpha_2 \frac{z_2^{2K}}{1-z_2^2} = \bar{\xi}_2. \tag{19}$$

The consistency of eqs. (17) and (19) can be written as:

$$\begin{vmatrix} \bar{I} & \bar{\xi}_1 & \bar{\xi}_2 \\ \bar{\xi}_1 & \dfrac{z_1^{2K}}{1-z_1^2} & \dfrac{(z_1 z_2)^K}{1-z_1 z_2} \\ \bar{\xi}_2 & \dfrac{(z_1 z_2)^K}{1-z_1 z_2} & \dfrac{z_2^{2K}}{1-z_2^2} \end{vmatrix} \ge 0. \tag{20}$$

This can be readily extended to the case of N constraints:

$$\begin{vmatrix} \bar{I} & \bar{\xi}_1 & \bar{\xi}_2 & \cdots & \bar{\xi}_N \\ \bar{\xi}_1 & \dfrac{z_1^{2K}}{1-z_1^2} & \dfrac{(z_1 z_2)^K}{1-z_1 z_2} & \cdots & \dfrac{(z_1 z_N)^K}{1-z_1 z_N} \\ \bar{\xi}_2 & \dfrac{(z_1 z_2)^K}{1-z_1 z_2} & \dfrac{(z_2)^{2K}}{1-z_2^2} & \cdots & \dfrac{(z_2 z_N)^K}{1-z_2 z_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\xi}_N & \dfrac{(z_1 z_N)^K}{1-z_1 z_N} & \dfrac{(z_2 z_N)^K}{1-z_2 z_N} & \cdots & \dfrac{z_N^{2K}}{1-z_N^2} \end{vmatrix} \ge 0. \tag{21}$$

Alternatively, the solution can be obtained by introducing Lagrange multipliers also for the given coefficients g_k, $k = 0, ..., K-1$ in (13), as was done in ref. [10]. This leads to the inequality, equivalent to (21):

$$\begin{vmatrix} I & g_0 & g_1 & \cdots & g_{K-1} & \xi_1 & \xi_2 & \cdots & \xi_N \\ g_0 & 1 & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ g_1 & 0 & 1 & 0 & 0 & z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ g_{K-1} & 0 & 0 & \cdots & 1 & z_1^{K-1} & z_2^{K-1} & \cdots & z_N^{K-1} \\ \xi_1 & 1 & z_1 & \cdots & z_1^{K-1} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \cdots & \frac{1}{1-z_1 z_N} \\ \xi_2 & 1 & z_2 & \cdots & z_2^{K-1} & \frac{1}{1-z_1 z_2} & \frac{1}{1-z_2^2} & \cdots & \frac{1}{1-z_2 z_N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \xi_N & 1 & z_N & \cdots & z_N^{K-1} & \frac{1}{1-z_1 z_N} & \frac{1}{1-z_2 z_N} & \cdots & \frac{1}{1-z_N^2} \end{vmatrix} \geq 0. \tag{22}$$

This condition is expressed in a straightforward way in terms of the values of the form factor $F(t)$ at $t_i = t(z_i)$ and the derivatives at $t = 0$, using eqs. (4) and (6). It can be shown that (22) defines a convex domain in the space of the input parameters.

3.2 Analytic interpolation theory

Instead of using Lagrange multipliers, one can implement the constraints (10) by expanding the function $g \in \mathcal{G}$ in the most general way as [37,38]

$$g(z) = \sum_{k=0}^{K-1} g_k z^k + z^K \sum_{n=1}^{N} A_n B_n(z) + z^K B_{N+1}(z) h(z). \tag{23}$$

Here the functions $B_n(z)$ are products of Blaschke factors[1] defined recurrently as

$$B_1(z) = 1, B_n(z) = \frac{z - z_{n-1}}{1 - z z_{n-1}} B_{n-1}(z), 2 \leq n \leq N+1, \tag{24}$$

and the numbers A_n are obtained by solving the system of equations

$$\sum_{n=1}^{m} A_n B_n(z_m) = \frac{1}{z_m^K} \left[\xi_m - \sum_{k=0}^{K-1} g_k z_m^k \right], \quad 1 \leq m \leq N, \tag{25}$$

where we took into account the fact that $B_n(z_m) = 0$ for $n > m$. We recall that an expansion equivalent, but slightly different from (23), was proposed in [4].

The function $h(z)$ is analytic in $|z| < 1$ and is free of constraints. Expressed in terms of h, the minimum norm problem (12) becomes:

$$\mu_0^2 = \min_{\{h\}} ||h - H||_{L^2}^2, \tag{26}$$

where H is a function defined on the boundary of the unit disc $\zeta = \exp(i\theta)$ as

$$H(\zeta) = -\frac{1}{\zeta^K B_{N+1}(\zeta)} \left[\sum_{k=0}^{K-1} g_k \zeta^k + \zeta^K \sum_{n=1}^{N} A_n B_n(\zeta) \right], \tag{27}$$

and is meromorphic in $|z| < 1$.

