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Semirigid Geometry

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We provide an intrinsic description of N-super Riemann surfaces and TN-semirigid surfaces. Semirigid surfaces occur naturally in the description of topological gravity as well as topological supergravity. We show that such surfaces are obtained by an integrable reduction of the structure group of a complex supermanifold. We also discuss the super-moduli spaces of TN-semirigid surfaces and their relation to the moduli spaces of N-super Riemann surfaces.

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1. Introduction

Semirigid surfaces have been shown [1][2] to provide a geometric framework to describe $2d$ topological gravity and supergravity. For example, in the simplest theory the dilaton as well as the puncture equations have been proven using the semirigid formalism [3][4]. In this paper, we provide an intrinsic or coordinate invariant definition of semirigid super Riemann surfaces ($SSRS$) as well as ordinary super Riemann surfaces (SRS). The discussion of SRS is a natural extension to similar discussions provided in [5] and applied in [6] for the case of $N = 1$ SRS and in [7] for $N = 2$; the framework follows Cartan's theory of G-structures. (For an introduction to G-structures, see for example [8][9][6][10].) We show that these structures subject to some conditions called "torsion constraints" are integrable, which relates our intrinsic definition to the coordinate dependent definitions.

We will first discuss the various definitions and illustrate G-structures via two examples in sect. 2. We also find the appropriate group G for superconformal and semirigid surfaces and the corresponding torsion constraints. Sect. 3 deals with showing that the G-structures we impose are integrable provided the constraints are satisfied. Briefly the results are as follows. If we begin with a complex supermanifold, then N-SRS have no essential torsion constraints, generalizing Baranov, Frolov, and Schwarz [11], who considered $N = 1$.[1] We will refer to semirigid surfaces with N-supersymmetry as "topological N-SRS," or TN for short. $TN = 0$ surfaces have a rather trivial essential constraint while $TN = 1$ surfaces have several. Both in the usual and in the topological case the category of surfaces with appropriate G-structures, integrable in the sense we will specify, is equivalent to the corresponding category of surfaces with appropriate patching data. (Actually we will limit ourselves to proving this for $N \leq 3$ and $TN \leq 1$ to keep the algebra simple.) In particular there are no second-order conditions for flatness, just as for ordinary $N = 0$ conformal structures. Throughout this paper we will consider only untwisted superconformal and semirigid structures, since our focus is primarily on local properties. The integrability results we prove will also apply to the study of twisted surfaces.

We should comment on the relation of this work to [1][2]. In these papers the coordinate definition of semirigid surfaces was used. The interpretation of such surfaces as having a special G-structure was crucial for finding the right patching maps, but no attempt was made to prove the equivalence of the two approaches, *i.e.* the theorem that every integrable G-structure gave a semirigid surface. That is what we do here.

[1] This generalization was asserted in the appendix to [12]. The constraints found in [13] and discussed in [6] arise when we begin with a *real* supermanifold.

2. G-structures on manifolds and supermanifolds

We begin by stating the problem, then recall the general idea of G-structures with some examples.

2.1. Patch definition of SRS and SSRS

One way of defining SRS or $SSRS$ is to cut a supermanifold into patches, put coordinates on them and sew them back together with transition functions given by superconformal or semirigid coordinate transformation. Let us begin with SRS. Generalizing the $N=1$ superconformal transformation [14][11][15][12], we start with $C^{1|N}$ and define for $i=1,\ldots,N$

$$D_i = \frac{\partial}{\partial\theta^i} + g_{ij}\theta^j \, \frac{\partial}{\partial z} \tag{2.1}$$

where $g_{ij}=\delta_{ij}$. We impose the condition that $\{D_i\}$ transform linearly among themselves (not mix with $\frac{\partial}{\partial z}$) under a superconformal coordinate transformation $(z,\theta^i) \to (\widetilde{z},\widetilde{\theta}^i)$. This condition resembles the one for a complex manifold, where the good coordinate transformations do not mix the ∂_{z^i} with the $\partial_{\bar{z}^i}$. Thus,

$$D_i = F_i{}^j \widetilde{D}_j \quad ; \qquad F_i{}^j = D_i \widetilde{\theta}^{\,j} \quad , \tag{2.2}$$

where $\widetilde{D}_i = \frac{\partial}{\partial\tilde{\theta}^i} + g_{ij}\tilde{\theta}^j \frac{\partial}{\partial\tilde{z}}$ and F is some invertible matrix of functions. It follows that the superconformal transformations are those for which

$$D_i \widetilde{z} = g_{jk}\widetilde{\theta}^j D_i \widetilde{\theta}^k. \tag{2.3}$$

An N-superconformal surface is then just a supermanifold patched together from pieces of $C^{1|N}$ related by N-superconformal transition functions.

Semirigid surfaces (or SSRS) are patched together by restricted superconformal transition functions. The restriction imposed is that θ^+ be global, where $\theta^\pm \equiv \frac{1}{\sqrt{2}}(\theta^1 \pm i\theta^2)$. For instance, to obtain the $TN=0$ semirigid coordinate transformations, we start with $N=2$ superconformal coordinate transformations and impose $\widetilde{\theta}^+ = \theta^+$. This restriction together with (2.3) fixes the coordinate transformations on the rest of the coordinates. Such restricted coordinate transformations then provide the transition functions to build a $TN=0$ $SSRS$ [1]. One can similarly obtain $TN=1$ semirigid coordinate transformations from $N=3$ superconformal coordinate transformations by the same method.

Although this method of deriving SRS and $SSRS$ is adequate for doing physics, there are at least two features that are buried in them. One would like to classify the superconformal or semirigid coordinate transformations as being coordinate transformations which preserve some geometrical object. This object is not obvious using the above patch construction. In addition, to find the superconformal or semirigid moduli space, one would like to have a coordinate invariant definition of SRS or $SSRS$ so that it is clear that deformations of their structure are not artifacts of coordinate transformations. This is of interest when one studies the moduli space of these surfaces, where one's interest is to find deformations which cannot be undone by allowed coordinate transformations.

We will provide such an invariant description in the sequel by means of G-structures. To prove that the patch definition is equivalent to the intrinsic definition (*i.e.* the one using G-structures), we will show that a manifold constructed by the above patching functions implies a G-structure. To invert this correspondence and so establish equivalence we will ask whether *every* G-structure arises by this construction. In general this last step requires that the given G-structure be "integrable," a concept whose meaning we will recall in the following examples. We will find the appropriate integrability conditions in sect. 2.3 and show that they really do lead to an equivalence between the patch and G-structure definitions. While this is not too difficult for $TN = 0$, it does require some work for $TN = 1$, *i.e.* for topological supergravity.

2.2. Two examples

In this subsection, we will illustrate G-structures and the question of their integrability [9][16]. We will also demonstrate how one obtains coordinate transformations which preserve the G-structure chosen. This enables us to relate this definition to the patch definition once integrability is proved.

Suppose we are given a smooth manifold. Then its tangent space can be locally spanned by a field of frames $\{e_a\}$. However, there are in general no global frames. In order to obtain a global structure, we define an equivalence *class* of frames. The equivalence relation is given by a group G of matrices whose elements act on the frames, that is, $\{K_a{}^b e_b\}$ is defined to be equivalent to $\{e_a\}$, where K is a function with values in G. Without any extra structure beyond smoothness, all we can say about the matrices $K_a{}^b$ is that they belong to the group $GL(n, R)$. However, with additional structures, the structure group can be reduced to a subgroup of $GL(n, R)$. The structure group can be thought of as the local symmetry group of a physical theory defined on the manifold. In general, not all manifolds admit a reduction of structure group due to possible global obstructions [9][2]. Also, there are geometrical structures like connections and projective structures that are not G-structures. What we will see in this paper is that SRS and $SSRS$ as defined in sect. 2.1 *do* arise as reductions of the structure group of a supermanifold.

