Spatial and Temporal Correlation of the Interference in ALOHA Ad Hoc Networks

Radha Krishna Ganti and Martin Haenggi
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, USA
{rganti,mhaenggi}@nd.edu

Abstract

Interference is a main limiting factor of the performance of a wireless ad hoc network. The temporal and the spatial correlation of the interference makes the outages correlated temporally (important for retransmissions) and spatially correlated (important for routing). In this letter we quantify the temporal and spatial correlation of the interference in a wireless ad hoc network whose nodes are distributed as a Poisson point process on the plane when ALOHA is used as the multiple-access scheme.

I. Introduction

Interference in a wireless ad hoc network is a spatial phenomenon which depends on the set of transmitters, the path loss, and the fading. The presence of common randomness in the locations of the interferers induces temporal and spatial correlations in the interference, even for ALOHA. These correlations affect the retransmission strategies and the routing. In the literature, these correlations are generally neglected for the purpose of analytical tractability and because these correlations do not change the scaling behavior of an ad hoc wireless network. For example, in [1] and [2], the spatial correlations are neglected for the purpose of routing. Also extending results like the transmission capacity [3] from a single-hop to a multi-hop scenario requires taking the spatio-temporal correlations into account. In this letter we quantify the spatial and temporal correlations of the interference and the link outages for ALOHA.

II. System Model

We model the location of the nodes (radios) as a Poisson point process (PPP) $\phi = \{x_1, x_2, \ldots\} \subset \mathbb{R}^2$ of density λ. We assume that all the nodes transmit with unit power and that the fading is spatially and temporally independent with unit mean. The (power) fading coefficient between two pairs of nodes x and y at time instant n is denoted by $h_{xy}(n)$. The large scale path loss function is denoted by $g(x)$ and is assumed to have the following properties:

1) Depends only on $\|x\|$.

2) Monotonically decreases with $\|x\|$.
3) Integrable:

$$\int_0^\infty x g(x) \mathrm{d}x < \infty. \tag{1}$$

For example, a valid path loss model is given by

$$g_\epsilon(x) = \frac{1}{\epsilon + \|x\|^\alpha}, \ \epsilon \in (0, \infty), \ \alpha > 2. \tag{2}$$

We can model the standard singular path loss model $g(x) = \|x\|^{-\alpha}$ by considering the limit $\lim_{\epsilon \to 0} g_\epsilon(x)$. The interference at time instant m and (spatial) location z is given by

$$I_k(z) = \sum_{x \in \phi} \mathbf{1}(x \in \phi_k) h_{xz}(k) g(x - z). \tag{3}$$

where ϕ_k denotes the transmitting set at time k. We assume that the MAC protocol used is ALOHA where each node decides to transmit independently with probability p in each slot.

III. Spatio-Temporal Correlation of Interference

In a wireless system the transmitting set changes at every time slot because of the MAC scheduler. Since the transmitting sets at different time slots are chosen from ϕ (a common source of randomness), the interference exhibits temporal and spatial correlation. Since ALOHA chooses the transmitting sets identically across time, $I_k(u)$ is identically distributed for all k. Since nodes transmit independently of each other in ALOHA, the transmitting set $\phi_k \subset \phi$ is also spatially stationary, and hence $I_k(u) \stackrel{d}{=} I_k(o)$ where $\stackrel{d}{=}$ denotes equality in distribution and o denotes the origin in $\mathbb{R}^2$. Hence we have

$$\begin{aligned}
\mathbb{E} I_k(u) &= \mathbb{E} I_k(o) \\
&\stackrel{(a)}{=} \mathbb{E} \sum_{x \in \phi} \mathbf{1}(x \in \phi_k) h_{xo}(k) g(x) \\
&\stackrel{(b)}{=} p\lambda \int_{\mathbb{R}^2} g(x) \mathrm{d}x,
\end{aligned} \tag{4}$$

where (a) follows from Campbell's theorem [4] and (b) follows since $\mathbb{E}[h] = 1$. The second moment of the interference is given by