The solution of (26) is straightforward. We expand:

$$h(\zeta) = \sum_{l=0}^{\infty} h_l \zeta^l, \qquad H(\zeta) = \sum_{l=-\infty}^{\infty} H_l \zeta^l, \tag{28}$$

where H_l are known real numbers defined as

$$H_l = \frac{1}{2\pi i} \int_{|\zeta|=1} \zeta^{-l} H(\zeta) \frac{\mathrm{d}\zeta}{\zeta}, \qquad -\infty < l < \infty. \tag{29}$$

Using the expansions (28), we write (26) as:

$$\mu_0^2 = \min_{\{h_l\}} \left[\sum_{l=0}^{\infty} (h_l - H_l)^2 + \sum_{l=-\infty}^{-1} H_l^2 \right], \tag{30}$$

where the minimization is taken upon the numbers h_l. The minimum is reached for

$$h_l = H_l, \qquad l \geq 0, \tag{31}$$

which leads to the minimum

$$\mu_0^2 = \sum_{l=-\infty}^{-1} H_l^2. \tag{32}$$

The coefficients H_l for $l \leq -1$ are calculated by inserting into (29) the expression of H from (27) and applying the residues theorem. The poles are produced by the factors B_{N+1} and ζ^{K+l-k}, for $K + l - k \geq 0$. The contribution of B_{N+1} to H_l is written as,

$$-\sum_{n=1}^{N} \left[\frac{z - z_n}{B_{N+1}(z)} \right]_{z=z_n} z_n^{-l-1} \left[\sum_{k=1}^{K-1} g_k z_n^{k-K} + \sum_{m=1}^{n} A_m B_m(z_n) \right], \tag{33}$$

where in the parenthesis we recognize from (25) the numbers ξ_n / z_n^K. The factor ζ^{K+l-k} contributes to H_l as

$$-\sum_{k=0}^{K-1} \theta(K+l-k) \frac{g_k}{(K+l-k)!} \frac{\mathrm{d}^{K+l-k}}{\mathrm{d}z^{K+l-k}} \left[\frac{1}{B_{N+1}(z)} \right]_{z=0}. \tag{34}$$

Collecting the terms (33) and (34) we obtain, for $l \leq -1$,

$$H_l = -\sum_{n=1}^{N} \frac{Y_n \xi_n}{z_n^{l+K+1}} - \sum_{k=0}^{K-1} \theta(K+l-k)\, g_k \beta_{kl}, \tag{35}$$

where we denoted

$$Y_n = \left[\frac{z - z_n}{B_{N+1}(z)} \right]_{z=z_n},$$
$$\beta_{kl} = \frac{1}{(K+l-k)!} \frac{d^{K+l-k}}{dz^{K+l-k}} \left[\frac{1}{B_{N+1}(z)} \right]_{z=0}. \tag{36}$$

[1] A Blaschke factor is a function $B(z)$ analytic in $|z| < 1$, which satisfies the conditions $B(z_0) = 0$ for a value $|z_0| < 1$ and $|B(e^{i\theta})| = 1$ [14].

Using (35) it is easy to calculate the sum required in (32). Due to the θ function, only the values $l \geq k-K$ in the second term of H_l give non-vanishing contributions. The result is written in a compact form as:

$$\mu_0^2 = \sum_{m,n=1}^{N} \mathcal{A}_{mn}\xi_n\xi_m + \sum_{j,k=0}^{K-1} \mathcal{B}_{jk} g_j g_k + 2\sum_{n=1}^{N}\sum_{k=0}^{K-1} \mathcal{C}_{kn} g_k \xi_n, \tag{37}$$

where we defined

$$\mathcal{A}_{mn} = \frac{Y_n Y_m}{z_n^K z_m^K}\frac{1}{1-z_n z_m}, \quad \mathcal{B}_{jk} = \sum_{l=L}^{-1}\beta_{jl}\beta_{kl},$$
$$\mathcal{C}_{kn} = \frac{Y_n}{z_n^K}\sum_{l=k-K}^{-1}\frac{\beta_{kl}}{z_n^{l+1}}, \tag{38}$$

and $L = \max(k-K,\, j-K)$. Inserting (37) into the inequality (11) gives the allowed domain of the input values appearing in (10):

$$\sum_{m,n=1}^{N} \mathcal{A}_{mn}\xi_n\xi_m + \sum_{j,k=0}^{K-1} \mathcal{B}_{jk} g_j g_k + 2\sum_{n=1}^{N}\sum_{k=0}^{K-1} \mathcal{C}_{kn} g_k \xi_n \leq I. \tag{39}$$

It can be checked that the domains given by (21) and (39) are equivalent.

4 Inclusion of the phase

In this section we shall impose the condition that the phase of the form factor is known (modulo π) along the elastic part of the cut from the phase of the associated elastic amplitude by Watson's theorem [36]. We start by defining the Omnès function

$$\mathcal{O}(t) = \exp\left(\frac{t}{\pi}\int_{t_+}^{\infty} dt \frac{\delta(t')}{t'(t'-t)}\right), \tag{40}$$

where $\delta(t)$ is known for $t \leq t_{\rm in}$, and is an arbitrary function, sufficiently smooth (*i.e.* Lipschitz continuous) for $t > t_{\rm in}$. From (3) and (40) it follows that

$$\mathrm{Im}\left[\frac{F(t+i\epsilon)}{\mathcal{O}(t+i\epsilon)}\right] = 0, \qquad t_+ \leq t \leq t_{\rm in}. \tag{41}$$

