TANNAKIAN CLASSIFICATION OF EQUIVARIANT PRINCIPAL BUNDLES ON TORIC VARIETIES

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Abstract. Let X be a complete toric variety equipped with the action of a torus T and G a reductive algebraic group, defined over an algebraically closed field K. We introduce the notion of a compatible Σ–filtered algebra associated to X, generalizing the notion of a compatible Σ–filtered vector space due to Klyachko, where Σ denotes the fan of X. We combine Klyachko's classification of T–equivariant vector bundles on X with Nori's Tannakian approach to principal G–bundles, to give an equivalence of categories between T–equivariant principal G–bundles on X and certain compatible Σ–filtered algebras associated to X, when the characteristic of K is 0.

Contents

1. Introduction

Let X be a toric variety, with fan Σ, under the action of a torus T, and let G be a reductive algebraic group; all are defined over an algebraically closed field K. A T–equivariant vector bundle E on X is a vector bundle on X endowed with a lift of the T–action which is linear on fibers. The T–equivariant vector bundles over a nonsingular toric variety were first classified by Kaneyama [21]. This classification result for toric vector bundles is up to isomorphism and it involves both combinatorial and linear algebraic data modulo an equivalence relation. Recently this work has been generalized for T–equivariant principal G–bundles [2, 3]; also see [4, 8], when K is the field $\mathbb{C}$ of complex numbers.

In a foundational paper Klyachko gave an alternative description of equivariant vector bundles on arbitrary toric varieties (possibly non-smooth) defined over any algebraically closed field [13]. His correspondence gives an equivalence between the category $\mathfrak{Vec}^T(X)$ of equivariant vector bundles on X and the category $\mathfrak{Cvec}(\Sigma)$ of finite dimensional vector

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spaces with collection of decreasing $\mathbb{Z}$–graded filtrations, indexed by the rays of Σ, satisfying a certain compatibility condition. Klyachko used his classification theorem to compute the Chern characters and sheaf cohomology of equivariant vector bundles. As a major application, later he used his classification of equivariant vector bundles over $\mathbb{P}^2$ to prove Horn's conjecture on eigenvalues of sums of Hermitian matrices [14]. Another interesting and more recent application is a theorem of Payne [27] that the moduli space of rank 3 toric vector bundles satisfy Murphy's law. Klyachko's classification theorem has also been generalized for equivariant torsion-free and equivariant pure sheaves by Perling [28] and Kool [15] respectively.

In this paper, our aim is to prove an analogue of Klyachko's result for T–equivariant principal G–bundles over a toric variety X. For some technical reasons (see Lemma 5.1), we need the group G to be linearly reductive. Hence, for the main result (Theorem 6.6), we find it natural to restrict ourselves to the case where the characteristic of K is zero. However, the result may be generalized to some fields of positive characteristic; see [24, Chapter 6, Theorem 2].

The first step in our formulation is an equivariant Nori theorem (Theorem 2.2), where we identify T–equivariant principal G–bundles with functors from the category of finite dimensional G–modules to the category of T–equivariant vector bundles over X, satisfying Nori's four conditions; see Section 2. In fact this theorem holds not only for T, but for any affine algebraic group Γ acting on an algebraic variety X.

Next we introduce the notion of a compatible Σ–filtered K–algebra which is a K–algebra endowed with a collection of decreasing $\mathbb{Z}$–graded filtrations indexed by the rays of Σ, that satisfy certain additive and multiplicative compatibility conditions; see Definitions 3.5 and 3.6. Let $\mathfrak{Calg}_G(\Sigma)$ be the category of such filtered K–algebras, G–equivariantly isomorphic to $K[G]$ (under the standard action), that satisfy the following: For every top dimensional cone $\sigma \in \Sigma$, the K–algebra admits an action of T which is compatible with the filtrations and commutes with the G–action.

Then we invoke a crucial fact, that a T–equivariant principal G–bundle over any affine toric variety is equivariantly trivializable. In the complex case, a proof of this is given in [3, Theorem 2.1] using an equivariant Oka-Grauert principle [10]. See [16, 17, 18, 19] for stronger versions of equivariant Oka-Grauert principle and related results. The proof presented here (Theorem 4.1), which works for arbitrary characteristic, is due to one of the referees. It is based on a characteristic-free version of Luna's étale slice theorem proved in [5].

Now, assume that every maximal cone in the fan Σ of X is of top dimension. This holds, in particular, when X is complete. Using the two results mentioned above, we prove an equivalence between the category $\mathfrak{Pbun}_G^T(X)$ of T–equivariant principal G–bundles over X, and the category $\mathfrak{Calg}_G(\Sigma)$ (Theorem 6.6). The most intriguing step in our proof is the commutativity of the T and G actions on the K–algebras in the definition of $\mathfrak{Calg}_G(\Sigma)$; see Lemma 5.6. As a corollary to Theorem 6.6, we obtain a necessary and sufficient condition for an equivariant reduction of structure group (Theorem 7.1). When $G = \mathrm{GL}(n, K)$, Klyachko's filtration data for equivariant vector bundle may be recovered from our filtered algebra description (see the proof of Lemma 6.4).

In a recent work [11], Ilten and Süss have obtained a Klyachko–type classification of torus equivariant vector bundles over T–varieties, and related it to Hartshorne's conjecture on splitting of rank two bundles over projective spaces. It seems natural that our classification of equivariant principal G-bundles should generalize for T-varieties.

Recently Kaveh and Manon shared with us another interesting approach to toric principal G–bundles [12].

2. Nori's correspondence

Let X be a separated, integral, finite type scheme over an arbitrary field K. We denote the ring of K-valued regular functions on X by $K[X]$.

The category of finite dimensional vector spaces over K will be denoted by $\mathfrak{Vec}$. Let G be an affine algebraic group over K. By an algebraic group we mean a smooth finitely generated group scheme over K; see [23, Definition 0.2, pp. 2].

The category of algebraic left representations of G that are finite dimensional K–vector spaces will be denoted by G–mod.

For convenience of readers, we recall the following equivalent notions of categorical equivalence which will be useful later (see [22, Chapter IV, section 4] for details).

(1) Two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent, i.e. there exists functors $F_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that $F_2 \circ F_1$ and $F_1 \circ F_2$ are naturally isomorphic to identity functors $\mathrm{Id}_{\mathcal{C}_1}$ and $\mathrm{Id}_{\mathcal{C}_2}$ respectively. F_1 and F_2 are called quasi-inverses of each other.
(2) There exists a functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, which is full, faithful and essentially surjective.

Let $\mathfrak{T}$ be a tensor category over K (see [6, Chapter II, section 1] for definition of tensor category). A functor

$$\mathbf{H} : G\text{–mod} \longrightarrow \mathfrak{T}$$

is said to satisfy properties F1–F4 if the following hold (see [25, Chapter 1] for more detailed description of these properties):

(1) F1: $\mathbf{H}$ is a K–additive exact functor,
(2) F2: $\mathbf{H} \circ \otimes = \otimes \circ (\mathbf{H} \times \mathbf{H})$,
(3) F3: furthermore,
   (a) $\mathbf{H}$ respects associativity of tensor products,
   (b) $\mathbf{H}$ respects commutativity of tensor products,
   (c) $\mathbf{H}$ takes the identity object of (G–mod, $\otimes$), namely the trivial G–module K, to the identity object of ($\mathfrak{T}$, $\otimes$), and
(4) F4: the functor $\mathbf{H}$ is faithful.

Let $\mathfrak{Vec}(X)$ be the category of vector bundles over X. Whenever convenient, we shall identify a vector bundle on X with the locally free coherent sheaf on X given by its local sections. Consider the category $\mathfrak{Nor}(X)$ of "Nori functors" whose

- objects are functors $\mathbf{E} : G\text{–mod} \longrightarrow \mathfrak{Vec}(X)$ that satisfy F1-F4, and
- morphisms are natural isomorphisms of functors.

Let $\mathfrak{Pbun}_G(X)$ denote the category of principal G–bundles over X. Given a principal G–bundle E_G, the functor which sends a G-module V to the associated vector bundle $E_G \times^G V$ satisfies properties F1-F4. We may therefore refer to it as the Nori functor associated to E_G. Let

$$\mathbf{N}_0 : \mathfrak{Pbun}_G(X) \longrightarrow \mathfrak{Nor}(X) \tag{2.1}$$

be the functor that sends any principal G–bundle E_G to its associated Nori functor.

Let $\mathfrak{Qco}(X)$ be the category of quasi-coherent sheaves of $\mathcal{O}_X$–modules. In [25, Lemma 2.2], Nori showed that any functor $\mathbf{E} \in \mathfrak{Nor}(X)$ admits a unique and natural extension to a functor $\overline{\mathbf{E}}$ from affine G–schemes to $\mathfrak{Qco}(X)$. He showed that $\overline{\mathbf{E}}(G)$ is a principal G–bundle over X. This defines a functor

$$\mathbf{N}_1 \,:\, \mathfrak{Nor}(X) \longrightarrow \mathfrak{Pbun}_G(X) \tag{2.2}$$

that sends a Nori functor $\mathbf{E}$ to the principal G–bundle $\overline{\mathbf{E}}(G)$. He went on to show that $\mathbf{N}_0$ and $\mathbf{N}_1$ are quasi-inverses, proving that the categories $\mathfrak{Nor}(X)$ and $\mathfrak{Pbun}_G(X)$ are equivalent.

In this section, we shall establish an equivariant analogue of the above equivalence. Let Γ be an affine algebraic group defined over K, and let

$$\eta \,:\, \Gamma \times X \longrightarrow X$$

be an algebraic left action of Γ on X (see [23, Definition 0.3, pp. 2] for definition of group scheme action on an arbitrary scheme). A Γ–*equivariant* vector bundle on X is a pair $(W\,,\widetilde{\eta})$, where W is an algebraic vector bundle on X and

$$\widetilde{\eta} \,:\, \Gamma \times W \longrightarrow W$$

is an algebraic left action of Γ on the total space of W such that

- $\widetilde{\eta}$ is a lift of η, and
- $\widetilde{\eta}$ preserves the linear structure on W, in particular, it is fiberwise linear.

We refer the reader to [29, section 1.2] for a detailed definition of an equivariant sheaf.

Similarly, a Γ–*equivariant* principal G–bundle on X is a pair $(E_G\,,\widetilde{\eta})$, where E_G is an algebraic principal bundle on X and

$$\widetilde{\eta} \,:\, \Gamma \times E_G \longrightarrow E_G$$

is an algebraic left action of Γ on the total space of E_G such that

- $\widetilde{\eta}$ is a lift of η, and
- $\widetilde{\eta}$ commutes with the right action of G on E_G.