$$\begin{aligned}
\mathbb{E}[I_k(o)^2] &= \mathbb{E}\left[\left(\sum_{x \in \phi_k} h_{xo}(k) g(x)\right)^2\right] \\
&= \mathbb{E} \sum_{x \in \phi_k} h_{xo}^2(k) g^2(x) \\
&\quad + \mathbb{E} \sum_{x, y \in \phi_k}^{x \neq y} h_{xo}(k) h_{yo}(k) g(x) g(y) \\
&\stackrel{(a)}{=} p\mathbb{E}[h^2]\lambda \int_{\mathbb{R}^2} g^2(x) \mathrm{d}x \\
&\quad + p^2 \mathbb{E}[h]^2 \lambda^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x) g(y) \mathrm{d}x \mathrm{d}y.
\end{aligned} \tag{5}$$

where (a) follows from the independence of $h_{xo}(k)$ and $h_{yo}(k)$ and the second-order product density formula of the Poisson point process [4]. When the fading follows a Nakagami-m[1] distribution and the path loss model is given by $g_\epsilon(x)$, the variance of the interference follows from (4) and (5) and is given by

$$\text{Var}\left[I_k(o)\right] = \frac{2\pi^2(\alpha-2)p\lambda}{\epsilon^{2-2/\alpha}\alpha^2\sin(2\pi/\alpha)}\frac{m+1}{m}, \tag{7}$$

and the mean product of $I_k(u)$ and $I_l(v)$ at times k and $l,\ k\neq l$ is given by

$$\begin{aligned}
&\mathbb{E}[I_k(u)I_l(v)] \\
=\ &\mathbb{E}\left[\sum_{x\in\phi_k} h_{xu}(k)g(x-u)\sum_{y\in\phi_l} h_{yv}(l)g(y-v)\right] \\
=\ &p^2\mathbb{E}[h]^2\lambda\int_{\mathbb{R}^2} g(x-u)g(x-v)\mathrm{d}x \\
&+\mathbb{E}\sum_{x,y\in\phi}^{x\neq y}\mathbf{1}(x\in\phi_k)\mathbf{1}(y\in\phi_l)h_{xu}(k)h_{yv}(l)g(x)g(y).
\end{aligned}$$

By Campbell's theorem and the second order product density of a PPP, we have

$$\begin{aligned}
\mathbb{E}[I_k(u)I_l(v)] &= p^2\mathbb{E}[h]^2\lambda\int_{\mathbb{R}^2} g(x-u)g(x-v)\mathrm{d}x \\
&\quad+\lambda^2p^2\mathbb{E}[h]^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x)g(y)\mathrm{d}x\mathrm{d}y \qquad (8)\\
&= p^2\lambda\int_{\mathbb{R}^2} g(x-u)g(x-v)\mathrm{d}x \qquad (9)\\
&\quad+\lambda^2p^2\left(\int_{\mathbb{R}^2} g(x)\mathrm{d}x\right)^2. \qquad (10)
\end{aligned}$$

Lemma 1: The spatio-temporal correlation coefficient of the interferences $I_k(u)$ and $I_l(v), k\neq l$, when the path loss function $g(x)$ satisfies (1) is given by

$$\zeta(u,v) = \frac{p\int_{\mathbb{R}^2} g(x)g(x-\|u-v\|)\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2} g^2(x)\mathrm{d}x}. \tag{11}$$

Proof: Since $I_k(u)$ and $I_l(v)$ are identically distributed, we have

$$\zeta(u,v) = \frac{\mathbb{E}[I_k(u)I_l(v)] - \mathbb{E}[I_k(u)]^2}{\mathbb{E}[I_k(u)^2] - \mathbb{E}[I_k(u)]^2}. \tag{12}$$

Since $I_k(u) \stackrel{d}{=} I_k(o)$ and by substituting for the above quantities we have,

$$\begin{aligned}
\zeta(u,v) &= \frac{p\int_{\mathbb{R}^2} g(x-u)g(x-v)\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2} g^2(x)\mathrm{d}x} \\
&\stackrel{(a)}{=} \frac{p\int_{\mathbb{R}^2} g(x)g(x-\|u-v\|)\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2} g^2(x)\mathrm{d}x}, \qquad (13)
\end{aligned}$$

where (a) follows by using the substitution $y=x-u$ and the fact that $g(x)$ depends only on $\|x\|$. ■

[1]The distribution is given by

$$F(x) = 1 - \frac{\Gamma_{\text{ic}}(m,mx)}{\Gamma(m)}, \tag{6}$$

where Γ_{ic} denotes the incomplete gamma function.