We first consider a smooth manifold with additional structure provided by a metric

$$g = g_{ab} e^a \otimes e^b \quad , \tag{2.4}$$

where e^a is the dual to the frame e_a. Since a metric provides information about the length of a vector, it selects out from the classes of frames $\{e_a\}$ acted on by elements of the group $GL(n, R)$ those that are orthonormal, that is, $g_{ab} = \delta_{ab}$. The structure group that acts on the family of orthonormal frames is the group $O(n)$ leaving δ_{ab} invariant. Thus we have a reduction of structure group from $GL(n, R)$ to $O(n)$ imposed by the additional structure, the metric. Conversely, given a reduction of structure group to $O(n)$, it induces a metric on the manifold: we simply substitute any good frame into (2.4). Like the metric, the imposition of a G-structure on a manifold is an intrinsic concept. Note that the more structures one imposes, the smaller the class of good frames. For example, imposing in addition an orientation lets us restrict further to the class of *oriented* orthonormal frames; these are related by the smaller group $SO(n)$.

For our second example consider the case of a $2n$-dimensional manifold M endowed with an almost complex structure, specified by a tensor J similar to the metric. The tensor is given at a point P by $J_P : T_P M \to T_P M$ everywhere satisfying $J_P^2 = -I$. When diagonalized, J splits the complexified tangent $T_c M$ into holomorphic (with eigenvalue i) and antiholomorphic (with eigenvalue $-i$) tangent spaces. We can use J to define good frames $\{e_a, e_{\bar{a}}\}$ as those for which e_a are $+i$ eigenvectors and $e_{\bar{a}}$ are the complex conjugates of e_a, $a = 1, \ldots, n$. Then

$$J = i(e_a \otimes e^a - e_{\bar{a}} \otimes e^{\bar{a}}) \quad . \tag{2.5}$$

J thus selects out from the class of frames related by $GL(2n, R)$ a smaller class related by $GL(n, C)$, since J is invariant only under $GL(n, C)$ transformation of frames. Conversely, given a reduction of structure group to $GL(n, C)$, which gives us the class of good frames $\{e_a, e_{\bar{a}}\}$, we can obtain J by substituting any good frame in (2.5). Thus an almost complex

[2] We will not consider such obstructions because they are not relevant in establishing the equivalence between the patch and intrinsic definitions.

structure is nothing but a $GL(n, C)$ structure, an equivalence class of frames $\{e_a, e_{\bar{a}}\}$ where any two frames are related by a complex matrix of the form

$$\begin{pmatrix} e'_a \\ e'_{\bar{a}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \begin{pmatrix} e_a \\ e_{\bar{a}} \end{pmatrix} . \tag{2.6}$$

$\bar{A}$ is the complex conjugate of the invertible matrix A.

We have given a coordinate invariant characterization of a G-structure. But sometimes it is convenient to use coordinates. Since a G-structure makes sense even locally, let us first consider the problem of specifying one on an open set U of R^n. For any choice of coordinates $\{x^a\}$ on U we first choose a standard frame given by some universal rule. For example in Riemannian geometry we choose $\hat{e}_a^{\{x\}} = \frac{\partial}{\partial x^a}$. (We will choose a more complicated standard frame in the superconformal and semirigid cases.) If we begin with a different set of coordinates $\{y^a\}$, in general the two frames $\hat{e}_a^{\{x\}}$, $\hat{e}_a^{\{y\}}$ do not agree. However if we arrange for them to agree *modulo* a G-transformation then they do define the same G-structure. This happens when

$$\hat{e}_a^{\{y\}}|_P = K(P)_a{}^b \hat{e}_b^{\{x\}}|_P \tag{2.7}$$

for some function K in G. Since G is a group, the set of all coordinate transformations $y(x)$ defined by (2.7) is a group too; we call it the group of G-coordinate transformations, or simply the "good" transformations.

Thus one way to specify a G-structure on a manifold M is to present an atlas of coordinate charts U_α with coordinates x_α all related on patch overlaps by G-transformations.

Let us illustrate the above discussion with our two examples. In Riemannian geometry the only coordinate transformations preserving the standard frame up to $O(n)$ are the ones preserving the standard metric, *i.e.* the rigid Euclidean motions. For the almost-complex structure example things are more interesting. Given a choice of real coordinates $\{u^a, v^a\}$, $a = 1, \dots, n$ we let $z^a = u^a + \mathrm{i}v^a$ and take the standard frame to be $\hat{e}_a^{\{z\}} = \frac{\partial}{\partial z^a}, \hat{e}_{\bar{a}}^{\{z\}} = \frac{\partial}{\partial \bar{z}^a}$. Let $\{w^a, \bar{w}^a\}$ be another complex local coordinate with standard frame $\{\partial_{w^a}, \partial_{\bar{w}^a}\}$. On the overlap, let w and z be related by a coordinate transformation $w^a = w^a(z^b, \bar{z}^b)$ so that

$$\begin{pmatrix} \partial_{z^a} \\ \partial_{\bar{z}^a} \end{pmatrix} = M \begin{pmatrix} \partial_{w^a} \\ \partial_{\bar{w}^a} \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} \partial_{z^a} w^b & \partial_{z^a} \bar{w}^b \\ \partial_{\bar{z}^a} w^b & \partial_{\bar{z}^a} \bar{w}^b \end{pmatrix} . \tag{2.8}$$

For w and z to be complex coordinates for the same complex structure, we need M to be of the form (2.6). This means that the "good" coordinate transformations preserving the complex structure are *holomorphic* maps.

More generally, a manifold obtained by patching together coordinate charts by a class of G-transformations gets a G-structure. Clearly if we replace each local coordinate x^a_α by $y^a_\alpha = \psi_\alpha(x^b_\alpha)$ where ψ_α is itself a G-transformation, we determine exactly the same G-structure.

We would also like to show the converse: a manifold equipped with a G-structure can *always* be constructed from a set of "good" transition functions. In fact this converse is not always true. To find out when it is so, we introduce coordinate patches on the manifold with the G-structure. We seek coordinates $\{x_\alpha\}$ on a local patch U_α such that the standard frame $\{\hat{e}_a^{\{x_\alpha\}}\}$ determines the given G-structure. Since a G-structure is given by an equivalence class of good frames we are thus seeking a local coordinate whose standard frame belongs to the same equivalence class as the given $\{e_a\}$. If we can find such a coordinate system, we then call the G-structure *integrable*. However, this is in general not possible unless the frames belonging to the G-structure satisfy certain constraints. After all, $\{x^a\}$ contains only $n = \dim M$ degrees of freedom, while the given $\{e_a = e^\mu_a \partial_\mu\}$ has n^2 minus the dimension of G. This counting also makes it clear that different G-structures impose different integrability constraints. For instance, we will see that the superconformal structure does not need any such conditions while the semirigid case needs some first order constraints. Of course there is more to do than just count conditions. The statement that a set of local constraints on a G-structure really does suffice to find local coordinates inducing that structure is called an *integrability theorem*.