Let $\mathfrak{Vec}^\Gamma(X)$ (respectively, $\mathfrak{Pbun}^\Gamma_G(X)$) be the category of Γ–equivariant vector bundles (respectively, Γ–equivariant principal G–bundles) over X. Let

$$\mathfrak{Nor}^\Gamma(X) \tag{2.3}$$

be the category whose

- objects are functors $\mathbf{E} \,:\, G\text{–mod} \longrightarrow \mathfrak{Vec}^\Gamma(X)$ satisfying F1–F4, and
- morphisms are natural isomorphisms of functors.

Take any $\mathbf{E} \in \mathfrak{Nor}^\Gamma(X)$. For any $V \in G$–mod, let $E(V)$ denote the underlying vector bundle of $\mathbf{E}(V)$, and let $\widetilde{\eta}(V)$ denote the action of Γ on $E(V)$. For any homomorphism of G–modules $\phi \,:\, V \longrightarrow W$, the following diagram is commutative:

$$\begin{array}{ccc} \Gamma \times E(V) & \xrightarrow{\widetilde{\eta}(V)} & E(V) \\ {\scriptstyle \mathrm{id}\times\mathbf{E}(\phi)}\big\downarrow & & {\scriptstyle \mathbf{E}(\phi)}\big\downarrow \\ \Gamma \times E(W) & \xrightarrow{\widetilde{\eta}(W)} & E(W) \end{array} \tag{2.4}$$

Also, we have $E(V\otimes W) \,=\, E(V)\otimes E(W)$ and $\widetilde{\eta}(V\otimes W) \,=\, \widetilde{\eta}(V)\otimes\widetilde{\eta}(W)$.

We say that a (possibly infinite dimensional) G–module $\overline{V}$ is *locally finite* if given any vector $v \,\in\, \overline{V}$, there exists a finite dimensional G–submodule $V \,\subset\, \overline{V}$ with $v \,\in\, V$. Let $G\text{–}\overline{\text{mod}}$ denote the category whose objects are locally finite G–modules and morphisms are G–module homomorphisms.

It is well-known that for any affine algebraic group G and any affine G–scheme X, the G–module $K[X]$ is locally finite (cf. [20, Proposition 8.6], [30, Theorem, section 3.3]).

Lemma 2.1. *Let $\mathfrak{Alg}^{\Gamma}(X)$ denote the category of Γ–equivariant sheaves of commutative associative $\mathcal{O}_X$–algebras, and let $G-\mathrm{sch}$ be the category of affine G–schemes. Let $\mathbf{E}$ be an object of $\mathfrak{Nor}^{\Gamma}(X)$. Then there exists a unique extension of $\mathbf{E}$ to a functor*

$$\overline{\mathbf{E}} \,:\, G{-}\mathrm{sch} \longrightarrow \mathfrak{Alg}^{\Gamma}(X)\,.$$

Proof. Let $\mathfrak{Qco}^{\Gamma}(X)$ be the category of all Γ–equivariant quasicoherent sheaves of $\mathcal{O}_X$–modules. First, observe that there is a unique extension $\overline{\mathbf{E}} \,:\, G\text{–}\overline{\mathrm{mod}} \longrightarrow \mathfrak{Qco}^{\Gamma}(X)$ that satisfies properties F1-F4 (cf. [25, Lemma(2.1)]). For $\overline{V} \in G\text{–}\overline{\mathrm{mod}}$, denote the underlying sheaf of $\overline{\mathbf{E}}(\overline{V})$ by $\overline{E}(\overline{V})$. Note that $\overline{E}(\overline{V})$ is the direct limit of $E(V)$, where V varies over all finite dimensional G–submodules of $\overline{V}$.

Use (2.4) to take direct limit of the morphisms $\widetilde{\eta}(V) \,:\, \Gamma \times E(V) \longrightarrow E(V)$ as V varies over all finite dimensional G–submodules of $\overline{V}$. In this way we obtain an action

$$\widetilde{\eta}(\overline{V}) \,:\, \Gamma \times \overline{E}(\overline{V}) \longrightarrow \overline{E}(\overline{V})\,. \tag{2.5}$$

Suppose $\overline{\phi} \,:\, \overline{U} \longrightarrow \overline{V}$ is a morphism of locally finite G–modules. To define $\overline{\mathbf{E}}(\overline{\phi})$, consider any $u \in \overline{U}$. There exists a finite dimensional G–module $U \subset \overline{U}$ such that $u \in U$. Let V denote the image $\overline{\phi}(U)$ with $i_V \,:\, V \longrightarrow \overline{V}$ being the inclusion map. Note that V is a finite dimensional G–module. Let $\psi \,:\, U \longrightarrow V$ be the unique homomorphism such that $\overline{\phi}|_U \,=\, i_V \circ \psi$. Define

$$\mathbf{E}(\overline{\phi})(u) \,=\, [\mathbf{E}(\psi)(u)]\,,$$

to be the equivalence class of $\mathbf{E}(\psi)(u) \in V$ in the direct limit $\overline{V}$. It is straightforward to check that this is indeed well-defined. Since the operation of direct limit commutes with tensor product, the extension preserves tensor product.

Following Nori, consider a commutative G–algebra A as a locally finite G–module together with a homomorphism $m \,:\, A \otimes A \longrightarrow A$. Then $\overline{\mathbf{E}}(m)$ defines the structure of a Γ–equivariant commutative, associative $\mathcal{O}_X$–algebra on $\overline{E}(A)$.

A similar argument shows that if $\phi \,:\, A \longrightarrow B$ is a homomorphism of G–algebras, then $\overline{\mathbf{E}}(\phi)$ is a homomorphism of Γ–equivariant sheaves of $\mathcal{O}_X$–algebras. □

It was shown by Nori, [25, Lemma 2.3], that $\overline{\mathbf{E}}(K[G])$ is a sheaf of $\mathcal{O}_X$–algebras that corresponds to a principal G–bundle over X. We denote by $\overline{E}(K[G])$ the principal G–bundle on X corresponding to $\overline{\mathbf{E}}(K[G])$.

The right G–action on $\overline{E}(K[G])$ is constructed as follows. Consider G' to be a copy of G with trivial G–action. Note that $\overline{E}(K[G'])$ is the trivial principal G'–bundle $X \times G' \longrightarrow X$ with trivial Γ–action on fibers. Let

$$a \,:\, G \times G' \longrightarrow G \tag{2.6}$$

be the multiplication map of G. This a produces an action of G' on G. Then $\overline{\mathbf{E}}(a)$ induces a morphism

$$\overline{E}(a) \,:\, \overline{E}(K[G]) \times_X \overline{E}(K[G']) \,=\, \overline{E}(K[G]) \times G' \longrightarrow \overline{E}(K[G])\,. \tag{2.7}$$

This induces the required fiber-wise action of G' on $\overline{E}(K[G])$. Note that $\overline{\mathbf{E}}(a)$ is a morphism of Γ–equivariant sheaves. Therefore, the actions of Γ and G' on $\overline{E}(K[G])$ commute. Consequently, we have $\overline{E}(K[G]) \in \mathfrak{Pbun}_G^{\Gamma}(X)$.

It follows that $\mathbf{N_1}$ in (2.2) produces a functor

$$\mathbf{N}_1^{\Gamma} : \mathfrak{Nor}^{\Gamma}(X) \longrightarrow \mathfrak{Pbun}_G^{\Gamma}(X)\,,\ \mathbf{E} \longmapsto (\overline{E}(K[G])\,, \widetilde{\eta}(K[G]))\,,$$

where $\widetilde{\eta}$ is constructed in (2.5). On the other hand, the functor $\mathbf{N}_0$ in (2.1) produces a functor $\mathbf{N}_0^{\Gamma} : \mathfrak{Pbun}_G^{\Gamma}(X) \longrightarrow \mathfrak{Nor}^{\Gamma}(X)$.

An analogue of the following result when Γ is a finite group has appeared before in [1].

Theorem 2.2. *The above two functors $\mathbf{N}_0^{\Gamma}$ and $\mathbf{N}_1^{\Gamma}$ are mutually quasi-inverses that induce an equivalence of categories between $\mathfrak{Pbun}_G^{\Gamma}(X)$ and $\mathfrak{Nor}^{\Gamma}(X)$.*

Proof. The proof follows verbatim from [25, Proposition 2.5 and Lemmas 2.6, 2.7, 2.8] once one replaces the category of sheaves of quasi-coherent $\mathcal{O}_X$-modules with its Γ-equivariant analogue. □

3. Filtration functor for vector bundles

Let X be a toric variety defined over K, corresponding to a fan Σ in a lattice N (see [7, 9, 26] for details). Let T denote the algebraic torus whose one-parameter subgroups are indexed by N. Then X admits an action of T with an open dense T–orbit O. Let $n = \dim(X) = \dim(T)$. Denote the set of all d–dimensional cones of Σ by $\Sigma(d)$. Let $|\Sigma(1)|$ be the set of primitive integral generators of elements of $\Sigma(1)$.

Define $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Then M is isomorphic to the group of characters of T. For any $\sigma \in \Sigma$, denote the corresponding affine toric subvariety of X by X_σ; also define

$$\sigma^{\perp} = \{u \in M \mid u(n) = 0 \ \forall\, n \in \sigma\}$$

and $M_\sigma := M/\sigma^{\perp}$. Then M_σ is the character group of the maximal sub-torus $T_\sigma \subset T$ that has a fixed point in X_σ. Let $\overline{\mathfrak{Vec}}$ be the category of K–vector spaces of countable dimension; the morphisms are K–linear homomorphisms.

Definition 3.1. A *decreasing filtration* $\mathcal{V}$ on a K–vector space $V \in \overline{\mathfrak{Vec}}$ is a collection $\{V(i) \mid i \in \mathbb{Z}\}$ of subspaces of V such that $V(i) \supseteq V(i+1)$ for each i. We say $\mathcal{V}$ is full if given any $v \in V$ there exists an integer i depending on v such that $v \in V(i)$.

Definition 3.2. A Σ–*filtration* on a vector space $V \in \overline{\mathfrak{Vec}}$ is a collection of full decreasing filtrations

$$\mathcal{V}^{\rho} : \cdots \supseteq V^{\rho}(i-1) \supseteq V^{\rho}(i) \supseteq V^{\rho}(i+1) \supseteq \cdots\,,$$

on V, where $\rho \in |\Sigma(1)|$. We denote the data $(V, \{V^{\rho}(i)\})$ by $\mathcal{V}^{\bullet}$ and say that $\mathcal{V}^{\bullet}$ is a Σ–filtered vector space on X. If the vector space V is finite dimensional then $\mathcal{V}^{\bullet}$ is said to be finite dimensional.