We have the following result about the temporal correlation by setting $\|u-v\|=0$.

Corollary 2: The temporal correlation coefficient with ALOHA as the MAC protocol and is given by

$$\zeta_t = \frac{p}{\mathbb{E}[h^2]}. \tag{14}$$

When the fading is Nakagami-m, the correlation coefficient is $\zeta_t = \frac{pm}{m+1}$. In particular, for $m=1$ (Rayleigh fading), the temporal correlation coeffecient is $p/2$ and for $m \to \infty$ (no fading), the temporal correlation coeffecient is p. We first observe that the correlation increases with increasing m, i.e., fading decreases correlation which is intuitive. Observe that in the above derivation, $\int_{\mathbb{R}^2} g^2(x)\mathrm{d}x$ is not defined when $g(x)=\|x\|^{-\alpha}$, but we can use $g_\epsilon(x)$ and take $\epsilon \to 0$. We now find the correlation for the singular path-loss model as a limit of $g_\epsilon(x)$.

Corollary 3: Let the path loss model be given by $g_\epsilon(x) = 1/(\epsilon + \|x\|^\alpha)$. We then have

$$\lim_{\epsilon\to 0} \zeta(u,v) = 0, \quad u \neq v. \tag{15}$$

Proof: We have

$$\begin{aligned}
\zeta(u,v) &= \lim_{\epsilon \to 0} \frac{p\int_{\mathbb{R}^2} g_\epsilon(x-u)g_\epsilon(x-v)\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2} g_\epsilon^2(x)\mathrm{d}x} \\
&\overset{(a)}{=} \lim_{\epsilon \to 0} \frac{p\int_{\mathbb{R}^2} \frac{1}{1+\|x-u\epsilon^{-1/\alpha}\|^\alpha} \frac{1}{1+\|x-v\epsilon^{-1/\alpha}\|^\alpha}\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2}\left(\frac{1}{1+\|x\|^\alpha}\right)^2\mathrm{d}x} \\
&= 0,
\end{aligned}$$

where (a) follows from change of variables. ∎

The correlation coefficient being 0 is an artifact of the singular path loss model. When the path loss is $\|x\|^{-\alpha}$, the nearest transmitter is the main contributor to the interference. So for $u \neq v$, the interference as viewed by u is dominated by transmitters in a disc $B(u,\delta), \delta > 0$ of radius δ centered at u and for v dominated by transmitters in $B(v,\delta)$ for small δ. The transmitters locations being independent in $B(v,\delta)$ and $B(u,\delta)$ for a PPP, makes the correlation-coefficient go to zero. A more powerful metric like mutual information would be better able to capture the dependence of interference for the singular path loss model. In Figure 1, the spatial correlation is plotted as a function of $\|u-v\|$ for different ϵ.

IV. Temporal Correlation of Link Outages

In the standard analysis of retransmissions in a wireless ad hoc system, the link failures are assumed to be uncorrelated across time. But this is not so, since the interference is temporally correlated. We now provide the conditional probability of link formation assuming a successful transmission.

We assume that a transmitter at the origin has a destination located at $z \in \mathbb{R}^2$. Let A_k denote the event that the origin is able to connect to its destination z at time instant k, i.e.,