Let us illustrate these ideas in the two examples given above. For the case of Riemannian geometry, $G = O(n)$, it turns out that a G-structure is integrable iff its Riemann curvature tensor R vanishes (see for example [17]). That is, if $R \equiv 0$ in the neighborhood of a point, then there exist local coordinates (called inertial) such that the metric is in the standard form $g = \delta_{ab} \mathrm{d}x^a \otimes \mathrm{d}x^b$. Comparing this metric with the one specified by the given $O(n)$-structure $g = \delta_{ab} e^a \otimes e^b$, we see that the frames are related by $e_a = K_a{}^b \frac{\partial}{\partial x^b}$, where $K \in O(n)$. Thus the frame defining the G-structure e_a is G-equivalent to the standard frame of some coordinates, which is what we called integrability earlier. Notice that the integrability condition is given by constraining the curvature, a function involving up to second order derivatives of the original frame. We thus call this a second order constraint. The condition $R = 0$ implies flat space; thus integrability conditions are sometimes called flatness conditions, even though they may be given by first order constraints in other cases.

Instead of Riemannian geometry we can enlarge $O(n)$ somewhat to the group of matrices with $K^{\mathrm{t}} g K \propto g$ — the conformal group. The obstruction to flatness is now just a part of the Riemann curvature, namely the Weyl tensor [18]. An important case is two dimensions, where there is no Weyl tensor at all and *every* conformal (or $C^{\times}$)-structure is integrable.

In the case of an almost complex structure, the counterpart of the curvature is the Nijenhuis tensor [19], given in terms of J by

$$\mathcal{N}(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]. \tag{2.9}$$

where X and Y are arbitrary vector fields. The integrability theorem [20] says if $\mathcal{N} \equiv 0$, then there exists a local complex coordinate system $\{z^a\}$, $i = 1, \ldots, n$ such that J is of the form (2.5) with the frames given by $\hat{e}_a^{\{z\}} = \frac{\partial}{\partial z^a}, \hat{e}_{\bar{a}}^{\{z\}} = \frac{\partial}{\partial \bar{z}^a}$. Thus $\mathcal{N} = 0$ becomes the flatness condition. It is however a first order condition unlike the $O(n)$ case, since (2.9) clearly involves at most first derivatives of J. As mentioned above, the "good" coordinate transformations (those preserving J) are the holomorphic maps.

Given an *integrable* G-structure on M, we can now return to the question of whether it can be constructed via patching maps. On each coordinate patch choose a coordinate inducing the given G-structure. Then on patch overlaps the chosen coordinates are related by what we have called a "good" or G-transformation: $x_\beta = \phi_{\alpha\beta}(x_\alpha)$. Hence we can construct M with its G-structure from patching coordinate charts with the "good" coordinate transformations. Of course on each patch we have some freedom to redefine the good coordinate x_α^a by some G-transformation $y_\alpha^a = \psi_\alpha(x_\alpha^b)$. This simply corresponds to replacing the $\{\phi_{\alpha\beta}\}$ by the equivalent family $\{\psi_\alpha \circ \psi_{\alpha\beta} \circ \psi_\beta^{-1}\}$ as discussed above.

To summarize, given G and a choice of standard frames we may define a G-manifold as a collection of patching G-transformations modulo the substitution $\{\phi_{\alpha\beta}\} \mapsto \{\psi_\alpha \circ \phi_{\alpha\beta} \circ \psi_\beta^{-1}\}$, where ψ_α are themselves G-transformations. Or we may define a G-manifold as a smooth manifold with a collection of frames defined modulo G satisfying appropriate integrability conditions. We have seen that *these two definitions are equivalent once the appropriate integrability theorem is established.*

For the case of specifying the $N \geq 1$ superconformal structure, a coordinate invariant tensor analogous to the metric g or the tensor J is not known. However, one can still choose a group G and specify a G-structure by giving a frame defined up to transformations by elements of G. Without the analog of g or J, we cannot define a tensor like R or $\mathcal{N}$ measuring the local obstruction to integrability. Thus, one has to find another way to give the flatness condition for the case of superconformal structures or else prove that there is no such condition, that is, all G-structures are flat. The situation is similar for semirigid structures.

Let us once again use the case of an almost complex structure on a $2n$ dimensional real manifold to clarify how first-order flatness conditions can come about. The flatness condition $\mathcal{N}=0$ can be replaced by a condition similar to the one used in the Frobenius integrability theorem, namely ${t_{ab}}^{\bar{c}}=0$, where

$$[e_A, e_B] = {t_{AB}}^C e_C \tag{2.10}$$

and A denotes either $a, \bar{a}$. In other words, the Lie bracket of the holomorphic tangent frames stays in the same subspace. Conditions of this type are sometimes called "essential torsion constraints" [6].

We now recall a general prescription [6] to obtain the torsion constraints with the above example in mind and see that they are necessary conditions for integrability. In our examples the structure constants ${\hat{t}_{ab}}{}^{c}$ all vanish when we use the standard frame $\{\hat{e}_a^{\{x\}}\}$ in $[\hat{e}_a^{\{x\}}, \hat{e}_b^{\{x\}}] = {\hat{t}_{ab}}{}^c \hat{e}_c^{\{x\}}$. (More generally they will at least all be *constants* in the cases of interest.) Of course the same may not be true when we substitute some other equivalent frame $\{e_a\}$ to get ${t_{ab}}^c$. We obtain an arbitrary representative of the standard G-structure by letting an arbitrary function in G act on the standard frame. Those ${t_{ab}}^c$ that remain equal to ${\hat{t}_{ab}}{}^c$ clearly have the same values in *any* good frame. Thus we have found some conditions on ${t_{ab}}^c$ which follow from the assumption that our frame is equivalent to some standard frame. These conditions may be overcomplete; for example some may be related to others by Jacobi identities.

In other words given a frame we have found some conditions which must be met if the corresponding G-structure is to be integrable. These "torsion constraints" are first order conditions on any frame representing the given structure since the Lie bracket entering t contains one derivative. If we find that they are also *sufficient* for flatness, then we have an integrability theorem with *only* first order constraints. This is the case for $G = GL(n, C)$ since here the torsion constraints amount to the vanishing of the Nijenhuis tensor; it will also be true for superconformal and semirigid geometry. (And as we have mentioned, for superconformal geometry there will be no essential torsion constraints at all.) However as we have seen it is false for Riemannian geometry. It is sometimes convenient to impose further G-invariant "inessential" torsion constraints corresponding to normalization conditions [6], as we will recall below.

We will now apply all these ideas to the cases of N superconformal and TN semirigid structure.

2.3. Intrinsic Definitions of SRS and SSRS

We now provide an intrinsic definition of N superconformal structures [12] generalizing [11][21]. Below we will propose a similar intrinsic definition of semirigid structures. Let $\hat{M}$ be a complex supermanifold of dimension $1|N$ equipped with a holomorphic distribution (subbundle of TM) $\mathcal{E}$ of dimension $0|N$. Given $(\hat{M},\mathcal{E})$, one can always define a symmetric bilinear $B : \mathcal{E}\otimes\mathcal{E}\to\mathcal{T}/\mathcal{E}$, where $\mathcal{T}$ is the holomorphic tangent bundle. The bilinear is given by $B(E_i,E_j)\ \equiv\ [E_i,E_j] \bmod \mathcal{E}$, where $[\ ,\ ]$ is the graded Lie bracket and $E_i\in\mathcal{E}$. Following [11][21][12], we will call $(\hat{M},\mathcal{E})$ an N-*SRS* if B is non-degenerate.