A *morphism* of Σ–filtered vector spaces $\phi : \mathcal{V}^{\bullet} \longrightarrow \mathcal{W}^{\bullet}$ is a homomorphism of vector spaces $\phi : V \longrightarrow W$, such that $\phi(V^{\rho}(i)) \subseteq W^{\rho}(i)$ for each i and ρ. Such a morphism is injective (respectively, surjective) if the underlying homomorphism of vector spaces is injective (respectively, surjective).

The category of Σ–filtered vector spaces is a tensor category with the following tensor product:

$$\mathcal{V}^{\bullet} \otimes \mathcal{W}^{\bullet} = \{V \otimes W, (V \otimes W)^{\rho}_{\dagger}(j)\} \tag{3.3}$$

where

$$(V \otimes W)^{\rho}_{\dagger}(j) = \sum_{p+q=j} V^{\rho}(p) \otimes W^{\rho}(q)\,. \tag{3.4}$$

Definition 3.5. Let $(A\,, m)$ be a commutative, associative, K–algebra, where $A \in \overline{\mathfrak{Vec}}$ is the underlying K–vector space of the algebra, and $m\,:\, A \otimes A \longrightarrow A$ is the multiplication operation. A Σ–filtration on $(A,\, m)$ is a Σ–filtration $\mathcal{A}^\bullet \,=\, (A,\, \{A^\rho(i)\})$ on the vector space A such that

$$m(A^\rho(i) \otimes A^\rho(j)) \,\subseteq\, A^\rho(i+j)$$

for every $\rho \,\in\, |\Sigma(1)|$ and $i\,, j \,\in\, \mathbb{Z}$.

The above data $(A,\, \{A^\rho(i)\},\, m)$ is denoted by $(\mathcal{A}^\bullet,\, m)$, and is called a Σ–filtered algebra on X.

A morphism of Σ–filtered algebras $(\mathcal{A}_1^\bullet,\, m_1) \,\longrightarrow\, (\mathcal{A}_2^\bullet,\, m_2)$ is a homomorphism of underlying K–algebras that respects the filtrations. Equivalently, it is a morphism of Σ–filtered vector spaces

$$\phi \,:\, \mathcal{A}_1^\bullet \,\longrightarrow\, \mathcal{A}_2^\bullet$$

such that $\phi \circ m_1 \,=\, m_2 \circ (\phi \otimes \phi)$.

Definition 3.6. A *compatible* Σ–filtered vector space on X is a Σ–filtered vector space $\mathcal{F}^\bullet \,=\, (F,\, \{F^\rho(i)\})$ such that for every $\sigma \,\in\, \Sigma$, there exists a decomposition of the vector space

$$F \,=\, \bigoplus_{[u]\in M_\sigma} F^\sigma_{[u]}\,, \tag{3.7}$$

with the following property: For each σ and for each $\rho \,\in\, \sigma \bigcap |\Sigma(1)|$

$$F^\rho(i) \,=\, \bigoplus_{u(\rho)\geq i} F^\sigma_{[u]}\,. \tag{3.8}$$

Similarly a compatible Σ–filtered algebra on X is a Σ–filtered algebra $(\mathcal{F}^\bullet,\, m)$ whose underlying Σ–filtered vector space $\mathcal{F}^\bullet$ is compatible, and the subspaces $F^\sigma_{[u]}$ in (3.7) satisfy

$$\sum_{[u]+[v]\,=\,[w]} m(F^\sigma_{[u]} \otimes F^\sigma_{[v]}) \,\subseteq\, F^\sigma_{[w]}\,. \tag{3.9}$$

A *morphism* between compatible Σ–filtered vector spaces (respectively, algebras) is simply a morphism between the underlying Σ–filtered vector spaces (respectively, algebras).

Remark 3.10. In the above definition it is enough to require that a decomposition (3.7) satisfying (3.8) exists for every maximal cone σ in the fan Σ. A decomposition corresponding to a maximal cone induces decompositions corresponding to its subcones.

Remark 3.11. A decomposition as in (3.7) corresponds to an action of T_σ on F.

Given a Σ–filtered vector space $(F,\, \{F^\rho(i)\})$, a decomposition (3.7) that satisfies (3.8), will be called a *compatible decomposition.*

Let $\overline{\mathfrak{Fvec}}(\Sigma)$ and $\overline{\mathfrak{Cvec}}(\Sigma)$ denote the categories of Σ–filtered vector spaces and compatible Σ–filtered vector spaces on X respectively. Their finite dimensional counterparts are denoted by

$$\mathfrak{Fvec}(\Sigma) \quad \text{and} \quad \mathfrak{Cvec}(\Sigma) \tag{3.12}$$

respectively. These are additive categories (see [31, Section12.3] for a definition).

The category $\overline{\mathfrak{Cvec}}(\Sigma)$ is a tensor category with product as in (3.4): Suppose $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$ are compatible Σ–filtered vector spaces. Let $V \,=\, \bigoplus_{[u]\in M_\sigma} V^\sigma_{[u]}$ and $W \,=\, \bigoplus_{[u]\in M_\sigma} W^\sigma_{[u]}$ be compatible decompositions for $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$ respectively. Define

$$(V \otimes W)^\sigma_{[u]} \,=\, \bigoplus_{[u_1]+[u_2]=[u]} V^\sigma_{[u_1]} \otimes W^\sigma_{[u_2]}\,.$$

Then

$$V \otimes W \,=\, \bigoplus_{[u]\in M_\sigma} (V\otimes W)^\sigma_{[u]}$$

is a compatible decomposition for $\mathcal{V}^\bullet \otimes \mathcal{W}^\bullet$.

Let $\mathfrak{Vec}^T(X)$ (respectively, $\mathfrak{Pbun}^T_G(X)$) denote the category of T–equivariant vector bundles (respectively, T–equivariant principal G–bundles) on X. There exists a fully faithful, surjective functor

$$\mathbf{F} \,:\, \mathfrak{Vec}^T(X) \,\longrightarrow\, \mathfrak{Cvec}(\Sigma)$$

(see [13, Theorem 2.2.1]). We shall sketch the construction of $\mathbf{F}$. Let $\xi \in \mathfrak{Vec}^T(X)$ be a bundle of rank r. Fix a closed point x_0 in the open T–orbit $O \subset X$. Denote by F the fiber $\xi(x_0)$. Let σ be a cone of Σ and X_σ the corresponding affine toric variety. Denote by ξ_σ the restriction of ξ to X_σ. Consider the action of T on the space of sections of ξ_σ defined by

$$(t\cdot s)(x) \,=\, ts(t^{-1}x)$$

for any point $x \in X_\sigma$, any element $t \in T$, and any section s of ξ_σ. A section s is said to be semi-invariant if $t \cdot s \,=\, u(t)s$ for some character u of T.

It was shown by Klyachko, [13, Proposition 2.1.1], that there exists a framing (which is not unique) of ξ_σ by semi-invariant sections. Fix such a framing $(s_1, \ldots, s_r)$. Let S_σ be the T–submodule of $H^0(X_\sigma,\, \xi_\sigma)$ generated by the semi-invariant sections $s_1, \ldots, s_r$. Evaluation at x_0 gives an isomorphism of vector spaces $ev_0 \,:\, S_\sigma \,\longrightarrow\, F$. This isomorphism induces a T–module structure on F, or equivalently, a decomposition

$$F \,=\, \bigoplus_{u\in M} F^\sigma_u\,. \tag{3.13}$$

Restricting to the action of T_σ on ξ_σ, we similarly get a decomposition

$$F \,=\, \bigoplus_{[u]\in M_\sigma} F^\sigma_{[u]}\,. \tag{3.14}$$

The decompositions (3.13) and (3.14) may depend on the choice of the semi-invariant framing of ξ_σ. However, for each $\rho \in |\Sigma(1)|$, the subspaces

$$F^\rho(i) \,:=\, \bigoplus_{[u]\in M_\sigma, u(\rho)\geq i} F^\sigma_{[u]}\,, \quad \text{where } \sigma \text{ is such that } \rho \,\in\, |\Sigma(1)|\bigcap \sigma\,,$$

are independent of the choice of σ containing ρ as well as the framing (see [13]).

Then $\mathbf{F}(\xi)$ is defined to be the compatible Σ–filtered vector space $\mathcal{F}^\bullet \,=\, (F,\, \{F^\rho(i)\})$ on X.

Lemma 3.1. *The functor* $\mathbf{F} \,:\, \mathfrak{Vec}^T(X) \,\longrightarrow\, \mathfrak{Cvec}(\Sigma)$ *satisfies*

$$\mathbf{F}(\xi_1)\otimes\mathbf{F}(\xi_2) \,=\, \mathbf{F}(\xi_1\otimes\xi_2)$$

for all $\xi_1,\, \xi_2 \,\in\, \mathfrak{Vec}^T(X)$.