Expressed in terms of the function $g(z)$ this condition becomes

$$\mathrm{Im}\left[\frac{g(e^{i\theta})}{W(\theta)}\right] = 0, \qquad \theta \in (-\theta_{\rm in}, \theta_{\rm in}). \tag{42}$$

Here $\theta_{\rm in}$ is defined by $z(t_{\rm in}) = \exp(i\theta_{\rm in})$ and the function $W(\theta)$ is defined as:

$$W(\theta) = w(e^{i\theta})O(e^{i\theta}), \tag{43}$$

where $w(z)$ is the outer function and

$$O(z) = \mathcal{O}(t(z)). \tag{44}$$

As shown in [5,26,30], the constraint (42) can be imposed by means of a generalized Lagrange multiplier. The constraints at interior points can be treated either with Lagrange multipliers as in subsection 3.1, or by their explicit implementation as in subsection 3.2. Below we shall briefly present these two approaches.

4.1 Lagrange multipliers

The Lagrangian of the minimization problem (12) with the constraints (10) and (42) reads

$$\mathcal{L} = \frac{1}{2}\sum_{k=0}^{\infty} g_k^2 + \sum_{n=1}^{N}\alpha_n(\xi_n - \sum_{k=0}^{\infty} g_k z^k) \tag{45}$$
$$+ \frac{1}{\pi}\sum_{k=0}^{\infty} g_k \lim_{\rm r\to 1}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} \lambda(\theta')|W(\theta')|\mathrm{Im}[[W(\theta')]^{-1} r^k e^{ik\theta'}]d\theta'.$$

The Lagrange multiplier $\lambda(\theta)$ is an odd function, $\lambda(-\theta) = -\lambda(\theta)$ and, as in [26,30], the factor $|W(\theta)|$ was introduced for convenience.

We minimize $\mathcal{L}$ by brute force method with respect to the free parameters g_k with $k \geq K$. The Lagrange multipliers $\lambda(\theta)$ and α_n are found in the standard way by imposing the constraints (10) and (42). This leads to a system of coupled equations, which can be solved numerically.

The calculations are straightforward (see for instance [26]). In order to write the equations in a simple form, it is convenient to define the phase $\Phi(\theta)$ of the function $W(\theta)$ by

$$W(\theta) = |W(\theta)|e^{i\Phi(\theta)}. \tag{46}$$

From (43) we have

$$\Phi(\theta) = \phi(\theta) + \delta(t(e^{i\theta})), \tag{47}$$

where $\phi(\theta)$ is the phase of the outer function $w(e^{i\theta})$ and $\delta(t)$ is the elastic scattering phase shift. We introduce also the functions β_n for $n = 1, ...N$, by

$$\beta_n(\theta) = z_n^K \frac{\sin[K\theta - \Phi(\theta)] - \sin[(K-1)\theta - \Phi(\theta)]}{1 + z_n^2 - 2z_n\cos\theta}. \tag{48}$$

Then the equations for the Lagrange multipliers $\lambda(\theta)$ and α_n take the form:

$$\sum_{k=0}^{K-1} g_k \sin[k\theta - \Phi(\theta)] = \lambda(\theta) - \sum_{n=1}^{N}\alpha_n\beta_n(\theta) \tag{49}$$
$$-\frac{1}{2\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} \mathrm{d}\theta'\lambda(\theta'))\mathcal{K}_\Phi(\theta,\theta'), \qquad \theta \in (-\theta_{\rm in}, \theta_{\rm in}),$$

$$-\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} \lambda(\theta)\beta_n(\theta)\mathrm{d}\theta + \sum_{n'=1}^{N}\alpha_{n'}\frac{(z_n z_{n'})^K}{1 - z_n z_{n'}} = \bar{\xi}_n, \tag{50}$$

where $n = 1, \dots N$.

The integral kernel in (49), defined as

$$\mathcal{K}_\Phi(\theta,\theta') \equiv \frac{\sin[(K-1/2)(\theta-\theta') - \Phi(\theta) + \Phi(\theta')]}{\sin[(\theta-\theta')/2]}, \quad (51)$$

is of Fredholm type if the phase $\Phi(\theta)$ is Lipschitz continuous [30]. Then the above system can be solved numerically in a straightforward manner. Finally, the inequality (11) takes the form:

$$\frac{1}{\pi}\sum_{k=0}^{K-1} g_k \int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta \lambda(\theta) \sin\left[k\theta - \Phi(\theta)\right] + \sum_{n=1}^{N} \alpha_n \bar{\xi}_n \le \bar{I}, \quad (52)$$

with $\bar{I}$ defined in (18). Using the relation (6), the above inequality defines an allowed domain for the values of the form factor and its derivatives at the origin. Note that removing the phase constraint gives back the results of subsection 3.1. The results when phase alone is included ($N = 0$) as in [26] (and references therein) and the case $N = 1$ discussed in [30,28] are also readily reproduced. It must be emphasized that the theory for arbitrary number of constraints is being presented here for the first time.

It is easy to see that, if $t_{\rm in}$ is increased, the allowed domain defined by the inequality (52) becomes smaller. The reason is that by increasing $t_{\rm in}$ the class of functions entering the minimization (12) becomes gradually smaller, leading to a larger value for minimum μ_0^2 entering the definition (11) of the allowed domain.