A *SRS* can also be regarded as a reduction of the structure group on $\hat{M}$. We simply declare a frame $\{E_0,\vec{E}\,\}$ as "good" if E_0 is even and E_i are an (odd) frame for the given $\mathcal{E}$. Then all good frames are related to one another by elements of a supergroup as follows:

$$\begin{pmatrix} E_0' \\ \vec{E}' \end{pmatrix} = \left(\begin{array}{c|c} a^2 & \vec{\omega} \\ \hline \vec{0} & a\overleftrightarrow{M} \end{array}\right) \begin{pmatrix} E_0 \\ \vec{E} \end{pmatrix} \quad , \tag{2.11}$$

where a is an invertible even function, $\vec{\omega}$ are odd functions, and $\overleftrightarrow{M}$ is an invertible matrix of even functions. In order for the set of frames $\{E_i'\}$ to span the same distribution $\mathcal{E}$ as $\{E_i\}$, we have required the column $\vec{0}$.

We can always put a *SRS* in a more canonical form. The non-degeneracy condition above implies that the bilinear B is diagonalizable. Thus we can always use a transformation of the form (2.11) to get from a frame $\{E_0,\vec{E}\,\}$ to a normalized frame with

$$[E_i,E_j] = 2g_{ij}E_0 \mod \mathcal{E} \tag{2.12}$$

where $g_{ij}=\delta_{ij}$. Such normalized good frames are then all related by a smaller group than (2.11), in which M is in the orthogonal group $O(N,C)$. This residual group we will call G^N, and we will call a G^N-structure an almost superconformal structure. Since we can always pass to normalized frames, and the new frame is unique modulo the residual group, we find that *an N-SRS in the above sense is precisely a reduction of the structure group of $\hat{M}$ to G^N*. We will prove in sect. 3 that this reduction $N\le 3$ is always integrable.

We would like to point out that E_+ and E_-(in a complex basis) in the $N=2$ case are preserved up to a multiplicative factor on a *SRS* because in this basis matrices in $O(2,C)$ are diagonal. Hence the distribution $\mathcal{E}$ is split into two line bundles. This is not true for $N\ge 3$, a fact related to the existence of a nonabelian current algebra in the superconformal algebra starting at $N=3$.

What are the “good” coordinate transformations for this superconformal structure? To answer this, and to make precise what we wish to prove in the integrability theorem, we must specify the standard frames associated to a coordinate patch. We choose $\hat{E}_0^{\{\mathbf{z}\}} = \frac{\partial}{\partial z}$, $\hat{E}_i^{\{\mathbf{z}\}} = D_i$ where $\mathbf{z} \equiv (z, \vec{\theta})$ and D_i are defined in (2.1). We can then identify the N-superconformal coordinate transformations as those complex coordinate transformations that leave this structure unchanged along the lines similar to the discussion below (2.8). Then the “good” coordinate transformations preserving the standard G structure will take $\mathbf{z}$ to $\widetilde{\mathbf{z}}$ with

$$\begin{pmatrix} \partial_{\tilde{z}} \\ \vec{\tilde{D}} \end{pmatrix} = \left(\begin{array}{c|c} a^2 & \vec{\omega} \\ \hline \vec{0} & a\overset{\leftrightarrow}{M} \end{array}\right) \begin{pmatrix} \partial_z \\ \vec{D} \end{pmatrix} \quad . \tag{2.13}$$

The set of coordinate transformations in the form of (2.13) are given by the N-superconformal transformations defined by (2.2)–(2.3). As in the general analysis above, this leads to a patch definition of super Riemann surfaces. Once the integrability theorem is proved in sect. 3 we thus have that every N-SRS in the above sense is also a SRS in the sense of sect. 2.1.

Next we turn to the semirigid case. An almost TN-structure is obtained by reduction of the structure group from an $(N+2)$-superconformal structure. Consider the set of frames spanning $\mathcal{E}$, $\{E_i\} = \{E_+, E_r, E_-\}$ where $E_\pm = \frac{1}{\sqrt{2}}(E_1 \pm iE_2)$ and $r = 3, \ldots, N+2$. The metric g_{ij} in this frame is

$$g_{ij} = \begin{pmatrix} 0 & \vec{0} & 1 \\ \vec{0} & \overset{\leftrightarrow}{\widetilde{g}_{rs}} & \vec{0} \\ 1 & \vec{0} & 0 \end{pmatrix} \quad , \tag{2.14}$$

where $\widetilde{g}_{rs} = \delta_{rs}$. The reduction from (2.11) is specified by the G-structure where now the group consists of matrices of the form

$$K = \begin{pmatrix} a^2 & \omega_- & \vec{\omega} & \omega_+ \\ 0 & 1 & \vec{Y} & -\frac{1}{2}Y\widetilde{g}Y^t \\ \vec{0} & \vec{0} & a\overset{\leftrightarrow}{M} & -aM\widetilde{g}Y^t \\ 0 & 0 & \vec{0} & a^2 \end{pmatrix} \quad . \tag{2.15}$$

Here $\vec{Y}$, $\overset{\leftrightarrow}{M}$ have even elements, $M\widetilde{g}M^t = \widetilde{g}$, a is invertible and the ω are odd functions. It can be verified that matrices of type (2.15) form a supergroup G^{TN}, which is a subgroup of G^{N+2}. In fact this structure group arises by a reduction from G^{N+2} by imposing extra structure: we have chosen a $1d$ subbundle $\mathcal{D}_- \subset \mathcal{E} \subset \mathcal{T}$. $\mathcal{D}_-$ is not trivial; indeed we also choose a (parity-reversing) isomorphism $\mathcal{D}_- \cong \mathcal{T}/\mathcal{E}$. The good frames are those good superconformal frames for which E_- spans $\mathcal{D}_-$ and corresponds to E_0 mod $\mathcal{E}$ under the chosen isomorphism. These frames are then all related by (2.15).

The motivation for this construction is simple for $TN = 0$. Any kind of topological field theory should have a superspace formulation involving a global, spinless odd coordinate for bookkeeping. For us this coordinate will be θ^+. For $TN = 0$ (2.15) says that "good" coordinate transformations take D_+ to itself, and hence they also take θ^+ to itself as desired. For $N > 0$ this may not be so clear, but in fact (2.15) again ensures that the "good" TN-coordinate transformations are just N-superconformal transformations which keep θ^+ fixed [2]. Note that the N-superconformal structure group is embedded in that of TN semirigid geometry, $G^N \subset G^{TN} \subset G^{N+2}$ by comparing with (2.11). This is seen by setting $\vec{Y} = \omega_+ = \omega_- = 0$. This is why the TN-coordinate transformations include the N-superconformal group and give rise to topological supergravity.

In sect. 4 we will find first order constraints which are sufficient flatness conditions for the existence of a coordinate system with the standard frames G^{TN}-equivalent to the frames E_a defining the semirigid structure. Hence as in our general discussion a complex supermanifold with an integrable G^{TN}-structure is glued together by semirigid transition functions, which recovers the patch definition of semirigid surfaces given in sect. 2.1.

3. Superconformal Integrability

In sect. 2.3 we defined an almost superconformal structure. We shall prove that this reduction is always integrable for $N = 3$; there are no flatness conditions to impose in this case. The cases $N < 3$ are much easier and can easily be obtained from our derivation. We expect $N = 4$ to be similar.

We are given a distribution $\mathcal{E}$ which satisfies the non-degeneracy condition. As we have discussed above we can always choose a frame $\{E_0, \vec{E}\}$ with $\vec{E}$ spanning $\mathcal{E}$ and satisfying (2.12), or in the notation of (2.10)

$$t_{ij}{}^0 = 2g_{ij} \equiv 2\delta_{ij} \quad , \tag{3.1}$$

and any two such frames are related by (2.11) with the matrix M orthogonal. Indeed (2.11) shows that we have a lot of freedom with E_0; modifying it by adding any linear combination of the $\vec{E}$ does not change the superconformal structure. Given a normalized frame we can thus discard E_0 and focus on $\vec{E}$, regenerating E_0 when needed by $\frac{1}{2}[E_+, E_-]$ or some other convenient variant.