Proof. Let $V \,=\, \xi_1(x_0)$ and $W \,=\, \xi_2(x_0)$ with $r_i \,=\, \dim(\xi_i(x_0))$. Clearly $V\otimes W \,=\, (\xi_1\otimes\xi_2)(x_0)$. Denote,

$\mathbf{F}(\xi_1) \,=\, (V, \{V^\rho(j)\})$, $\mathbf{F}(\xi_2) \,=\, (W, \{W^\rho(j)\})$, and $\mathbf{F}(\xi_1\otimes\xi_2) \,=\, (V\otimes W, \{(V\otimes W)^\rho(j)\})\,.$

By (3.3) and (3.4), we need to show that

$$(V\otimes W)^\rho(j) \,=\, \sum_{p+q=j} V^\rho(p)\otimes W^\rho(q)\,. \tag{3.15}$$

Consider any $\rho \in |\Sigma(1)|$. Let σ be any cone that contains ρ. Fix semi-invariant frames $s^i_1, \cdots, s^i_{r_i}$ of $(\xi_i)_\sigma$. Let $[u^i_k] \in M_\sigma$ be the character corresponding to the action of T_σ on s^i_k. We have compatible decompositions

$$V = \bigoplus_{1\le k\le r_1} V^\sigma_{[u^1_k]} \quad \text{and} \quad W = \bigoplus_{1\le l\le r_2} W^\sigma_{[u^2_l]}$$

induced by these frames. Note that $\{s^1_k \otimes s^2_l\}$ is a semi-invariant frame of $(\xi_1)_\sigma \otimes (\xi_2)_\sigma$, which induces a compatible decomposition

$$V\otimes W = \bigoplus_{[u]\in M_\sigma} (V\otimes W)^\sigma_{[u]}\,,$$

where

$$(V\otimes W)^\sigma_{[u]} = \bigoplus_{[u^1_k]+[u^2_l]=[u]} V^\sigma_{[u^1_k]} \otimes W^\sigma_{[u^2_l]} \tag{3.16}$$

Note that

$$V^\rho(p) = \bigoplus_{u^1_k(\rho)\ge p} V^\sigma_{[u^1_k]}\,,\ W^\rho(q) = \bigoplus_{u^2_l(\rho)\ge q} W^\sigma_{[u^2_l]}\,,\ (V\otimes W)^\rho(j) = \bigoplus_{u(\rho)\ge j} (V\otimes W)^\sigma_{[u]}\,.$$

Therefore, by (3.16) we get,

$$V^\rho(p)\otimes W^\rho(j-p) \subset (V\otimes W)^\rho(j) \text{ for any } p\in\mathbb{Z}\,.$$

To satisfy (3.15), we need to verify that

$$(V\otimes W)^\rho(j) \subset \sum_{p+q=j} V^\rho(p)\otimes W^\rho(q)\,.$$

Take any $w \in (V\otimes W)^\rho(j)$. Then $w = \sum w_t$, where each $w_t \in (V\otimes W)^\sigma_{[u_t]}$ for some $[u_t]\in M_\sigma$ such that $u_t(\rho) \ge j$.

It suffices to show that each $w_t \in \sum_{p+q=j} V^\rho(p)\otimes W^\rho(q)$. Since $\{s^1_k(x_0)\otimes s^2_l(x_0)\}$ is a basis for $V\otimes W$, we have

$$w_t = \sum_{[u^1_k]+[u^2_l]=[u_t]} a^t_{kl}\, s^1_k(x_0)\otimes s^2_l(x_0) \text{ where } a^t_{kl}\in K\,.$$

Note that $[u^1_k]+[u^2_l] = [u_t]$ implies $u^1_k(\rho)+u^2_l(\rho) = u_t(\rho) \ge j$. Since $u^1_k(\rho)$ and $u^2_l(\rho)$ are integers, there exist integers p and q such that $u^1_k(\rho)\ge p$, $u^2_l(\rho)\ge q$, and $p+q=j$. It follows that

$$s^1_k(x_0)\otimes s^2_l(x_0) \in V^\rho(p)\otimes W^\rho(q)\,, \quad \text{and} \quad w_t \in \sum_{p+q=j} V^\rho(p)\otimes W^\rho(q)\,.$$

This completes the proof. □

Since $\mathbf{F} : \mathfrak{Vec}^T(X) \longrightarrow \mathfrak{Cvec}(\Sigma)$ is an equivalence of additive categories, it is an exact functor. The quasi-inverse $\mathbf{K} : \mathfrak{Cvec}(\Sigma) \longrightarrow \mathfrak{Vec}^T(X)$ of $\mathbf{F}$, constructed by Klyachko in [13], respects direct sums and tensor products. Being an equivalence of categories, it is also exact and faithful.

Consider the category $\mathfrak{Cnor}(\Sigma)$ whose objects are functors

$$\mathbf{M} : G\text{–mod} \longrightarrow \mathfrak{Cvec}(\Sigma)$$

(see (3.12)) that satisfy properties F1-F4, and whose morphisms are natural isomorphisms of functors.

Theorem 3.2. *There exists an equivalence of categories between* $\mathfrak{Cnor}(\Sigma)$ *(defined above) and* $\mathfrak{Pbun}_G^T(X)$ *(the category of* T*–equivariant principal* G*–bundles on the toric variety* X*).*

Proof. Let

$$\mathfrak{Nor}^T(X) \tag{3.17}$$

be the category in (2.3) obtained by substituting T in place of Γ. Consider the functor $\mathbf{F}_* : \mathfrak{Nor}^T(X) \longrightarrow \mathfrak{Cnor}(\Sigma)$ defined by composition with $\mathbf{F}$,

$$\mathbf{F}_*(\mathbf{E}) = \mathbf{F} \circ \mathbf{E} \text{ for any } \mathbf{E} \in \mathfrak{Nor}^T(X)\,.$$

Similarly, composition with $\mathbf{K}$ gives a functor $\mathbf{K}_* : \mathfrak{Cnor}(\Sigma) \longrightarrow \mathfrak{Nor}^T(X)$,

$$\mathbf{K}_*(\mathbf{M}) = \mathbf{K} \circ \mathbf{M} \text{ for any } \mathbf{M} \in \mathfrak{Cnor}(\Sigma)\,.$$

It is easily observed from the construction of $\mathbf{K}$ that $\mathbf{F} \circ \mathbf{K} = 1_{\mathfrak{Cvec}(\Sigma)}$. Therefore,

$$\mathbf{F}_* \circ \mathbf{K}_* = 1_{\mathfrak{Cnor}(\Sigma)}\,.$$

Since $\mathbf{F}$ and $\mathbf{K}$ are fully faithful, so are $\mathbf{F}_*$ and $\mathbf{K}_*$. Hence, they induce an equivalence of categories between $\mathfrak{Cnor}(\Sigma)$ and $\mathfrak{Nor}^T(X)$. Then, by Theorem 2.2, $\mathfrak{Cnor}(\Sigma)$ and $\mathfrak{Pbun}_G^T(X)$ are equivalent categories. □

4. Equivariant trivialization on affine toric variety

The credit for the following result goes to one of our referees.

Theorem 4.1. *Let* $f : E_G \longrightarrow Y$ *be a* T*-equivariant principal* G*-bundle where* Y *is an affine toric variety, defined over* K*, under the action of the torus* T*. Assume* G *is a reductive algebraic group. Then* E_G *is equivariantly trivializable.*

Proof. $G \times T$ acts on E_G with an open orbit, the preimage under f of the open T-orbit in Y. Also, the variety E_G is affine, as the morphism f is so. Thus, E_G contains a unique closed $(G \times T)$–orbit, $(G \times T) \cdot e_0$: the image of the closed T–orbit, say $T \cdot y_0$, in Y.

Consider first the case where y_0 is fixed by T. Then the (scheme-theoretic) isotropy group $(G \times T)_{e_0} \subset G \times T$ intersects G trivially since the latter acts freely on E_G. Moreover, $(G \times T)_{e_0}$ is sent onto T by the second projection. Indeed, for any $t \in T$, we have $f(t \cdot e_0) = t \cdot f(e_0) = f(e_0)$, hence $t \cdot e_0 \in G \cdot e_0$. Thus the second projection yields an isomorphism $(G \times T)_{e_0} \cong T$. So there exists a unique homomorphism $\rho : T \longrightarrow G$ such that

$$(G \times T)_{e_0} = \{(\rho(t), t) \mid t \in T\}\,.$$

In particular, the closed $(G \times T)$–orbit is separable and its isotropy group is linearly reductive. We may now apply [5, Proposition 8.5] which yields an equivariant isomorphism

$$E_G \cong (G \times T) \times^T F$$

for some affine T–variety F. Here $(G \times T) \times^T F$ denotes the associated fiber bundle to the principal T–bundle $G \times T \longrightarrow (G \times T)/T$ (where T is viewed as a subgroup of $G \times T$ via $t \mapsto (\rho(t), t)$) and the T–variety F. Thus, $Y = E_G/G \cong F$, and $E_G \cong (G \times T) \times^T Y$. This yields the desired isomorphism

$$E_G \cong G \times^T Y\,,$$

where T acts on G via left multiplication through ρ and on Y via the given action.

The general proof reduces to the former one as in proof of Corollary 2.3 in [3]. □

5. Filtered algebra associated to an equivariant principal bundle

Henceforth, we assume that K has characteristic zero. Then the reductive group G is linearly reductive. Let E_G be a T–equivariant principal G–bundle over X.

Given any $\mathbf{E} \in \mathfrak{Nor}^T(X)$, define $\mathbf{E}_\sharp \in \mathfrak{Cnor}(\Sigma)$ by

$$\mathbf{E}_\sharp = \mathbf{F} \circ \mathbf{E}.$$

It is easily checked that $\mathbf{E}_\sharp$ is faithful. Moreover, it preserves tensor products as a consequence of Lemma 3.1.

Let $\mathbf{O} : \mathfrak{Fvec}(\Sigma) \longrightarrow \mathfrak{Vec}$ be the forgetful functor that maps a Σ–filtered vector space to its underlying vector space. Define $E_\sharp := \mathbf{O} \circ \mathbf{E}_\sharp$. Note that $E_\sharp(V) = \mathbf{E}(V)(x_0)$. It is evident that $E_\sharp$ preserves tensor products.

Let $\phi_1, \phi_2 : V \longrightarrow W$ be two morphisms of G–modules such that $E_\sharp(\phi_1) = E_\sharp(\phi_2)$. Then $\mathbf{E}_\sharp(\phi_1) = \mathbf{E}_\sharp(\phi_2)$. By faithfulness of $\mathbf{E}_\sharp$, $\phi_1 = \phi_2$. Therefore, $E_\sharp$ is faithful.

Lemma 5.1. *There exists a unique extension of $\mathbf{E}_\sharp$ (respectively, $E_\sharp$) to a functor $\overline{\mathbf{E}}_\sharp : G\text{–}\overline{\text{mod}} \longrightarrow \overline{\mathfrak{Cvec}}(\Sigma)$ (respectively, $\overline{E}_\sharp : G\text{–}\overline{\text{mod}} \longrightarrow \overline{\mathfrak{Vec}}$) that preserves direct limits and tensor products.*

Proof. It is easily observed that the category of Σ–filtered vector spaces over X admits direct limits. For any $\overline{V}$ in $G\text{–}\overline{\text{mod}}$, define $\overline{\mathbf{E}}_\sharp(\overline{V})$ (respectively, $\overline{E}_\sharp(\overline{V})$) to be the direct limit of $\mathbf{E}_\sharp(V)$ (respectively, $E_\sharp(V)$) as V varies over all finite dimensional G–submodules of $\overline{V}$. Note that direct limit commutes with tensor product. So it follows from Lemma 3.1 that $\overline{\mathbf{E}}_\sharp$ preserves tensor products.