4.2 Analytic interpolation theory

Alternatively, we shall implement the constraints in the interior points by expressing the function $g(z)$ as in eq. (23) of subsection 3.2, in terms of a function $h(z)$ free of constraints. The Lagrangian can be expressed in terms of the coefficients h_l defined in (28) as:

$$\mathcal{L} = \sum_{l=0}^{\infty}(h_l - H_l)^2 + \sum_{l=-\infty}^{-1} H_l^2 - 2\sum_{l=0}^{\infty} h_l\, c_l + \cdots \quad (53)$$

where

$$c_l = \frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta' \lambda(\theta')\, |W(\theta')|\, {\rm Im}\left[\frac{e^{i\theta'(K+l)} B_{N+1}(e^{i\theta'})}{W(\theta')}\right]. \quad (54)$$

The minimization of the Lagrangian given in (53) with respect to h_l has the simple solution

$$h_l = H_l + c_l, \qquad l \ge 0 \quad (55)$$

leading to the minimum

$$\mu_0^2 = \sum_{l=0}^{\infty} c_l^2 + \sum_{l=-\infty}^{-1} H_l^2. \quad (56)$$

The second sum in the r.h.s. was already evaluated in subsection 3.2. The first term involves the function $\lambda(\theta)$, which we determine by imposing the condition (42), written in terms of the function $h(z)$ expanded as in (28). Using h_l given by (55) as the sum $H_l + c_l$, we obtain by a straightforward calculation the integral equation for a function $\lambda(\theta)$ in the interval $\theta \in (-\theta_{\rm in}, \theta_{\rm in})$:

$$\lambda(\theta) - \frac{1}{2\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta' \lambda(\theta')\mathcal{K}_\Psi(\theta,\theta') = V(\theta), \quad (57)$$

where the kernel $\mathcal{K}_\Psi$ is defined as in (51) in terms of the known function

$$\Psi(\theta) = \arg[B_{N+1}(\exp(i\theta))] - \arg[W(\theta)], \quad (58)$$

and V is a known function defined as:

$$V(\theta) = \sum_{n=1}^{N} \frac{Y_n \xi_n}{z_n^K} \frac{z_n \sin[\Psi(\theta)] - \sin[\Psi(\theta) - \theta]}{1 + z_n^2 - 2z_n\cos\theta} + \sum_{k=0}^{K-1} \frac{g_k}{(K-k-1)!}\left[\frac{{\rm d}^{K-k-1}U(z)}{{\rm d}z^{K-k-1}}\right]_{z=0}, \quad (59)$$

with

$$U(z) = \frac{z\sin[K\theta + \Psi(\theta)] - \sin[(K-1)\theta + \Psi(\theta)]}{B_{N+1}(z)(1 + z^2 - 2z\cos\theta)}. \quad (60)$$

Using the expression (54) of c_l and the integral equation (57), it is straightforward to evaluate the sum in the first term of (56):

$$\sum_{l=0}^{\infty} c_l^2 = \frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta \lambda(\theta) V(\theta). \quad (61)$$

Collecting all the terms in (56), (11) can be written as:

$$\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}} {\rm d}\theta \lambda(\theta) V(\theta) + \sum_{m,n=1}^{N} \mathcal{A}_{mn}\xi_n\xi_m + \sum_{j,k=0}^{K-1} \mathcal{B}_{jk} g_j g_k + 2\sum_{n=1}^{N}\sum_{k=0}^{K-1} \mathcal{C}_{kn} g_k \xi_n \le I. \quad (62)$$

This inequality gives the allowed domain for the input values appearing in the conditions (10) when the phase is known in the elastic region. We note that the arbitrary function $\delta(t)$ for $t > t_{\rm in}$ entering the Omnès function (40) does not appear in the result. The first term in (62) represents the improvement brought by the information on the phase, as can be seen by comparing with (39). It can be checked numerically that the allowed domains described by (52) and (62) are equivalent.

5 Inclusion of phase and modulus

In some cases, information on the modulus of the form factor along an interval of the unitarity cut is available from an independent source. As we mentioned in the Introduction, a rigorous implementation of this information is difficult. In this section, we shall present an approach proposed in [26], which leads to an independent constraint that should be satisfied by the inputs.

We start with the remark that the knowledge of the phase was implemented in the previous section by the relation (41), which says that the function $f(t)$ defined through

$$F(t) = f(t)\mathcal{O}(t), \tag{63}$$

is real in the elastic region. In fact, since the Omnès function $\mathcal{O}(t)$ fully accounts for the elastic cut of the form factor, the function $f(t)$ has a larger analyticity domain, namely the complex t-plane cut only for $t > t_{\rm in}$. Implementing this fact leads to a modified version of the unitarity bounds, proposed in [26]. The method requires also the modulus of the form factor in the elastic region. Indeed, (1) implies that f satisfies the condition

$$\int_{t_{\rm in}}^{\infty} {\rm d}t \rho(t) |\mathcal{O}(t)|^2 |f(t)|^2 \leq I' \tag{64}$$

where

$$I' = I - \int_{t_+}^{t_{\rm in}} {\rm d}t \rho(t) |F(t)|^2. \tag{65}$$

If the modulus $|F(t)|$ is known for $t_+ \leq t \leq t_{\rm in}$, the quantity I' is known. Then (64) leads, through the techniques presented in section 3, to constraints on the values of f inside the analyticity domain.