Recall that a complex structure has been given on the manifold and that $\{E_0, \vec{E}\}$ are holomorphic. Hence in an arbitrary complex coordinate system with coordinates given by w and λ^i, we can represent $\{E_i\}$ by

$$E_i = M_i{}^j \partial_j + \alpha_i \partial_w \quad , \tag{3.2}$$

where $\partial_i \equiv \frac{\partial}{\partial \lambda^i}$, $\partial_w \equiv \frac{\partial}{\partial w}$ and $M_i{}^j$ and α_i are holomorphic functions of w and λ^i.

We would like to show that we can find a coordinate system in which $\{E_i\}$ is G^N-equivalent to the standard frame $\{D_i\}$. We shall proceed in four steps, order by order in the odd coordinate λ.

Step 1: We shall first find a coordinate system in which

$$E_i = \partial_i + \mathcal{O}(\lambda) \quad . \tag{3.3}$$

Let $M_i{}^j = m_i{}^j + \mathcal{O}(\lambda)$ and $\alpha_i = \alpha_{i0} + \mathcal{O}(\lambda)$ in (3.2). We make the following complex coordinate transformation:

$$\widetilde{\lambda}^i = \lambda^j [m^{-1}]_j{}^i \quad ; \quad \widetilde{w} = w \quad . \tag{3.4}$$

Under coordinate transformation (3.4), we obtain that

$$E_i = \widetilde{\partial}_i + \alpha_{i0} \widetilde{\partial}_w + \mathcal{O}(\widetilde{\lambda}) \quad .$$

We can now drop the tildes for convenience. We make another complex coordinate transformation

$$\widetilde{\lambda}^i = \lambda^i \quad ; \quad \widetilde{w} = w + \lambda^r \beta_r \quad , \tag{3.5}$$

and obtain

$$E_i = \widetilde{\partial}_i + (\beta_i + \alpha_{i0}) \widetilde{\partial}_w + \mathcal{O}(\lambda) \quad .$$

Choosing $\beta_i = -\alpha_{i0}$, we obtain (after dropping the tildes again) (3.3).

Step 2: Restoring λ terms in E_i, we have

$$E_i = \{\delta_i{}^k + \lambda^r \mu_{ri}{}^k\} \partial_k + \lambda^r a_{ri} \partial_w + \mathcal{O}(\lambda^2) \quad , \tag{3.6}$$

where we have introduced two functions ${\mu_{ri}}^k$ and a_{ri}. The normalization conditions (3.1) are easily seen to imply that

$$a_{(ij)} = \delta_{ij} a_0$$

where a_0 is some invertible function. The antisymmetric part of a_{ij} can now be removed by a coordinate transformation of the form

$$\widetilde{w} = w + \tfrac{1}{2}\lambda^s\lambda^t b_{st} \quad ,$$

while the trace bit can be set to one by a further transformation of the form $\widetilde{w} = \widetilde{w}(w)$ with $\frac{\partial\widetilde{w}}{\partial w} = a_0^{-1}$. We will now use our freedoms to put ${\mu_{ri}}^k$ into more canonical form.

Again we perform coordinate transformations. Let

$$\widetilde{\lambda}^i = \lambda^i + \tfrac{1}{2!}\lambda^r\lambda^s{\rho_{sr}}^i \quad ; \quad \widetilde{w} = w \quad , \tag{3.7}$$

where ${\rho_{sr}}^i = {\rho_{[sr]}}^i$. In this coordinate system

$$E_i = \{{\delta_i}^k + \widetilde{\lambda}^r({\rho_{ri}}^k + {\mu_{ri}}^k)\}\widetilde{\partial}_k + \widetilde{\lambda}_i\partial_{\widetilde{w}} + \mathcal{O}(\widetilde{\lambda}^2) \quad . \tag{3.8}$$

We can also consider the G^N-equivalent frame $E_i' = (KE)_i$, where

$${K_i}^j = {\delta_i}^j + \lambda^r({\alpha_{ri}}^j + \xi_r{\delta_i}^j) + \mathcal{O}(\lambda^2) \quad . \tag{3.9}$$

Here $\alpha_{rij} = \alpha_{r[ij]}$ is a generator of $SO(3,C)$. Together with (3.7) we see that we can shift μ by

$$\mu \to \mu_{rik} + \rho_{rik} + \alpha_{rik} + \xi_r\delta_{ik} \quad .$$

To begin simplifying this we see we may without loss of generality use ρ to get $\mu = \mu_{(ri)k}$, symmetric on the first two indices. A little algebra then shows that with an appropriate choice of further ρ, α transformations we may take $\mu = \mu_{(rik)}$, and moreover using ξ we can get ${\mu_{ij}}^i = 0$.

Step 3: Thus we have

$$E_i = D_i + \lambda^r{\mu_{ri}}^k\partial_k + \tfrac{1}{2}\lambda^s\lambda^t\epsilon_{tsu}({M_{ui}}^k\partial_k + \Theta_{ui}\partial_w) + \mathcal{O}(\lambda^3) \quad ,$$

where again $\mu = \mu_{(rik)}$ and we have introduced the next order, coefficients ${M_{ui}}^k$ and Θ_{ui}.

Using our freedom to choose a convenient E_0 we now take

$$E_0 = \tfrac{1}{6}\sum_i [E_i, E_i] + F^\ell E_\ell \quad , \tag{3.10}$$

where F^ℓ is some function of w, λ of order λ. Imposing (3.1) to $\mathcal{O}(\lambda)$ now shows that $\mu \equiv 0$ and $\Theta_{ui} \propto \delta_{ui}$. But this means that we may remove Θ altogether by the coordinate transformation $\widetilde{w} = w + \lambda^3 \beta$, where

$$\lambda^3 \equiv \tfrac{1}{6} \lambda^s \lambda^t \lambda^u \epsilon_{uts} \quad .$$

Step 4: Thus we have

$$E_i = D_i + \tfrac{1}{2} \lambda^s \lambda^t \epsilon_{tsu} M_{ui}{}^k \partial_k + \lambda^3 (s_i \partial_w + \sigma_i{}^\ell \partial_\ell)$$

where $s_i, \sigma_i{}^\ell$ are new sets of coefficients. There remain the freedom to make coordinate transformations of the form $\widetilde{\lambda}^i = \lambda^i + \lambda^3 K^i$ as well as $SO(3,C) \times C^\times$ frame rotations. One readily sees that this freedom suffices to make σ traceless symmetric, $M = M_{u(ik)}$, $s_i \equiv 0$, and $M_{ik}{}^i = 0$.

We now make a convenient choice of F^ℓ in (3.10):

$$E_0 = \tfrac{1}{6} \sum_i [E_i, E_i] - \tfrac{1}{6} (2\lambda^s \epsilon_{siu} M_{ui}{}^\ell + \lambda^s \lambda^t \epsilon_{tsi} \sigma^{i\ell}) E_\ell \quad . \tag{3.11}$$

Then the condition (3.1) says $\sigma \equiv 0$, $M \equiv 0$. Thus we have

$$E_i = D_i$$

as was to be shown.

We close this section by remarking that superconformal integrability should be related to the conformal flatness of an appropriate supergravity theory. Indeed $N = 3, 4$ supergravity theories have been constructed using conformal flatness as a principle [22]. Perhaps the rather simple idea of superconformal geometry can shed some light on the structure of these theories.