To understand the compatibility condition, consider the isotypical decomposition

$$\overline{V} = \bigoplus_{i \in I} V_i \otimes Hom_G(V_i, \overline{V})$$

obtained by linear reductivity of G. Here I denotes the set of isomorphism classes of irreducible G–submodules of $\overline{V}$. Each of the V_i's is finite dimensional. Since G acts trivially on the module $Hom_G(V_i, \overline{V})$, it follows that $\overline{\mathbf{E}}_\sharp(Hom_G(V_i, \overline{V}))$ has trivial Σ–filtration. Therefore it may be assigned the trivial compatible decomposition comprising the subspaces

$$(\overline{E}_\sharp(Hom_G(V_i, \overline{V})))^\sigma_{[u]} = \begin{cases} \overline{E}_\sharp(Hom_G(V_i, \overline{V})) & \text{if } [u] = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now a choice of compatible decomposition for each $\mathbf{E}_\sharp(V_i)$ determines a compatible decomposition for $\overline{\mathbf{E}}_\sharp(\overline{V})$.

The construction of $\overline{\mathbf{E}}_\sharp(\overline{\phi})$ for a morphism $\overline{\phi} : \overline{U} \longrightarrow \overline{V}$ is similar to Lemma 2.1. □

Lemma 5.2. *The functors $\overline{\mathbf{E}}_\sharp$ (respectively, $\overline{E}_\sharp$) satisfy properties* F2, F3 *and* F4.

Proof. Since direct limit commutes with tensor product, $\overline{\mathbf{E}}_\sharp$ satisfies F2 and F3.

Suppose there exist morphisms $\overline{\phi}_j : \overline{U} \longrightarrow \overline{V}$, $j = 1, 2$, such that $\overline{\mathbf{E}}_\sharp(\overline{\phi}_1) = \overline{\mathbf{E}}_\sharp(\overline{\phi}_2)$. To prove $\overline{\mathbf{E}}_\sharp$ is faithful, it is enough to show that $\overline{\phi}_1 = \overline{\phi}_2$. Consider any element $u \in \overline{U}$. Then there exists a finite dimensional G–submodule U of $\overline{U}$ such that $u \in U$. Let $i_U : U \longrightarrow \overline{U}$ be the inclusion map. Let $\phi_j = \overline{\phi}_j \circ i_U$. Then

$$\overline{\mathbf{E}}_\sharp(\phi_1) = \overline{\mathbf{E}}_\sharp(\phi_2)\,. \tag{5.1}$$

There exists a finite dimensional G–module $V \subset \overline{V}$ such that $\phi_j(U) \subset V$ for each j. Let $i_V : V \longrightarrow \overline{V}$ be the inclusion map. Let $\psi_j : U \longrightarrow V$ be the unique map such that $\phi_j = i_V \circ \psi_j$. So, by (5.1),

$$\overline{\mathbf{E}}_\sharp(i_V) \circ \mathbf{E}_\sharp(\psi_1) = \overline{\mathbf{E}}_\sharp(i_V) \circ \mathbf{E}_\sharp(\psi_2) . \tag{5.2}$$

Since $\mathbf{E}_\sharp$ is faithful, $\overline{\mathbf{E}}_\sharp(i_V)$ is a direct limit of inclusion maps. Therefore, $\overline{\mathbf{E}}_\sharp(i_V)$ is injective. Hence it follows from (5.2) that

$$\mathbf{E}_\sharp(\psi_1) = \mathbf{E}_\sharp(\psi_2) . \tag{5.3}$$

Hence, by faithfulness of $\mathbf{E}_\sharp$ and (5.3), we have $\psi_1 = \psi_2$. It follows that $\phi_1 = \phi_2$, and hence $\overline{\phi}_1(u) = \overline{\phi}_2(u)$. Since u is arbitrary, $\overline{\phi}_1 = \overline{\phi}_2$.

The proof for $\overline{E}_\sharp$ is similar. □

Lemma 5.3. $\overline{\mathbf{E}}_\sharp$ *(respectively, $\overline{E}_\sharp$) defines a functor from affine G–schemes to Σ–filtered algebras (respectively, K-algebras). We denote this functor by $\overline{\mathbf{E}}_\sharp$ (respectively, $\overline{E}_\sharp$) as well.*

Proof. Suppose A is a finitely generated K–algebra on which G acts. Following Nori, we view A as a locally finite G–module, and its multiplication as a morphism

$$m_A : A \otimes A \longrightarrow A$$

in G–$\overline{\text{mod}}$.

Note that $\overline{E}_\sharp(A \otimes A) = \overline{E}_\sharp(A) \otimes \overline{E}_\sharp(A)$. Therefore we have a morphism

$$\overline{E}_\sharp(m_A) : \overline{E}_\sharp(A) \otimes \overline{E}_\sharp(A) \longrightarrow \overline{E}_\sharp(A) .$$

Since m_A is nontrivial and $\overline{E}_\sharp$ is faithful, $\overline{E}_\sharp(m_A)$ defines a nontrivial multiplication on $\overline{E}_\sharp(A)$. This is commutative and associative, since $\overline{E}_\sharp$ satisfies property F3.

For any two finite dimensional G–submodules V and W of A, by (3.4) we have

$$E_\sharp(V)^\rho(i) \otimes E_\sharp(W)^\rho(j) \subseteq (E_\sharp(V) \otimes E_\sharp(W))^\rho_\dagger(i+j) .$$

Since $\mathbf{E}_\sharp(V \otimes W) = \mathbf{E}_\sharp(V) \otimes \mathbf{E}_\sharp(W)$,

$$(E_\sharp(V) \otimes E_\sharp(W))^\rho_\dagger(i+j) = (E_\sharp(V \otimes W))^\rho(i+j) .$$

Then applying Lemma 5.1, we have

$$\overline{E}_\sharp(A)^\rho(i) \otimes \overline{E}_\sharp(A)^\rho(j) \subseteq (\overline{E}_\sharp(A \otimes A))^\rho(i+j) . \tag{5.4}$$

As $\overline{\mathbf{E}}_\sharp(m_A)$ is a morphism of Σ–filtered vector spaces, we have

$$\overline{\mathbf{E}}_\sharp(m_A)(\overline{E}_\sharp(A \otimes A))^\rho(i+j) \subseteq \overline{E}_\sharp(A)^\rho(i+j) . \tag{5.5}$$

By (5.4) and (5.5),

$$\overline{\mathbf{E}}_\sharp(m_A)(\overline{E}_\sharp(A)^\rho(i) \otimes \overline{E}_\sharp(A)^\rho(j)) \subseteq \overline{E}_\sharp(A)^\rho(i+j) .$$

Thus $\overline{\mathbf{E}}_\sharp(A)$ is a Σ–filtered algebra.

Now suppose that $\phi : A \longrightarrow B$ is a morphism of G–algebras, so that

$$\phi \circ m_A = m_B \circ (\phi \otimes \phi).$$

By functoriality,

$$\overline{\mathbf{E}}_\sharp(\phi) \circ \overline{\mathbf{E}}_\sharp(m_A) = \overline{\mathbf{E}}_\sharp(m_B) \circ (\overline{\mathbf{E}}_\sharp(\phi) \otimes \overline{\mathbf{E}}_\sharp(\phi)) .$$

Thus $\overline{\mathbf{E}}_\sharp(\phi)$ is a morphism of Σ–filtered algebras. □

Lemma 5.4. *The algebra $\overline{E}_\sharp(K[G])$ is the algebra of regular functions of the fiber $\overline{E}(K[G])(x_0)$ of the principal bundle $\overline{E}(K[G])$. Moreover, $\overline{E}_\sharp(K[G])$ admits a G–action which makes it equivariantly isomorphic to $K[G]$.*

Proof. Let $\overline{\mathbf{O}}$ be the forgetful functor that takes a Σ–filtered algebra to its underlying K–algebra. Then

$$\overline{E}_\sharp \,=\, \overline{\mathbf{O}} \circ \overline{\mathbf{E}}_\sharp\,. \tag{5.6}$$

Indeed, this follows from $E_\sharp \,=\, \mathbf{O} \circ \mathbf{E}_\sharp$ combined with Lemma 5.2 and the uniqueness of the extensions of $E_\sharp$ and $\mathbf{E}_\sharp$ to G–$\overline{\text{mod}}$.

Let $q \,:\, \mathcal{O}_X \longrightarrow K$ be the evaluation map corresponding to the closed point $x_0 \in X$. Using q, to any $\mathcal{O}_X$–module M we may associate a K–vector space $M \otimes_{\mathcal{O}_X} K$. Now recall that, for any $V \,\in\, G$–mod,

$$E_\sharp(V) \,=\, \mathbf{E}(V) \otimes_{\mathcal{O}_X} K\,.$$

Then by the uniqueness of extensions we have

$$\overline{E}_\sharp(K[G]) \,=\, \overline{\mathbf{E}}(K[G]) \otimes_{\mathcal{O}_X} K\,.$$

It is then clear that $\overline{E}_\sharp(K[G])$ is the algebra of regular functions of the fiber at x_0 of the principal bundle $\overline{E}(K[G])$. Note that, by (5.6), this $\overline{E_\sharp}(K[G])$ is the underlying algebra of $\overline{\mathbf{E}}_\sharp(K[G])$. This completes the proof of the first part of the lemma.

Next note that there is a natural G–action on the principal bundle $\overline{E}(K[G])$ which is free and transitive on each fiber. This yields the required G–action on $\overline{E}_\sharp(K[G])$ by the first part of the lemma. □

Lemma 5.5. *Let X be a toric variety. Then the Σ–filtered algebra $\overline{\mathbf{E}}_\sharp(K[G])$ is compatible.*

Proof. By Remark 3.10, it is enough to concentrate on a maximal cone σ. It follows from Theorem 4.1 (cf. [3]) that E_G admits a section s over X_σ, such that

- $ts(x) \,=\, s(tx)\rho_s(t)$ for every $x \,\in\, X_\sigma$ and $t \,\in\, T$, where $\rho_s \,:\, T \longrightarrow G$ is a group homomorphism.

Then for any locally finite G–module V, the homomorphism ρ_s and the given action of G induce a T–action on V which we denote by ρ_s again without confusion. An eigenvector v of this action with weight $\chi(t)$ gives rise to a semi-invariant section $[(s(x), v)]$ of $\overline{\mathbf{E}}(V)$ with the same weight. Such sections, corresponding to a choice of an eigen-basis of V, induce a compatible T–decomposition of $\overline{\mathbf{E}}_\sharp(V)$. Moreover, this decomposition does not depend on the choice of the eigen-basis.