The problem is brought into the canonical form by the transformation

$$\tilde{z}(t) = \frac{\sqrt{t_{\rm in}} - \sqrt{t_{\rm in} - t}}{\sqrt{t_{\rm in}} + \sqrt{t_{\rm in} - t}}, \tag{66}$$

which maps the complex t-plane cut for $t > t_{\rm in}$ on to the unit disc in the z-plane defined by $z = \tilde{z}(t)$. Then (64) can be written as

$$\frac{1}{2\pi} \int_0^{2\pi} {\rm d}\theta |g(\exp(i\theta))|^2 \leq I', \tag{67}$$

where the function g is now

$$g(z) = \tilde{w}(z)\, \omega(z)\, F(\bar{t}(z))\, [O(z)]^{-1}. \tag{68}$$

Here $\tilde{w}(z)$ is the outer function related to the weight $\rho(t)$ and the Jacobian of the new mapping (66) and $O(z)$ is defined as

$$O(z) = \mathcal{O}(\tilde{t}(z)), \tag{69}$$

where $\tilde{t}(z)$ is the inverse of $z = \tilde{z}(t)$ with $\tilde{z}(t)$ defined in (66), and

$$\omega(z) = \exp\left(\frac{\sqrt{t_{\rm in} - \tilde{t}(z)}}{\pi} \int_{t_{\rm in}}^{\infty} {\rm d}t' \frac{\ln|\mathcal{O}(t')|}{\sqrt{t' - t_{\rm in}}(t' - \tilde{t}(z))} \right). \tag{70}$$

The inequality (67) has exactly the same form as (5). Therefore, by using the techniques of section 3 we derive constraints on the function g at interior points, which can be written in the equivalent forms as in (21) or (39). Using (68), these constaints are expressed in terms of the physically interesting values of the form factor $F(t)$.

In fact, the Omnès function $\mathcal{O}(t)$ defined in (40) is not unique, as it involves the arbitrary function $\delta(t)$ for $t > t_{\rm in}$. We have seen that the results of section 4 are not affected by this arbitrariness. This is true also for the results of this section: the reason is that a change of the function $\delta(t)$ for $t > t_{\rm in}$ is equivalent with a multiplication of $g(z)$ by a function analytic and without zeros in $|z| < 1$ (*i.e.* an outer function). According to the general theory of analytic functions of Hardy class [14,16], the multiplication by an outer function does not change the class of functions used in minimization problems. In our case, the arbitrary function $\delta(t)$ for $t > t_{\rm in}$ enters in both the functions $O(z)$ and $\omega(z)$ appearing in (68), and their ambiguities compensate each other exactly. The independence of the results on the choice of the phase for $t > t_{\rm in}$ is confirmed numerically, for functions $\delta(t)$ that are Lipschitz continuous. It is important to emphasize that the method relies on the Omnès function making an appearance first through a related outer function and then through its inverse, while the function $f(t)$ is merely introduced at an intermediate stage and is subsequently eliminated.

The constraints provided by the technique of this section are expected to be quite strong since they result from a minimization on a restricted class of analytic functions, where the second Riemann sheet of the form factor is described explicitly by the Omnès function. On the other hand, it is easy to see that the fulfillment of the condition (64) does not automatically imply that the original condition (1) is satisfied. More exactly, the technique described here does not impose the knowledge of the modulus in addition and simultaneously with the bound (1) and the knowledge of the phase, but exploits only a consequence of the original conditions of the problem. Therefore, one must calculate separately the allowed domains of the parameters of interest given by the techniques of sections 4 and 5, and take as the final results the intersection of these domains.

6 Example: scalar $K\pi$ form factor

We consider as an example the scalar $K\pi$ form factor $f_0(t)$, presenting constraints on the slope λ_0' and curvature λ_0'', appearing in the expansion

$$f_0(t) = f_0(0) \left[1 + \lambda_0' \frac{t}{M_\pi^2} + \lambda_0'' \frac{t^2}{2M_\pi^4} + \cdots \right], \tag{71}$$

often used in the physical range of the semileptonic decay $K \to \pi l \nu$. We work in the isospin limits, adopting the convention that M_K and M_π are the masses of the charged mesons.

The scalar $K\pi$ form factor has been calculated at low energies in ChPT and on the lattice (for recent reviews see

[39,40,41]). At $t = 0$, the current value $f_0(0) = 0.962 \pm 0.004$ [40] shows that the corrections to the Ademollo-Gatto theorem are quite small. Other low energy theorems frequently used are [42,43]

$$f_0(\Delta_{K\pi}) = \frac{F_K}{F_\pi} + \Delta_{CT}, \quad f_0(\bar{\Delta}_{K\pi}) = \frac{F_\pi}{F_K} + \bar{\Delta}_{CT}, \qquad (72)$$

where $\Delta_{K\pi} = M_K^2 - M_\pi^2$ and $\bar{\Delta}_{K\pi} = -\Delta_{K\pi}$ are the first and second Callan-Treiman points, denoted below as CT_1 and CT_2, respectively. The lowest order values are known from $F_K/F_\pi = 1.193 \pm 0.006$ [40], and the corrections calculated to one loop are $\Delta_{CT} = -3.1 \times 10^{-3}$ and $\bar{\Delta}_{CT} = 0.03$ [44]. In the isospin limit, the higher corrections are negligible at the first point, but are expected to be quite large at the second one.