$$\text{SIR} = \frac{h_{oz}(k)g(z)}{I_k(z)} > \theta. \tag{16}$$

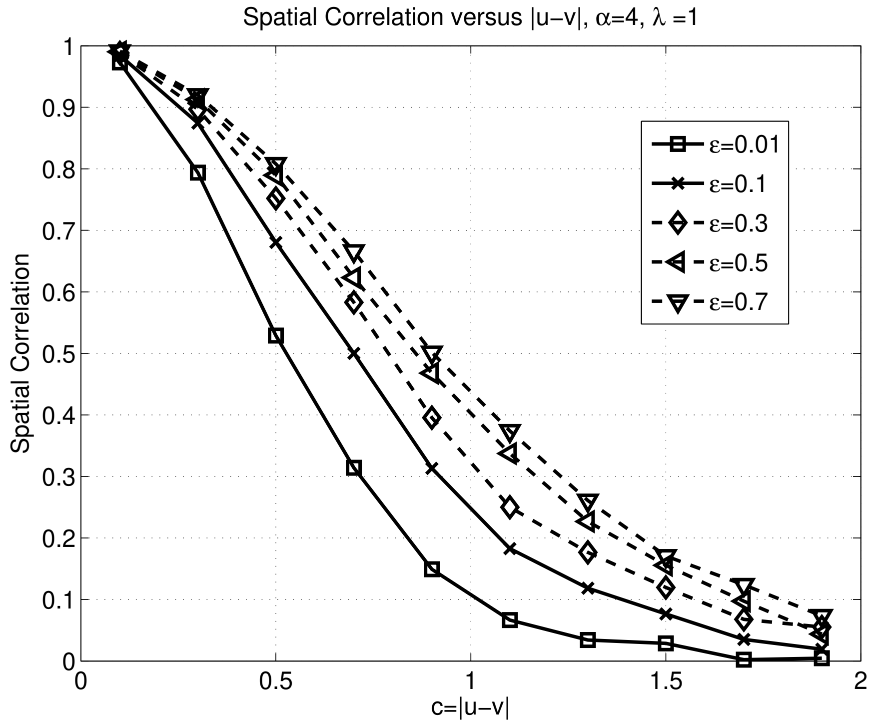

Fig. 1. Spatial correlation $\zeta(u,v)/p$ versus $\|u-v\|$, when the path-loss model is given by $g_\epsilon(x)$, $\lambda = 1$ and $\alpha = 4$. We observe that $\zeta_s(u,v) \to 0,\ u \neq v$, for $\epsilon \to 0$.

For simplicity we shall assume the fading is Rayleigh (similar methods can be used for Nakagami-m). We now provide the joint probability of success $\mathbb{P}(A_k, A_l),\ k \neq l$. We have

$$\begin{aligned}
\mathbb{P}(A_k, A_l) &= \mathbb{P}\left(h_{oz}(k) > aI_k(z), h_{oz}(l) > aI_l(z)\right) \\
&\overset{(a)}{=} \mathbb{E}\left[\exp(-aI_k(z))\exp(-aI_l(z))\right] \\
&= \mathbb{E}[\exp(-a\sum_{x\in\phi} g(x)[\mathbf{1}(x\in\phi_k)h_{xz}(k) \\
&\quad +\mathbf{1}(x\in\phi_l)h_{xz}(l)])] \\
&\overset{(b)}{=} \mathbb{E}\left[\prod_{x\in\phi}\left(\frac{p}{1+ag(x)}+1-p\right)^2\right] \\
&\overset{(c)}{=} \exp\left(-\lambda\int_{\mathbb{R}^2} 1-\left(\frac{p}{1+ag(x)}+1-p\right)^2 \mathrm{d}x\right),
\end{aligned} \tag{17}$$

where $a = \theta/g(z)$. (a) follows from the independence of $h_{oz}(k)$ and $h_{oz}(l), k \neq l$, (b) follows by taking the average with respect to $h_{xz}(k),\ h_{xz}(l)$ and the ALOHA, (c) follows from the probability generating functional of the PPP. Similarly we have

$$\mathbb{P}(A_l) = \exp\left(-\lambda\int_{\mathbb{R}^2} 1-\left(\frac{p}{1+ag(x)}+1-p\right)\mathrm{d}x\right).$$