Now consider the G–module $K[G]$. The action of T on $K[G]$ induced by ρ_s satisfies the condition that

$$\rho_s(t)f(\cdot) \,=\, f(\cdot\, \rho_s(t))\,.$$

It follows that if $f_1,\, f_2 \,\in\, K[G]$ are T-eigenvectors with weights $\chi_1(t)$ and $\chi_2(t)$ respectively, then the product $f_1 f_2$ is a T-eigenvector with weight $\chi_1(t)\chi_2(t)$. This implies that the compatible T–decomposition on on $\overline{\mathbf{E}}_\sharp(K[G])$ respects the multiplication of the algebra $K[G]$ (cf. (3.9)). □

Lemma 5.6. *For every n–dimensional cone σ, an action of T on $\overline{E}_\sharp(K[G])$ which is compatible with the Σ–filtration, commutes with the action of G.*

Proof. We revisit the G–action on $\overline{E}_\sharp(K[G])$. Recall the multiplication map $a\,:\, G\times G' \longrightarrow G$ of G defined in (2.6). Let $a^*\,:\, K[G] \longrightarrow K[G]\otimes K[G']$ be the algebra morphism corresponding to a. Then we have a map

$$\overline{\mathbf{E}}_\sharp(a^*)\,:\,\overline{\mathbf{E}}_\sharp(K[G]) \longrightarrow \overline{\mathbf{E}}_\sharp(K[G])\otimes\overline{\mathbf{E}}_\sharp(K[G'])\,. \tag{5.7}$$

of Σ–filtered algebras. Note that the underlying algebra $\overline{E}_\sharp(K[G'])$ of $\overline{\mathbf{E}}_\sharp(K[G'])$ is $K[G']$. This follows from the fact observed in Section 2 that $\overline{E}(K[G']) = X\times G'$, and the first part of Lemma 5.4. Thus the map $\overline{\mathbf{E}}_\sharp(a^*)$ induces an action of G on $\mathrm{Spec}(\overline{E}_\sharp(K[G])) = \overline{E}(K[G])(x_0)$, which agrees with the action of G given by $\overline{E}(a)$ (cf. (2.7)).

It follows from property F3(c) that the bundle $\overline{E}(K[G'])$ has trivial T–action for any $\mathbf{E}\in\mathfrak{Nor}^T(X)$. Therefore, the Σ–filtration on $\overline{\mathbf{E}}_\sharp(K[G'])$ satisfies

$$\overline{\mathbf{E}}_\sharp(K[G'])^\rho(i) \,=\, \begin{cases} K[G'] & \text{if}\, i\le 0\\ 0 & \text{if}\, i>0\end{cases} \tag{5.8}$$

for every $\rho\in|\Sigma(1)|$. Since the filtrations are decreasing, it follows (cf. (3.4)) that for every ρ,

$$\left(\overline{\mathbf{E}}_\sharp(K[G])\otimes\overline{\mathbf{E}}_\sharp(K[G'])\right)^\rho(i) \,=\, \overline{\mathbf{E}}_\sharp(K[G])^\rho(i)\otimes K[G']\,. \tag{5.9}$$

As $\overline{\mathbf{E}}_\sharp(a^*)$ respects the Σ–filtration, we have

$$\overline{\mathbf{E}}_\sharp(a^*)(\overline{\mathbf{E}}_\sharp(K[G])^\rho(i))\subset\overline{\mathbf{E}}_\sharp(K[G])^\rho(i)\otimes K[G']\,.$$

Let σ be an n–dimensional cone. Note that $T_\sigma = T$. Fix a decomposition (equivalently, a compatible T–action)

$$\overline{\mathbf{E}}_\sharp(K[G]) \,=\, \bigoplus_{\chi\in M}\overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi\,, \tag{5.10}$$

such that for any $\rho\in\sigma\bigcap\Sigma(1)$,

$$\overline{\mathbf{E}}_\sharp(K[G])^\rho(i) \,=\, \sum_{\chi(\rho)\ge i}\overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi\,.$$

We will show that

$$\overline{\mathbf{E}}_\sharp(a^*)(\overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi) \,\subseteq\, \overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi\otimes K[G'] \tag{5.11}$$

for every χ.

Suppose $f\in\overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi$. Since $K[G]\otimes K[G']$ is locally finite, we may write $\overline{\mathbf{E}}_\sharp(a^*)(f)$ as a finite sum

$$\overline{\mathbf{E}}_\sharp(a^*)(f) \,=\, \sum f_j\otimes b_j, \tag{5.12}$$

where $f_j\in\overline{\mathbf{E}}_\sharp(K[G])^\sigma_{\chi_j}$ and $b_j\in K[G']$. Note that

$$f\in\overline{\mathbf{E}}_\sharp(K[G])^\rho(\chi(\rho))\text{ for every }\rho\in\sigma\bigcap\Sigma(1)\,.$$

Since $\overline{\mathbf{E}}_\sharp(a^*)$ preserves the Σ–filtration, we must have $\chi_j(\rho)\ge\chi(\rho)$ for every $\rho\in\sigma\bigcap\Sigma(1)$.

Suppose, if possible, $\chi_{j_0}\neq\chi$ for some value j_0 of j. Then since $\sigma\bigcap\Sigma(1)$ spans $N\otimes\mathbb{R}$, there exists $\rho_0\in\sigma\bigcap\Sigma(1)$ such that $\chi_{j_0}(\rho_0)>\chi(\rho_0)$.

Given any $h\in G'$, consider the G–map

$$\phi_h\,:\,G\longrightarrow G\times G'$$

defines by $g\longmapsto (g,\,h)$. The induced map

$$\phi_h^*\,:\,K[G]\otimes K[G']\longrightarrow K[G]$$

satisfies $\phi_h^*(x \otimes y) = y(h)\, x$. Identifying $K[G]$ with $K[G] \otimes_K K$, we can write

$$\phi_h^* \;=\; \mathrm{id} \otimes ev_h$$

where

$$ev_h : K[G'] \longrightarrow K$$

is the evaluation map at h. Therefore,

$$\overline{\mathbf{E}}_\sharp(\phi_h^*) \;=\; \mathrm{id} \otimes \overline{\mathbf{E}}_\sharp(ev_h)\,.$$

Note that $\mathbf{E}(K)$ is the trivial line bundle with trivial T–action by property F3(c). Therefore, $\overline{\mathbf{E}}_\sharp(K) \;=\; K$. Moreover, for any two G–algebras A, B with trivial G–action, and any homomorphism $\theta : A \longrightarrow B$, $\overline{\mathbf{E}}_\sharp(\theta) \;=\; \theta$. Hence, $\overline{\mathbf{E}}_\sharp(ev_h) \;=\; ev_h$. Thus we have,

$$\overline{\mathbf{E}}_\sharp(\phi_h^*) \;=\; \mathrm{id} \otimes ev_h : \overline{\mathbf{E}}_\sharp(K[G]) \otimes K[G'] \;\longrightarrow\; \overline{\mathbf{E}}_\sharp(K[G])\,.$$

Hence,

$$\overline{\mathbf{E}}_\sharp(\phi_h^*)\,(\sum f_j \otimes b_j) \;=\; \sum b_j(h) f_j\,.$$

Then using (5.12) we have

$$\overline{\mathbf{E}}_\sharp(\phi_h^* \circ a^*)(f) \;=\; \sum b_j(h) f_j\,.$$

Since $b_{j_0} \neq 0$, there exists h_0 such that $b_{j_0}(h_0) \neq 0$. Let $i_0 = \chi(\rho_0)$. Then

$$b_{j_0}(h_0)\, f_{j_0} \in \overline{\mathbf{E}}_\sharp(K[G])^{\rho_0}(i_1) \text{ where } i_1 \;=\; \chi_{j_0}(\rho_0) \;>\; i_0\,.$$

Note that the composition of maps

$$(\phi^*_{h_0^{-1}} \circ a^*) \circ (\phi^*_{h_0} \circ a^*) = \mathrm{id}\,.$$

Since $\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)$ is also filtration preserving,

$$\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)(b_{j_0}(h_0)\, f_{j_0}) \;\in\; \overline{\mathbf{E}}_\sharp(K[G])^{\rho_0}(i_1)\,.$$

Therefore,

$$\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)(b_{j_0}(h_0)\, f_{j_0}) \notin \overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi$$

unless $\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)(b_{j_0}(h_0)\, f_{j_0}) \;=\; 0$. But

$$\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*) \circ \overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)(f) \;=\; f \in \overline{\mathbf{E}}_\sharp(K[G])^\sigma_\chi\,.$$

Therefore,

$$\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)(b_{j_0}(h_0)\, f_{j_0}) = 0\,.$$

This is a contradiction since $\overline{\mathbf{E}}_\sharp(\phi^*_{h_0^{-1}} \circ a^*)$ is an isomorphism. Thus no such j_0 exists. Therefore, using (5.12), we obtain (5.11). This implies that the actions of G and T on $\overline{\mathbf{E}}_\sharp(K[G])$, induced by (5.7) and (5.10) respectively, commute. The lemma follows. □

We may associate to any equivariant principal bundle E_G, the compatible Σ–filtered algebra $\overline{\mathbf{E}}_\sharp(K[G])$, where $\mathbf{E} \;=\; \mathbf{N}_0^T(E_G)$.

6. Correspondence between equivariant G–bundles and Σ–filtered algebras

Let X be a toric variety defined over an algebraically closed field K of characteristic 0. Assume that every maximal cone in the fan Σ of X is of top dimension. We note that this assumption is always satisfied when X is a complete toric variety.

Definition 6.1. Let $\mathfrak{Calg}_G(\Sigma)$ be the category whose objects are compatible Σ–filtered K–algebras B such that

- B admits a G–action with respect to which it is G–equivariantly isomorphic to the algebra $K[G]$,
- For every top dimensional cone in the fan, B admits a compatible action of T that commutes with the action of G on B.

The morphisms of $\mathfrak{Calg}_G(\Sigma)$ are G–equivariant isomorphisms of compatible Σ–filtered K–algebras.

Lemma 6.1. *The association* $\mathbf{E} \longmapsto \overline{\mathbf{E}}_\sharp(K[G])$ *induces a functor*

$$\mathbf{A} \,:\, \mathfrak{Nor}^T(X) \,\longrightarrow\, \mathfrak{Calg}_G(\Sigma)\,,$$

where $\mathfrak{Nor}^T(X)$ *and* $\mathfrak{Calg}_G(\Sigma)$ *are as in* (3.17) *and Definition 6.1 respectively.*

Proof. It follows from Section 4 that $\overline{\mathbf{E}}_\sharp(K[G])$ is an object in $\mathfrak{Calg}_G(\Sigma)$.