An inequality of the type (1) is obtained for $f_0(t)$ starting with a dispersion relation satisfied by a suitable correlator of the strangeness-changing current [30,31]:

$$\chi_0(Q^2) \equiv \frac{\partial}{\partial q^2}\left[q^2 \Pi_0\right] = \frac{1}{\pi}\int_{t_+}^{\infty} dt\, \frac{t \mathrm{Im}\Pi_0(t)}{(t+Q^2)^2}, \qquad (73)$$

where unitarity implies the inequality:

$$\mathrm{Im}\Pi_0(t) \geq \frac{3}{2}\frac{t_+ t_-}{16\pi}\frac{[(t-t_+)(t-t_-)]^{1/2}}{t^3}|f_0(t)|^2\,, \qquad (74)$$

with $t_\pm = (M_K \pm M_\pi)^2$. The quantity $\chi_0(Q^2)$ was calculated up to the order α_s^4 in perturbative QCD [45]. From (73) and (74) it follows that in this case the quantity I appearing in (5) is

$$I = \chi_0(Q^2). \qquad (75)$$

For illustration we give also the outer function $w(z)$ entering (6):

$$w(z) = \frac{\sqrt{3}}{32\sqrt{\pi}}\,\frac{M_K - M_\pi}{M_K + M_\pi}\,(1-z)\,(1+z)^{3/2} \times \frac{(1+z(-Q^2))^2}{(1-z\,z(-Q^2))^2}\,\frac{(1-z\,z(t_-))^{1/2}}{(1+z(t_-))^{1/2}}, \qquad (76)$$

Further, below the inelastic threshold $t_{\rm in}$ the phase is known from the $I = 1/2$ S-wave of $K\pi$ elastic scattering [46], while the modulus $|f_0(t)|$ was measured recently from the decay $\tau \to K\pi\nu_\tau$ [47]. For details of the input quantities see [33]. When the modulus is also included, as in section (5), the outer functions in (68) can be written as,

$$\tilde{w}(z) = \frac{\sqrt{3}(M_K^2 - M_\pi^2)}{16\sqrt{2\pi}t_{\rm in}}\,\frac{\sqrt{1-z}\,(1+z)^{3/2}(1+\tilde{z}(-Q^2))^2}{(1-z\,\tilde{z}(-Q^2))^2} \times \frac{(1-z\,\tilde{z}(t_+))^{1/2}\,(1-z\,\tilde{z}(t_-))^{1/2}}{(1+\tilde{z}(t_+))^{1/2}\,(1+\tilde{z}(t_-))^{1/2}}. \qquad (77)$$

In ref. [33] we derived stringent bounds on the slope λ_0' and curvature λ_0'', using as input the values of $f_0(t)$ at $t = 0$ and CT_1, and information on the phase and modulus included in the formalism of section 5. The additional input from the unitarity cut led to a dramatic improvement of the bounds obtained in [29] with the standard unitarity bounds of section 3. In [33] we obtained also a narrow allowed range for the ChPT correction $\bar{\Delta}_{CT}$ at the second Callan-Treiman point CT_2.

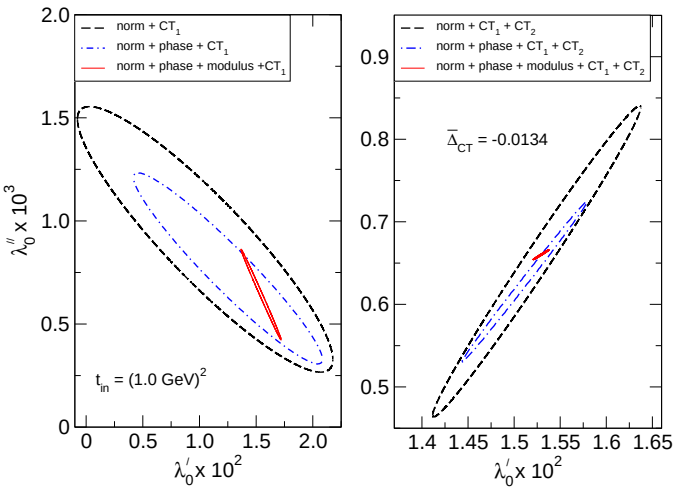

Fig. 1. Allowed regions in the slope-curvature plane of the scalar $K\pi$ form factor, obtained with the techniques of sections 3-5. For explanations see the text.

In the present paper we further illustrate the techniques presented in sections 3, 4 and 5, by comparing their constraining power for various inputs. In fig. 1 we present the allowed domain for the slope and curvature of $f_0(t)$, obtained with the methods described above: the large ellipse is obtained with standard unitarity bounds of section 3, the intermediate ellipse includes the phase up to $t_{\rm in} = 1.0\,\mathrm{GeV}^2$ with the method of subsection 4.1, and the small ellipse is obtained with the method of section 5, for the same $t_{\rm in}$. The domains shown in the left panel are derived using as input the normalization $f_0(0) = 0.962$ and $f_0(\Delta_{K\pi}) = 1.193$, those in the right panel use as input also a value at the second Callan-Treiman point $f_0(\bar{\Delta}_{K\pi}) = 1/1.193 + \bar{\Delta}_{CT}$ for a certain choice $\bar{\Delta}_{CT} = -0.0134$.