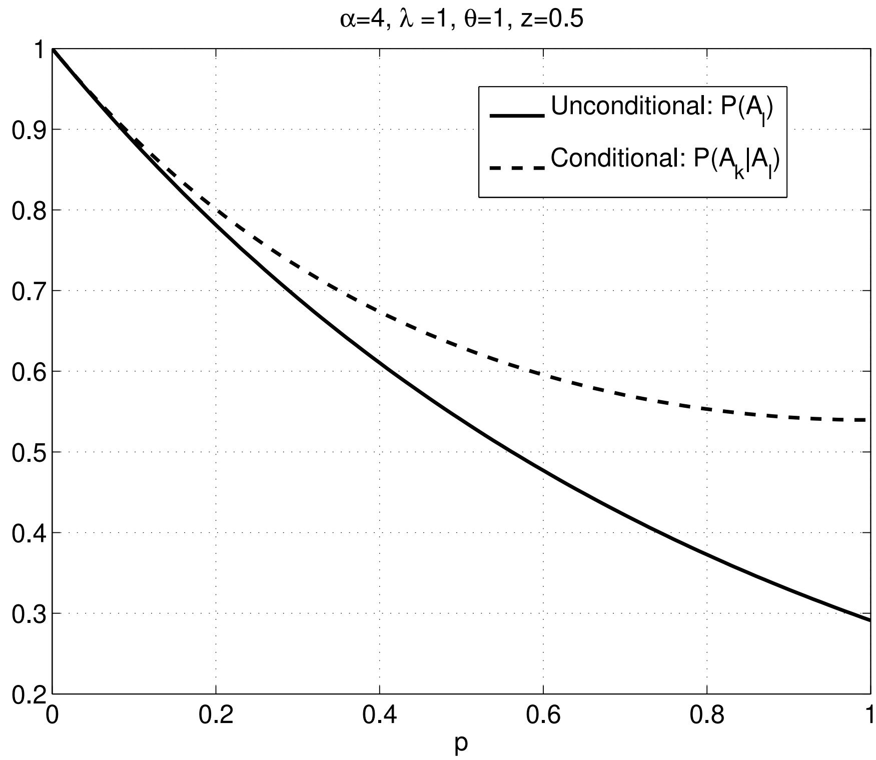

Fig. 2. $\mathbb{P}(A_k|A_l)$ and $\mathbb{P}(A_l)$ versus the ALOHA parameter p. $\lambda = 1$, $g(x) = \|x\|^{-4}$, $z = 0.5$, $\theta = 1$.

So the ratio of conditional and the unconditional probability is given by

$$\begin{aligned} \frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)} &= \frac{\mathbb{P}(A_k, A_l)}{\mathbb{P}(A_l)^2} \\ &= \exp\left(\lambda p^2 \int_{\mathbb{R}^2} \left(\frac{ag(x)}{1+ag(x)}\right)^2 \mathrm{d}x\right) \\ &> 1. \end{aligned} \tag{18}$$

When $g(x) = \|x\|^{-\alpha}$, we have

$$\frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)} = \exp\left(2\lambda a^{2/\alpha} p^2 \pi^2 \frac{(\alpha-2)}{\alpha^2} \csc\left(\frac{2\pi}{\alpha}\right)\right). \tag{19}$$

In Figure we plot the conditional and the unconditional link success probabilities. We make the following observations:

1) From (18), we observe that the link formation is correlated across time.
2) If a transmission succeeds at a time instant m, there is a higher probability that a transmission succeeds at a time instant n.
3) From (18), we also have $\mathbb{P}(A_k^c|A_l^c) > \mathbb{P}(A_l^c)$. So a link in outage is always more likely to be in outage and hence the retransmission strategy should reduce the rate of transmission or change the density of transmitters rather than retransmit "blindly".

4) We observe that $\frac{\mathbb{P}(A_k|A_l)}{\mathbb{P}(A_l)}$ always increases with θ, λ, p. The increase in λ and p is because of the larger transmit set due to which the probability of the same sub-set of nodes transmitting at different times increases, thereby causing more correlation. When θ is large, the outage is a result of the interfering transmissions caused by a larger number of nodes. Hence by a similar reasoning as above, the correlation increases.

V. Conclusions

In this paper, we have derived the spatial and temporal correlations of interference in an ALOHA wireless network. We also have proved that the link outages are temporally correlated. This fact should be taken into account when analyzing ad hoc performance and designing retransmission strategies.

Acknowledgments

The support of the NSF (grants CNS 04-47869, CCF 728763) and the DARPA/IT-MANET program is gratefully acknowledged.

References

[1] F. Baccelli, B. Blaszczyszyn, and P. Muhlethaler, “An ALOHA protocol for multihop mobile wireless networks,” *IEEE Transactions on Information Theory*, no. 2, Feb 2006.

[2] M. Haenggi, “On routing in random Rayleigh fading networks,” *IEEE Transactions on Wireless Communications*, vol. 4, no. 4, pp. 1553–1562, 2005.

[3] S. Weber, X. Yang, J. Andrews, and G. de Veciana, “Transmission capacity of wireless ad hoc networks with outage constraints,” *Information Theory, IEEE Transactions on*, vol. 51, no. 12, pp. 4091–4102, 2005.

[4] D. Stoyan, W. S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*, 2nd ed., ser. Wiley series in probability and mathematical statistics. New York: Wiley, 1995.