Let $\Psi : \mathbf{E}^1 \longrightarrow \mathbf{E}^2$ be a morphism in $\mathfrak{Nor}^T(X)$. For any morphism $f : V \longrightarrow W$ in G–mod, we have a commuting diagram.

$$\begin{array}{ccc}
\mathbf{E}^{\mathbf{1}}_\sharp(V) & \xrightarrow{\mathbf{F}\circ\Psi(V)} & \mathbf{E}^{\mathbf{2}}_\sharp(V) \\
{\scriptstyle \mathbf{E}^{\mathbf{2}}_\sharp(f)}\big\downarrow & & {\scriptstyle \mathbf{E}^{\mathbf{2}}_\sharp(f)}\big\downarrow \\
\mathbf{E}^{\mathbf{1}}_\sharp(W) & \xrightarrow{\mathbf{F}\circ\Psi(W)} & \mathbf{E}^{\mathbf{2}}_\sharp(W)
\end{array}$$

So the direct limit of the morphisms $\mathbf{F}\circ\Psi(V)$ exists as V runs over all finite dimensional G–submodules of $K[G]$. We denote this limit by $\mathbf{A}(\Psi) \,:\, \overline{\mathbf{E}}^{\mathbf{1}}_\sharp(K[G]) \,\longrightarrow\, \overline{\mathbf{E}}^{\mathbf{2}}_\sharp(K[G])$.

Let G' be a copy of G with trivial G–action as in (2.6). We will denote the limit of $\mathbf{F}\circ\Psi(V)$ as V varies over all finite dimensional G–submodules of $K[G']$ by $\mathbf{A}'(\Psi)$. Note that $\overline{\mathbf{E}}^{\mathbf{j}}_\sharp(K[G'])$ is the algebra $K[G']$ with the trivial filtration (5.8) and $\mathbf{A}'(\Psi) = \mathrm{id}$ using property F3(c).

Since the $\mathbf{F}\circ\Psi(V)$'s are morphisms of filtered vector spaces and the filtration on $\overline{\mathbf{E}}^{\mathbf{j}}_\sharp(K[G])$ is the direct limit of the filtrations on $\overline{\mathbf{E}}^{\mathbf{j}}_\sharp(V)$, it follows that $\mathbf{A}(\Psi)$ is a morphism of Σ–filtered vector spaces.

Since Ψ, by definition, respects F1-F4, it follows that $\mathbf{A}(\Psi)$ is a morphism of algebras. Regard the action of G on $K[G]$ as a morphism of algebras $a^* \,:\, K[G] \,\longrightarrow\, K[G]\otimes K[G']$. By (5.9), the G–action on $\overline{\mathbf{E}}^{\mathbf{j}}_\sharp(K[G])$ is given by

$$\overline{\mathbf{E}}^{\mathbf{j}}_\sharp(a^*) \,:\, \overline{\mathbf{E}}^{\mathbf{j}}_\sharp(K[G]) \,\longrightarrow\, \overline{\mathbf{E}}^{\mathbf{j}}_\sharp(K[G])\otimes K[G']\,.$$

Again, using functoriality, we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathbf{E}}^{\mathbf{1}}_{\sharp}(K[G]) & \xrightarrow{\overline{\mathbf{E}}^{\mathbf{1}}_{\sharp}(a^*)} & \overline{\mathbf{E}}^{\mathbf{1}}_{\sharp}(K[G]) \otimes K[G'] \\ {\scriptstyle \mathbf{A}(\Psi)}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(\Psi)\otimes \mathrm{id}} \\ \overline{\mathbf{E}}^{\mathbf{2}}_{\sharp}(K[G]) & \xrightarrow{\overline{\mathbf{E}}^{\mathbf{2}}_{\sharp}(a^*)} & \overline{\mathbf{E}}^{\mathbf{2}}_{\sharp}(K[G]) \otimes K[G'] \end{array}$$

This shows that $\mathbf{A}(\Psi)$ is G–equivariant. □

Lemma 6.2. *The functor* $\mathbf{A} : \mathfrak{Nor}^T(X) \longrightarrow \mathfrak{Calg}_G(\Sigma)$ *is faithful.*

Proof. Let $\Psi_j : \mathbf{E}^1 \longrightarrow \mathbf{E}^2$, $j = 1, 2$, be two morphisms. By Lemma 5.4, the underlying algebra of $\mathbf{A}(\mathbf{E}^j)$ is the coordinate algebra

$$K[\overline{E}^j(K[G])(x_0)] = K[\mathbf{N}_1^T(\mathbf{E}^j)(x_0)]$$

If $\mathbf{A}(\Psi_1) = \mathbf{A}(\Psi_2)$, then

$$\mathbf{N}_1^T(\Psi_1)|_{x_0} = \mathbf{N}_1^T(\Psi_1)|_{x_0} .$$

Now, by T–equivariance, $\mathbf{N}_1^T(\Psi_1)$ and $\mathbf{N}_1^T(\Psi_2)$ must agree over the open T–orbit. Hence, they must agree over X. Since $\mathbf{N}_1^T$ is faithful, we conclude that $\Psi_1 = \Psi_2$. □

Lemma 6.3. *Consider an object B in $\mathfrak{Calg}_G(\Sigma)$. Fix an embedding θ of G in* $\mathrm{GL}(V)$*. Then the Σ–filtration on B induces a compatible Σ–filtration on $(B \otimes K[V])^G$.*

Proof. Suppose σ is a maximal cone. Consider an action of $T_\sigma = T$ on B which is compatible with the $\Sigma|_\sigma$–filtration. Let

$$B = \bigoplus_{u \in M_\sigma} B_u^\sigma .$$

be the corresponding isotypical decomposition.

We first claim that

$$(B \otimes K[V])^G = \bigoplus_{u \in M_\sigma} (B_u^\sigma \otimes K[V])^G . \tag{6.2}$$

Let $x \in (B \otimes K[V])^G$. We may write x uniquely as a finite sum

$$x = \sum x_u , \tag{6.3}$$

where $x_u \in B_u^\sigma \otimes K[V]$. Since the actions of T and G on B commute, B_u^σ is G–invariant. This implies that $gx_u \in B_u^\sigma \otimes K[V]$ for any $g \in G$. Since $gx = x$, we have

$$x = \sum gx_u$$

which is another decomposition of x with components in $B_u^\sigma \otimes K[V]$. By the uniqueness of (6.3), we must have $gx_u = x_u$ for all u and g. This means that $x_u \in (B_u^\sigma \otimes K[V])^G$. Hence,

$$(B \otimes K[V])^G \subseteq \bigoplus_{u \in M_\sigma} (B_u^\sigma \otimes K[V])^G .$$

Clearly

$$(B \otimes K[V])^G \supseteq \bigoplus_{u \in M_\sigma} (B_u^\sigma \otimes K[V])^G .$$

Hence (6.2) follows. Using it we conclude that

$$((B \otimes K[V])^G)^\sigma_u \;=\; (B^\sigma_u \otimes K[V])^G\,. \tag{6.4}$$

By the compatibility of the Σ–filtration on B, the decomposition

$$B^\rho(i) = \bigoplus_{u(\rho)\geq i} B^\sigma_u$$

is independent of the choice of σ such that $\rho \in \sigma$. By (6.4),

$$((B \otimes K[V])^G)^\rho(i) \;=\; \bigoplus_{u(\rho)\geq i} (B^\sigma_u \otimes K[V])^G\,. \tag{6.5}$$

To show the independence of (6.5) of the choice of maximal cone σ, we want to show that

$$((B \otimes K[V])^G)^\rho(i) \;=\; (B^\rho(i) \otimes K[V])^G\,. \tag{6.6}$$

If $v(\rho) \geq i$, then

$$(B^\sigma_v \otimes K[V])^G \;\subseteq\; ((\bigoplus_{u(\rho)\geq i} B^\sigma_u) \otimes K[V])^G \;=\; (B^\rho(i) \otimes K[V])^G\,.$$

Therefore, by (6.5),

$$((B \otimes K[V])^G)^\rho(i) \;\subseteq\; (B^\rho(i) \otimes K[V])^G\,. \tag{6.7}$$

On the other hand, suppose $x \in (B^\rho(i) \otimes K[V])^G$. Then x admits a unique decomposition

$$x = \sum_{u(\rho)\geq i} x_u$$

where $x_u \in B^\sigma_u \otimes K[V]$. Then by using the G–invariance of x and the uniqueness of the decomposition as before, we conclude that $gx_u = x_u$ for all $g \in G$. Hence $x_u \in (B^\sigma_u \otimes K[V])^G$, and consequently,

$$x \;\in\; \bigoplus_{u(\rho)\geq i} (B^\sigma_u \otimes K[V])^G = ((B \otimes K[V])^G)^\rho(i)\,.$$

Hence,

$$((B \otimes K[V])^G)^\rho(i) \;\supseteq\; (B^\rho(i) \otimes K[V])^G\,. \tag{6.8}$$

By (6.7) and (6.8), equation (6.6) holds, concluding the proof. □

Lemma 6.4. *Assume all maximal cones in Σ are top dimensional. Then the functor $\mathbf{A} : \mathfrak{Vor}^T(X) \longrightarrow \mathfrak{Calg}_G(\Sigma)$ is essentially surjective.*

Proof. Consider an object B in $\mathfrak{Calg}_G(\Sigma)$. Consider a top-dimensional cone σ in Σ. Note that $T_\sigma = T$.

Fix a T–action on B which is compatible with the Σ–filtration and commutes with the G–action on B. Define $E^\sigma_G = X_\sigma \times \mathrm{Spec}(B)$. With the induced actions of T and G, this E^σ_G is a T–equivariant principal G–bundle over X_σ.

Fix a closed point y_0 in $\mathrm{Spec}(B)$. Recall the closed point $x_0 \in O$ used in the construction of $\mathbf{F}$. Let $e = (x_0, y_0) \in E^\sigma_G$. The T–action on $\mathrm{Spec}(B)$ may be represented by a homomorphism

$$\rho_\sigma : T \longrightarrow G, \quad \text{defined by} \quad ty_0 \;=\; y_0 \cdot \rho_\sigma(t) \text{ for any } t \,\in\, T\,.$$

We claim that for any two top-dimensional cones σ and τ, the functions

$$\rho_\sigma \rho_\tau^{-1} \;:\; T \;\longrightarrow\; G$$

extend to regular G–valued functions over $X_\sigma \bigcap X_\tau$ under the standard identification of T with the open orbit O in X.