As emphasized in section 4, the domain obtained with the inclusion of the phase should be contained entirely inside the domain obtained with the expressions of section 3, since it is related to a minimization on a smaller class of admissible functions. This is confirmed by the large and intermediate ellipses of fig. 1. On the other hand, the small ellipses are given by a minimization on an admissible class, defined in section 5, which is not *a priori* contained in the class defined in section 4. Therefore, the small ellipses need not to be contained entirely in the intermediate ones.

In the left panel of fig. 1, the small ellipse is contained inside the other two, which means that for points inside this allowed domain all the constraints are fulfilled. In the right panel, where we use as input also the value at CT_2, all the domains shrink, and the small ellipse has a part situated outside the intermediate one. To satisfy all the constraints, one should take the intersection of the small and intermediate ellipses.

Clearly, by increasing $t_{\rm in}$, *i.e.* the energy up to which the phase (and the modulus) are given, the system is more and more constrained and one may reach a situation when the inputs become inconsistent. This is illustrated in figs. 2 and 3, where we show the configuration of the allowed domains for two larger values of $t_{\rm in}$. The large ellipses are the same in all figures, since they are are independent of $t_{\rm in}$. As follows from the arguments of section 4, the intermediate ellipses become gradually smaller when $t_{\rm in}$ is increased. In the left panels, obtained with the normalization at $t = 0$ and the value at CT_1, the small ellipses are contained inside the intermediate ones, indicating that we can find an allowed domain that satisfies all the constraints. However, if we impose also the constraint at CT_2, the system becomes over-constrained, and it is impossible to find a domain that satisfies all the constraints. Indeed, in the right panels of figs. 2 and 3, the small ellipses are not inside the intermediate ones.

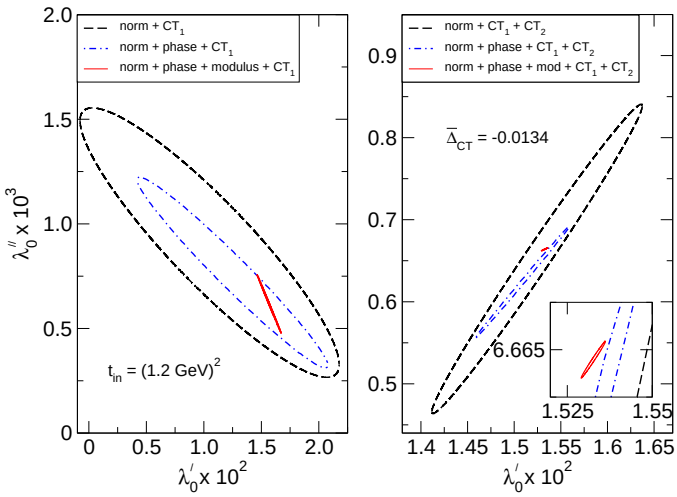

Fig. 2. As in fig. 1 for $t_{\rm in} = (1.2\,\mathrm{GeV})^2$. The inset zooms in on the smallest ellipse obtained using the techniques of section 5.

The same behavior is illustrated in figs. 4 and 5, where we show the allowed regions in the slope-curvature plane for various values of the inelastic threshold $t_{\rm in}$. The left panels show that by increasing $t_{\rm in}$ the ellipses obtained with the phase included as in section 4 become gradually smaller and are contained one within the other. This is not the case with the ellipses derived with the phase and modulus included as in section 5. In the particular case when both CT_1 and CT_2 are used as input, the ellipses obtained with various $t_{\rm in}$ have a zero intersection (see right panel of fig. 5). This signals an inconsistency of the various quantities used as input (normalization at $t = 0$, values at CT_1 and CT_2, the phase up to $t_{\rm in}$ and the modulus below $t_{\rm in}$). In particular, the assumption of neglecting the inelastic effects up to $t_{\rm in} = (1.4\,\mathrm{GeV})^2$ may be too strong. We note however that the above figures were obtained with the central values of the inputs described in [33]. Of course, in phenomenological analyses one should account for the errors of the various pieces of the input. We did

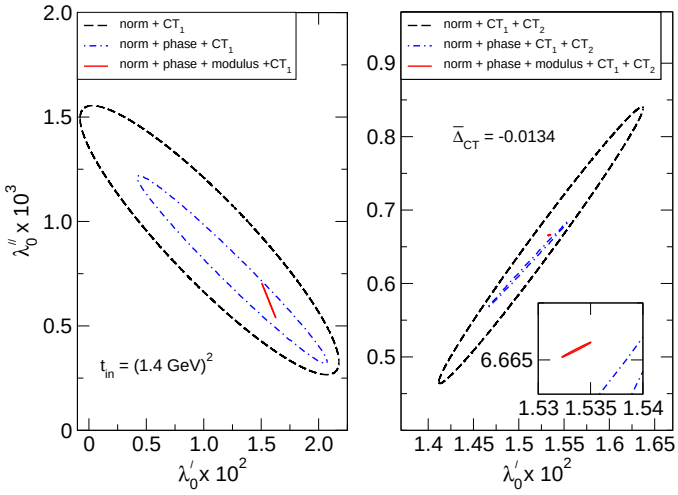

Fig. 3. As in fig. 1 for $t_{\rm in} = (1.4\,\mathrm{GeV})^2$. The inset zooms in on the smallest ellipse obtained using the techniques of section 5.

not consider this aspect here, because our purpose was only to illustrate the mathematical techniques in a definite framework. Thus we have provided a concrete illustration of the various techniques discussed in detail in the previous sections. The phenomenological implications of the results will be analyzed in a future work.