4. Semirigid Integrability

4.1. TN=0 Integrability

We now investigate the local integrability of semirigid structures. To begin suppose we have been given a $TN = 0$ (or "almost semirigid") structure specified by a frame

$\{E_0, E_+, E_-\}$ obeying (3.1). This is the same information as in the superconformal case, but now we do not consider two frames equivalent unless they are related by (2.15), *i.e.*

$$\begin{pmatrix} E_0' \\ E_+' \\ E_-' \end{pmatrix} = \begin{pmatrix} c & \cdots & \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} E_0 \\ E_+ \\ E_- \end{pmatrix} \quad . \tag{4.1}$$

Thus to integrate the frame we have a harder job than in sect. 3: find local coordinates such that the standard frame equals the given one modulo $G^{TN=0} \subset G^{N=2}$ (4.1), not just modulo $G^{N=2}$ (2.11).

We can again simplify the problem somewhat by noticing that E_0 will take care of itself once we put the $\vec{E}$ into the desired form. Accordingly we take $E_0 = \frac{1}{2}[E_+, E_-]$, since this choice is still normalized correctly and is related to the given one by a $G^{TN=0}$ transformation (4.1).

Following the procedure in sect. 2.2, we look for torsion constraints by taking a standard frame and applying an arbitrary transformation of the form (4.1): $E_+ = D_+$, $E_- = cD_-$ where c is a function. By the remark in the previous paragraph we then take $E_0 = \frac{1}{2}[E_+, E_-]$. Computing $t_{AB}{}^C$ we find that in addition to (3.1), preserved since we have maintained the normalization condition, we also preserve various other elements of t, including in particular

$$t_{++}{}^{+} = 0 \quad . \tag{4.2}$$

Thus (4.2) is necessary for a frame to be integrable.

Now suppose our given frame does satisfy (4.2). By the result of the previous section we can at least find superconformal coordinates, *i.e.* coordinates $\mathbf{z} = (z, \theta^{\pm})$ such that $\{E_0, \vec{E}\}$ is $G^{N=2}$-equivalent to $\{\partial_z, \vec{D}\}$. In particular

$$D_i = N_i{}^j E_j \qquad N = \begin{pmatrix} ab^{-1} & 0 \\ 0 & ab \end{pmatrix} \in SO(2, C) \tag{4.3}$$

for some invertible functions a, b. We would now like to find another set of coordinates $\widetilde{\mathbf{z}}(\mathbf{z})$ with

$$\widetilde{D}_i = (K^{-1})_i{}^j E_j \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \quad . \tag{4.4}$$

Since $\mathbf{z}$ is not a good semirigid coordinate, $\widetilde{\mathbf{z}}(\mathbf{z})$ is not a semirigid transformation. However (4.3)–(4.4) say $\mathbf{z}$ and $\widetilde{\mathbf{z}}$ are at least superconformal coordinates and so $\widetilde{\mathbf{z}}(\mathbf{z})$ will be a superconformal transformation. But we know how $\vec{D}$ transform under the latter (eqn. (2.2)).

Putting it all together, given a, b in (4.3) we need to choose $\widetilde{\mathbf{z}}$ and c in (4.4) such that

$$\begin{pmatrix} 1 & \\ & c^{-1} \end{pmatrix} \begin{pmatrix} b/a & \\ & (ab)^{-1} \end{pmatrix} \begin{pmatrix} D_+ \\ D_- \end{pmatrix} = \begin{pmatrix} D_+\widetilde{\theta}^+ & \\ & D_-\widetilde{\theta}^- \end{pmatrix}^{-1} \begin{pmatrix} D_+ \\ D_- \end{pmatrix} \quad . \tag{4.5}$$

In other words, while we can always adjust c to satisfy the second equation, we do need to find a superconformal transformation for which $D_+\widetilde{\theta}^+ = a/b$. As expected we see that in general there is no solution. Imposing (4.2), however, tells that $D_+(a/b) = 0$, which ensures that an appropriate function $\widetilde{\theta}^+$ exists. To see that there is a $\widetilde{\mathbf{z}}$ with this $\widetilde{\theta}^+$, we need to inspect the most general $N = 2$ superconformal coordinate transformation:

$$\begin{aligned} \widetilde{z} &= f + \theta^+ t\psi + \theta^- s\tau + \theta^+\theta^-\partial_z(\tau\psi) \\ \widetilde{\theta}^+ &= \tau + \theta^+ t + \theta^+\theta^-\partial_z\tau \\ \widetilde{\theta}^- &= \psi + \theta^- s - \theta^+\theta^-\partial_z\psi \quad , \end{aligned} \tag{4.6}$$

where $\partial_z f = ts - \tau\partial_z\psi - \psi\partial_z\tau$. Thus, we have

$$D_+\widetilde{\theta}^+ = t + 2\theta^-\partial_z\tau - \theta^+\theta^-\partial_z t \tag{4.7}$$

and we can choose t, τ to match this to any chiral superfield a/b.

4.2. TN=1 Integrability

In this subsection, we start with an $N = 3$ *SRS* endowed with the $TN = 1$ structure given by a frame $\{E_0, E_\pm, E_3\}$ normalized per (3.1), (2.14). As in the previous subsection we may discard E_0 and replace it by $E_0 = \frac{1}{2}[E_3, E_3]$ without changing the semirigid structure.

Proceeding as before we get the torsion constraints by acting on the standard frame $\vec{D}$ with[3]

$$K = \begin{pmatrix} 1 & x & -\frac{x^2}{2} \\ 0 & a & -ax \\ 0 & 0 & a^2 \end{pmatrix} \tag{4.8}$$

where a is invertible. Examining the commutators of $E_i = K_i{}^j D_j$ we find that in addition to (3.1) we have (among other things)

$$t_{ij}{}^+ = 0 \;, \quad t_{--}{}^3 = 0 \quad . \tag{4.9}$$

[3] Recall that in this basis the metric g_{ij} is antidiagonal.

We will show that these necessary conditions are sufficient for integrability.

Suppose then that we have a local frame $\vec{E}$ for $\mathcal{E}$ obeying (3.1) and (4.9) once we set $E_0 = \frac{1}{2}[E_3, E_3]$. Once again we can use the result in sect. 3 to choose superconformal coordinates, so that $E_i = (N^{-1})_i{}^j D_j$ where N belongs to $SO(3,C) \times C^{\times}$. Semirigid integrability means that there exists a superconformal coordinate transformation $\widetilde{\mathbf{z}}(\mathbf{z})$ such that the given frame E_i is $G^{TN=1}$-equivalent to the standard frame $\widetilde{D}_i$:

$$(K^{-1}E)_i = \widetilde{D}_i$$

with K some matrix function of the form (4.8). Analogous to (4.5) this requires us to solve

$$F = NK \quad \text{where} \quad F_j{}^k = D_j \widetilde{\theta}^k \quad . \tag{4.10}$$

In this equation we are seeking suitable $\widetilde{\mathbf{z}}(\mathbf{z})$ and K given N. Once again this is in general impossible until we impose the constraints (3.1), (4.9) on N.

We will subdivide our task by writing $K = K_1 K_2$ and choosing K_1 to put $\hat{N} = NK_1$ into the form of a lower triangular matrix L (when $N_+{}^+$ is invertible) or an "upper" triangular matrix U (when $N_+{}^+$ is not invertible; see below) with unit determinant. This puts our problem (4.10) into standard form: $F = LK_2$ or $= UK_2$. We will in the following concentrate on the case when $N_+{}^+$ is invertible, prove the integrability theorem and then comment on the other case.