Fix an embedding θ of G in $\mathrm{GL}(V)$. Let $E^\sigma = E^\sigma_G \times^G V$ be the associated T–equivariant vector bundle. The actions of T and G on $\mathrm{Spec}(B) = E^\sigma_G(x_0)$ commute and hence induce a T–action on $E^\sigma(x_0) = \mathrm{Spec}(B) \times^G V$. Using a specific isomorphism

$$\mathrm{Spec}(B) \times^G V \cong V \quad \text{induced by the rule} \quad [(y_0, v)] \longmapsto v\,, \tag{6.9}$$

we obtain an induced T–action and therefore a $\Sigma|_\sigma$–filtration on V. We claim that as σ varies, this induces a compatible Σ–filtration on V. We will derive this from the compatibility of the Σ–filtration on the algebra B.

The T–action on B, for any fixed σ, induces a T–action on $(B \otimes K[V])^G$. Here the action of G on $K[V]$ is induced by θ, and the action of T on $K[V]$ is assumed to be trivial. As σ varies, from Lemma 6.3 it follows that these actions yield a compatible Σ–filtration on $(B \otimes K[V])^G$.

Since $\mathrm{Spec}(B) \times V$ is affine and G is reductive, $\mathrm{Spec}(B) \times^G V = \mathrm{Spec}((B \otimes K[V])^G)$ [23, Theorem 1.1, page 27]. The isomorphism (6.9) induces a specific isomorphism $(B \otimes K[V])^G \cong K[V]$. This gives a Σ–filtration on $K[V]$. By linearity of the G–action on V, the cone-wise T–actions on $K[V]$ are determined by cone-wise T–actions on the dual V^* of V. These determine a Σ–filtration on V^*, which is compatible as it is the restriction of the compatible Σ–filtration on $K[V]$. We have an induced compatible dual Σ–filtration on V (see [11], section 6.3). This Σ–filtration on V agrees with the Σ–filtration on V derived immediately after (6.9) as the cone-wise T–actions match. This proves the claim regarding the compatibility of that Σ–filtration.

Note that the T–action on V, associated to the cone σ, is given by $\theta(\rho_\sigma)$. Since the Σ–filtration on V is compatible, by Klyachko [13], it gives rise to a T–equivariant vector bundle over X and the $\mathrm{GL}(V)$–valued functions $\theta(\rho_\sigma\rho_\tau^{-1})$ extend regularly over $X_\sigma \bigcap X_\tau$. It follows that the functions $\rho_\sigma\rho_\tau^{-1}$ extend regularly over $X_\sigma \bigcap X_\tau$. Since G is closed in $\mathrm{GL}(V)$, the extensions are G–valued. This allows us to construct a T–equivariant principal G–bundle E_G over X by gluing the bundles $\{E^\sigma_G\}$ using $\{\rho_\sigma\rho_\tau^{-1}\}$ as transition functions.

It is straightforward to show that $\mathbf{A}(\mathbf{N}^T_0(E_G)) \cong B$: As an algebra, $\mathbf{A}(\mathbf{N}^T_0(E_G)) = \mathrm{Spec}(B) \times^G K[G]$. We have an isomorphism

$$\alpha_* \,:\, \mathrm{Spec}(B) \times^G K[G] \longrightarrow K[G]\,,$$

induced by the G–equivariant isomorphism

$$\alpha \,:\, \mathrm{Spec}(B) \longrightarrow G\,, \quad \text{where } \alpha(y_0) = 1_G\,.$$

Moreover, the map α also induces an isomorphism

$$\alpha^* : K[G] \longrightarrow B\,.$$

Thus we have an isomorphism

$$\alpha^* \circ \alpha_* \,:\, \mathrm{Spec}(B) \times^G K[G] \longrightarrow B\,.$$

Given a right T–action ρ_σ on $\mathrm{Spec}(B)$, let ρ^*_σ be the induced T–action on B. These actions, as σ varies, produce the Σ–filtration on B. The $\Sigma|_\sigma$–filtration or T_σ–action on $\mathrm{Spec}(B) \times^G K[G]$ is induced from ρ_σ by α. Let us call this action $\alpha_*(\rho_\sigma)$. Note that the action on B induced from $\alpha_*(\rho_\sigma)$ by the isomorphism α^* is same as ρ^*_σ. Thus $\alpha^* \circ \alpha_*$ induces the required isomorphism of Σ–filtrations. □

Lemma 6.5. *Let X be a toric variety such that every maximal cone in its fan is top-dimensional. Then the functor*

$$\mathbf{A} \,:\, \mathfrak{Nor}^T(X) \longrightarrow \mathfrak{Calg}_G(\Sigma)$$

(see (3.17)*, Definition 6.1 and Lemma 6.1) is full.*

Proof. Let $\mathbf{E}^1,\, \mathbf{E}^2 \in \mathfrak{Nor}^T(X)$ and $\phi \,:\, \mathbf{A}(\mathbf{E}^2) \longrightarrow \mathbf{A}(\mathbf{E}^1)$ be a morphism in $\mathfrak{Calg}_G(\Sigma)$. We need to show that there exists a morphism $\Psi \,:\, \mathbf{E}^1 \longrightarrow \mathbf{E}^2$ in $\mathfrak{Nor}^T(X)$ such that $\mathbf{A}(\Psi) = \phi$.

Note that the underlying algebra of $\mathbf{A}(\mathbf{E}^j)$ is the algebra of functions on the fiber of the principal G–bundle $\overline{E}^j(K[G])$ at x_0. Then ϕ induces a G–equivariant morphism of varieties

$$\phi' \,:\, \overline{E}^1(K[G])(x_0) \longrightarrow \overline{E}^2(K[G])(x_0)\,.$$

The latter, by T–equivariance induces a morphism of principal bundles

$$\phi'_* \,:\, \overline{E}^1(K[G]) \longrightarrow \overline{E}^2(K[G])$$

over the open orbit O of X. This induces an isomorphism of the sheaves of $\mathcal{O}_O-$algebras

$$\phi_* \,:\, \overline{\mathbf{E}}^1(K[G]) \longrightarrow \overline{\mathbf{E}}^2(K[G])\,.$$

For any finite dimensional Σ–filtered subspace V of $\mathbf{A}(\mathbf{E}^2)$, the restriction

$$\phi|_V \,:\, V \longrightarrow \phi(V)$$

is an isomorphism of Σ–filtered vector spaces. The corresponding restriction of ϕ_* induces a T–equivariant isomorphism of vector bundles over O. But since $\phi|_V$ respects the filtrations, by the arguments of Klyachko [13, Assertion 2.2.4], this extends to a T–equivariant isomorphism of vector bundles over X. Then taking direct limit over all finite dimensional Σ–filtered subspaces of $\mathbf{A}(\mathbf{E}^2)$, we observe that ϕ_* extends over X as a T–equivariant isomorphism of quasi-coherent sheaves of $\mathcal{O}_X-$modules. We show that the extension is in fact an isomorphism of $\mathcal{O}_X-$algebras.

For $j = 1,\, 2$, let

$$m_j \,:\, K[\overline{\mathbf{E}}^j(K[G])] \otimes K[\overline{\mathbf{E}}^j(K[G])] \longrightarrow K[\overline{\mathbf{E}}^j(K[G])]$$

denote the multiplication in the sheaf $\overline{\mathbf{E}}^j(K[G])$. Then by construction of ϕ_*, the equality

$$m_2 \circ (\phi_* \otimes \phi_*) \,=\, \phi_* \circ m_1$$

of morphisms of quasi-coherent sheaves of $\mathcal{O}_X-$modules, holds over O. Therefore, it holds over the closure X of O.

We may similarly argue that ϕ_* is G–equivariant. Thus ϕ_* induces an isomorphism $\widehat{\phi}_* \,:\, \overline{E}^1(K[G]) \longrightarrow \overline{E}^2(K[G])$ of T–equivariant principal G–bundles. This induces an isomorphism

$$\mathbf{N}_0^T(\widehat{\phi}_*) \,:\, \mathbf{N}_0^T(\overline{E}^1(K[G])) \longrightarrow \mathbf{N}_0^T(\overline{E}^2(K[G]))\,.$$

By Theorem 2.2, there exists a functorial of isomorphism $\Phi \,:\, \mathbf{N}_0^T \circ \mathbf{N}_1^T \longrightarrow 1_{\mathfrak{Nor}^T(X)}$. Applying it we have the required morphism

$$\Psi \,=\, \Phi(\mathbf{N}_0^T(\widehat{\phi}_*)) \,:\, \overline{\mathbf{E}}^{\mathbf{1}} \longrightarrow \overline{\mathbf{E}}^{\mathbf{2}}\,.$$

□

The following theorem is a consequence of Theorem 2.2, and Lemmas 6.2, 6.4 and 6.5.

Theorem 6.6. *Let X be a toric variety such that every maximal cone in its fan is top-dimensional. Then there is an equivalence of categories between $\mathfrak{Pbun}_G^T(X)$ and $\mathfrak{Calg}_G(\Sigma)$, where G is a reductive group.*

7. Reduction of structure group

Theorem 7.1. *Suppose H is a reductive subgroup of G, and let E_G be a T–equivariant principal G–bundle over X. Let $S = \overline{\mathbf{E}}_\sharp(K[G])$, where $\mathbf{E} = \mathbf{N}_0^T(E_G)$. Then E_G admits a T–equivariant reduction of structure group to H if and only if there exists a filtered algebra $R \in \mathfrak{Calg}_H(\Sigma)$ such that $(R \otimes K[G])^H \cong S$ in $\mathfrak{Calg}_G(\Sigma)$.*

Proof. If E_G admits a T–equivariant reduction of structure group to $E_H \in \mathfrak{Pbun}_H^T(X)$, there exists an isomorphism

$$E_H \times_H G \cong E_G\,. \tag{7.1}$$

Let R be the Σ–filtered algebra in $\mathfrak{Calg}_H(\Sigma)$ associated to E_H. Then (7.1) yields an isomorphism

$$(R \otimes K[G])^H \cong S$$

in $\mathfrak{Calg}_G(\Sigma)$.

On the other hand, a bundle E_G with filtered algebra $(R \otimes K[G])^H$ is isomorphic to $E_H \times_H G$ where E_H is the bundle associated to R. This follows from the isomorphism at the level of fibers at the closed point x_0 in the open T–orbit O, together with the isomorphism of filtrations, as in the proof of Lemma 6.5. □

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