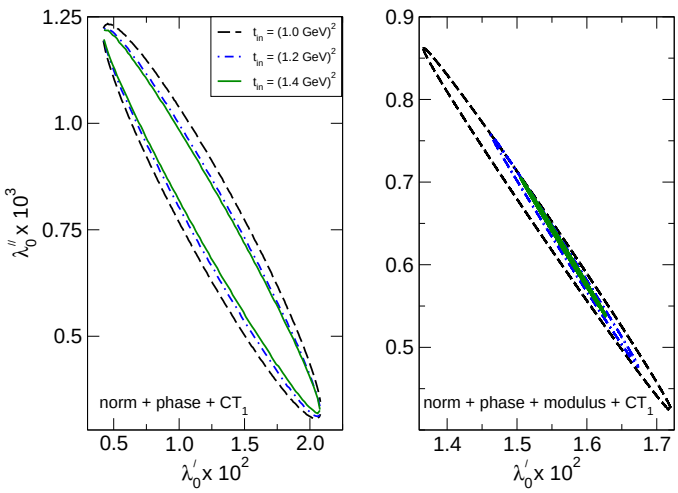

Fig. 4. Allowed region in the slope-curvature plane for various $t_{\rm in}$ using as input the normalization at $t = 0$ and the value at CT_1. Left: phase included as in section 4. Right: phase and modulus included as in section 5.

7 Conclusions

In this paper we reviewed the method of unitarity bounds, extended in order to include information on the phase and modulus of the form factor on the unitarity cut. We provided explicit formulas, easily implementable in *Mathe-*

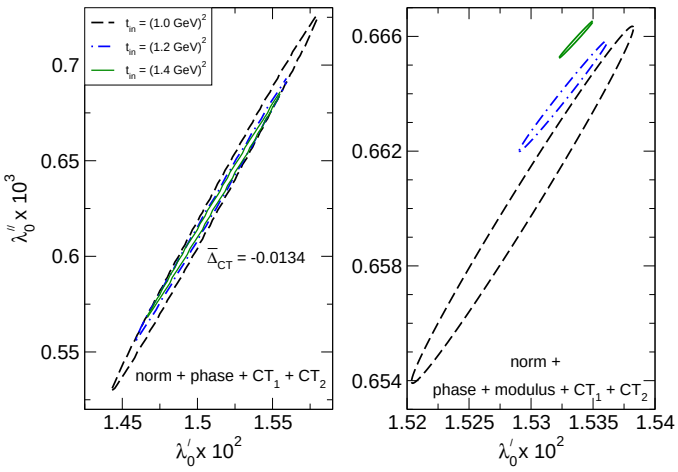

Fig. 5. Allowed region in the slope-curvature plane for various $t_{\rm in}$ using as input the normalization at $t = 0$ and the values at CT_1 and CT_2. Left: phase included as in section 4. Right: phase and modulus included as in section 5.

matica or C programs, for an arbitrary number of derivatives at $t = 0$ and an arbitrary number of interior points.

The method is very suitable for correlating through analyticity various pieces of information about the form factors: perturbative QCD, lattice calculations and effective theories like ChPT or SCET. It does not depend on specific assumptions usually adopted in the standard dispersion relations, like the absence of zeros, or the behavior of the form factor above the inelastic threshold.

As shown in section 6, the techniques presented here provide strong constraints on the shape parameters of the form factors, of interest for the parameterizations of the experimental data. The method can be used also to test the low energy theorems and put constraints on the higher order corrections of ChPT [33]. Moreover, it can be adapted in order to control theoretically the truncation error of the experimental parametrization [25,35].

Another possible application is the detection of the zeros of the form factors. The problem is of interest for the Omnès representations used recently for the parametrization of various form factors, which assume that the zeros are absent. For such a study, one has to assume that $F(t_0) = 0$ for a certain unknown t_0, include this condition among the interior constraints (10) and test the consistency of the inputs, in one of the versions presented in sections 3-5. The method then can give in an unambiguous way the points t_0 where zeros are excluded.

The method of unitarity bounds proved to be very useful for the description of the $B \to D$, $B \to \pi$ or $K \to \pi$ form factors. It can be applied also to other form factors, such as those describing $D \to \pi$ semileptonic decays, or the scalar form factors of the pion or kaon, for which bounds of the type (1) can be obtained from the dispersion relation of a suitable correlator calculated in perturbative QCD. The method is a valuable tool for increasing the precision of the predictions in low energy flavor physics, which has been discussed here in great detail and generality in an accessible manner.

Acknowledgements. BA thanks DST, Government of India, and the Homi Bhabha Fellowships Council for support. IC acknowledges support from CNCSIS in the Program Idei, Contract No. 464/2009, and from ANCS, project PN 09370102.

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