We organize the proof into four steps. First, we will show that NK_1 can be put into the form of L. Then we impose the semirigid essential torsion constraints on L (recall the constraints are $G^{TN=1}$-invariant). With this done, we will substitute L into (4.10), and solve for the $\widetilde{\theta}^+$ component of the superconformal coordinate transformation in terms of the unconstrained superfield components of L just as in sect. 4.1. The rest of the components, $\widetilde{\theta}^3$ and $\widetilde{\theta}^-$, can always be made to satisfy (4.10) by choosing K_2 appropriately. Finally, we will show by construction that there really does exist a superconformal coordinate transformation with the required $\widetilde{\theta}^+$.

The torsion constraint (3.1) implies that N belongs to $SO(3,C) \times C^{\times}$, meaning $NgN^t \propto g$. In particular we have

$$N_+{}^- = -\frac{(N_+{}^3)^2}{2N_+{}^+} \quad \textit{and} \quad N_3{}^- = \frac{N_3{}^+}{2}\left(\frac{N_+{}^3}{N_+{}^+}\right)^2 - \frac{N_+{}^3 N_3{}^3}{N_+{}^+} \quad . \tag{4.11}$$

${(NK_1)_+}^3$ is set to zero by choosing K_1 in (4.8) with

$$x_1 = -a_1 \frac{{N_+}^3}{{N_+}^+}. \tag{4.12}$$

Substituting (4.11) and (4.12) into the expressions ${(NK_1)_+}^-$ and ${(NK_1)_3}^-$, they too vanish. Furthermore, a_1 is chosen so that NK_1 has unit determinant, that is, $a_1 = (\det N)^{-\frac{1}{3}}$. Thus, NK_1 by construction is a lower triangular matrix given by

$$L = \begin{pmatrix} b & 0 & 0 \\ y & 1 & 0 \\ -\frac{y^2}{2b} & -\frac{y}{b} & \frac{1}{b} \end{pmatrix} \quad . \tag{4.13}$$

We can now let $\widetilde{E}_i = (K_1^{-1}E)_i = (L^{-1}D)_i$. While we have used our $G^{TN=1}$ freedom to put L into the standard form (4.13), still further restrictions come when we impose the torsion constraints (4.9). Since these torsion constraints are by construction $G^{TN=1}$-invariant, we can impose them on $\widetilde{E}_i$. The constraints give respectively

$$({L^{-1})_-}^k \, D_k ({L^{-1})_-}^l {L_l}^3 = 0 \quad \text{and} \tag{4.14}$$

$$\{ {(L^{-1})_{(i}}^k \, D_k {(L^{-1})_{j)}}^l - g_{ij} {(L^{-1})_3}^k \, D_k {(L^{-1})_3}^l \} \, {L_l}^+ = 0. \tag{4.15}$$

Substituting (4.13) into (4.14) and (4.15), we obtain the following four constraints on the two independent matrix elements b and y of the matrix L:

$$D_+ b = 0, \tag{4.16}$$

$$D_+ y + D_3 b = 0, \tag{4.17}$$

$$bD_- b - 2bD_3 y - yD_+ y = 0, \quad \text{and} \tag{4.18}$$

$$\frac{y^2}{2} D_+ y + byD_3 y - b^2 D_- y = 0. \tag{4.19}$$

In appendix A, we show that under this set of torsion constraints we obtain a unique odd superfield Ω satisfying

$$b = D_+ \Omega \ , \quad y = D_3 \Omega \ , \quad \text{and } g^{ij}(D_i \Omega)(D_j \Omega) = 0 \quad . \tag{4.20}$$

We are now ready to show that there exists a superconformal coordinate transformation and suitable K_2 for which $F = LK_2$. That is,

$$\begin{pmatrix} D_+ \widetilde{\theta}^+ & D_+ \widetilde{\theta}^3 & D_+ \widetilde{\theta}^- \\ D_3 \widetilde{\theta}^+ & D_3 \widetilde{\theta}^3 & D_3 \widetilde{\theta}^- \\ D_- \widetilde{\theta}^+ & D_- \widetilde{\theta}^3 & D_- \widetilde{\theta}^- \end{pmatrix} = \begin{pmatrix} b & bx_2 & \frac{-bx_2^2}{2} \\ y & x_2 y + a_2 & -x_2(a_2 + \frac{x_2 y}{2}) \\ -\frac{y^2}{2b} & -\frac{y}{b}(a_2 + \frac{x_2 y}{2}) & \frac{1}{b}(a_2 + \frac{x_2 y}{2}) \end{pmatrix}, \tag{4.21}$$

where a_2 and x_2 are the independent elements of K_2. Taking the determinant of both sides of (4.21), we see that we have to choose

$$a_2 = \det(F)^{\frac{1}{3}}. \tag{4.22}$$

As for x_2, we will choose it so that $bx_2 = D_+\widetilde{\theta}^3$. Eqn. (4.20) then shows that the first column of equations (4.21) are satisfied when we identify $\widetilde{\theta}^+$ as Ω. One can show that the remaining five components of the matrix equation (4.21) are then satisfied by the use of the superconformal conditions (2.3). These turn into two sets of readily applicable relations

$$FgF^t = g(\det F)^{\frac{2}{3}} \tag{4.23}$$

and the set of equations where we replace F by F^{-1}, since F^{-1} is also a superconformal transformation.

Finally, the question is if there exists an $N = 3$ superconformal coordinate transformation with $\widetilde{\theta}^+$ given by the function Ω. The answer is yes; details are given in appendix B. The point is that from the superconformal conditions $\widetilde{z}$ can be expressed in terms of the components of the transformation of $\widetilde{\theta}^i$, $i = +, 3, -$. The only requirement left for the coordinate transformation to be superconformal is that the $\widetilde{\theta}^i$ satisfy the superconformal conditions among themselves. In appendix B, we have expanded $\widetilde{z}$ and $\widetilde{\theta}^i$ in components. There are four even and four odd components in each of the superfields. We set out with $\widetilde{\theta}^+$ given, namely Ω, and there are sixteen degrees of freedom in the components of $\widetilde{\theta}^3$ and $\widetilde{\theta}^-$ to choose to satisfy the internal superconformal conditions. The superconformal conditions among $\widetilde{\theta}^i$ are linear in the components of $\widetilde{\theta}^3$ and $\widetilde{\theta}^-$ and there are sixteen such equations. We have shown in the appendix that indeed a solution exists. If all the even components of $\widetilde{\theta}^+$ are invertible, then we use all sixteen degrees of freedom to solve the sixteen equations. If one or more even components of $\widetilde{\theta}^+$ are noninvertible, then the linear matrix equations become singular and it implies that there are more variables than equations. Thus, there exists a family of solutions.

When ${N_+}^+$ is not invertible, from the fact that N belongs to $SO(3,C) \times C^\times$, we immediately obtain that ${N_-}^+$, ${N_+}^-$, and ${N_3}^3$ are invertible. Since ${N_-}^+$ is invertible, we can choose elements in K_1 so that (NK_1) takes the form

$$U = NK_1 = \begin{pmatrix} -\frac{y^2}{2b} & -\frac{y}{b} & \frac{1}{b} \\ y & 1 & 0 \\ b & 0 & 0 \end{pmatrix}. \tag{4.24}$$

All the essential torsion constraints are the same as before with the roles of D_+ and D_- interchanged. We again wish to find a superconformal coordinate transformation F so that (4.10) is satisfied. We then have $b = D_-\widetilde{\theta}^+$ and $y = D_3\widetilde{\theta}^+$. The rest of the proof is analogous to the previous case with the roles of the superfield components switched between the untilded $+$ and $-$ components and a sign change for the tilde components along with interchanging the $+$ and $-$ components (*e.g.* $s_- \to s_+$ and $\widetilde{\psi}_- \to -\widetilde{\psi}_+$).

5. Moduli space of semirigid surfaces

There exists a natural projection from the moduli space of TN-semirigid surfaces to that of N-SRS [1][2]. We will show that this is the case for $TN = 0, 1$. This can be easily extended for the case of arbitrary N. As explained earlier, an N-SRS is obtained by patching together pieces of $C^{1|N}$ by means of N-superconformal transformations:

$$\mathbf{z}_\alpha = f_{\alpha\beta}(\mathbf{z}_\beta; \vec{m}, \vec{\zeta}) \tag{5.1}$$

where $\mathbf{z} = (z, \vec{\theta})$ and $\vec{m}$ ($\vec{\zeta}$) are the even (odd) moduli. Following [1][3] we obtain *augmented* N-superconformal transformations by introducing a new global odd variable θ^+ and promoting all the functions given above to be arbitrary functions of θ^+ in addition to $\mathbf{z}$. Now an *augmented* N- superconformal surface is obtained by patching together pieces of $C^{1|N+1}$ by means of the augmented superconformal transformations. An augmented N-superconformal surface still has a distinguished distribution $\bar{\mathcal{E}}$ of dimension $0|N$ spanned by $\vec{D}$. This is seen by checking that under augmented superconformal transformations, $\bar{\mathcal{E}}$ is preserved.

The group of augmented N-superconformal transformations is isomorphic to TN-semirigid transformations. This has been proved for the cases of $TN = 0$ [1] and for $TN = 1, 2$ [2]. Since we may represent any $SSRS$ by a collection of semirigid patching functions, we can apply this isomorphism to obtain an augmented SRS and vice versa.[4] This isomorphism implies that the moduli spaces of TN-$SSRS$ and augmented N-SRS are identical. Hence, it suffices to study the moduli space of augmented SRS.

[4] Note however that as a complex supermanifold the TN-surface is of dimension $1|N+2$ while the corresponding augmented N-SRS is of dimension $1|N+1$. The missing θ^- carries no information, though it was crucial to get superfield formulas in [1].

The moduli of the augmented superconformal surfaces are obtained by replacing the moduli of the superconformal surfaces by functions of θ^+ in (5.1), that is

$$\begin{aligned} m^a &\mapsto \widetilde{m}^a + \theta^+ \hat{m}^a \\ \zeta^\mu &\mapsto \widetilde{\zeta}^\mu + \theta^+ \hat{\zeta}^\mu \end{aligned} \tag{5.2}$$

where we have introduced extra odd(even) moduli, $\hat{m}^a$ ($\hat{\zeta}^\mu$) and placed tildes on $\widetilde{m}^a$ ($\widetilde{\zeta}^a$) to avoid confusion with the m^a (ζ^a) on the original space. Hence, given any family of N-SRS, we obtain a family of augmented N-SRS with twice as many parameters. We lack global information regarding the moduli space of augmented superconformal surfaces. For example, we do not know if any of the new even coordinates $\hat{\zeta}^\mu$ are periodic. But we can easily argue that infinitesimally, (5.2) spans the full tangent to the moduli space when we vary $\widetilde{m}$, $\hat{m}$, $\widetilde{\zeta}$, $\hat{\zeta}$. First, we note that deformations of any augmented SRS involve small changes in the patching maps. These are generated by vector fields $V_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ with no θ^+ component (this follows from the global nature of θ^+). Expanding $V_{\alpha\beta}$ in a power series in θ^+, we get two identical copies of the deformation space of N-SRS, with opposite parity. Furthermore, given $V^A(z,\vec{\theta},\theta^+) = v^A(z,\vec{\theta}) + \theta^+ \nu^A(z,\vec{\theta})$ with $V^{\theta^+} = 0$, the vector field $\{v^A_{\alpha\beta}\}$ generates infinitesimal deformations in the moduli $\widetilde{m}$ and $\widetilde{\zeta}$ and the vector field $\{\nu^A_{\alpha\beta}\}$ generates infinitesimal deformations in $\hat{m}$ and $\hat{\zeta}$ from (5.2).

A projection down to the moduli space of N-SRS corresponds to forgetting the new moduli introduced, that is, given a point with coordinates $(\widetilde{m}^a, \hat{m}^a, \widetilde{\zeta}^\mu, \hat{\zeta}^\mu)$ in the augmented moduli space, we project down to the point with coordinates $(m^a = \widetilde{m}^a, \zeta^a = \widetilde{\zeta}^a)$ in the moduli space of SRS. We would like to show that the projection is natural.

Let us now discuss projections in general. We wish to define a map Π from a space $\widehat{\mathcal{M}}$ to $\mathcal{M}$. Let $(\widetilde{x}^a,\ \hat{x}^a)$ be a set of coordinates near $\widetilde{P}$ on $\hat{\mathcal{M}}$ and x^a be coordinates near P on $\mathcal{M}$. We can define a projection Π by taking $x^a(P) = \widetilde{x}^a(\widetilde{P})$ or in other words

$$\Pi^*(x^a) = \widetilde{x}^a$$

which we refer to as the "forgetful" map. Unfortunately, the definition of Π depends on the choice of coordinates. Let $(\widetilde{y}^a = \widetilde{F}^a(\widetilde{x}^b, \hat{x}^b),\ \hat{y}^a = \hat{F}^a(\widetilde{x}^b, \hat{x}^b))$ be another set of coordinates near $\widetilde{P}$. Also, let $y^a = F^a(\vec{x})$ be a new coordinate near P. The new coordinates will define the same map Π as the old ones only if

$$\widetilde{y}^a \ \equiv\ \widetilde{F}^a(\widetilde{x}^b, \hat{x}^b) = F^a(\widetilde{x}^b) \quad . \tag{5.3}$$

Of course arbitrary coordinates for $\widehat{\mathcal{M}}$ will not be related to $(\widetilde{x}^a, \hat{x}^a)$ by (5.3). But if $\widehat{\mathcal{M}}$ has some natural class of coordinates all related by (5.3) then we do obtain a global projection Π. We will now see that semirigid moduli space does have such a natural class of coordinates.

Begin with the case of $TN = 0$ following the discussion in [1]. A Riemann surface is obtained by patching together pieces of C^1 by means of the transition function

$$z_\alpha = f_{\alpha\beta}(z_\beta, m^a) \quad ,$$

where m^a are complex coordinates on the moduli space of complex dimension $(3g-3)$. We now obtain a class of augmented Riemann surfaces parametrized by $(\widetilde{m}^a, \hat{m}^a)$ using the augmented transition functions

$$\begin{aligned} z_\alpha &= f_{\alpha\beta}(z_\beta, \widetilde{m}^a + \theta^+_\beta \hat{m}^a) \quad , \\ &= f_{\alpha\beta}(z_\beta, \widetilde{m}^a) + \theta^+_\beta \partial_a f_{\alpha\beta}(z_\beta, \widetilde{m}^a)\hat{m}^a \quad ; \\ \theta^+_\alpha &= \theta^+_\beta \quad , \end{aligned} \tag{5.4}$$

where a point in the moduli space of augmented Riemann surfaces has coordinates $(\widetilde{m}^a$, $\hat{m}^a)$. Coordinates obtained in this way are not the most general ones, and indeed we will now show that they are all related by the special class of maps (5.3).

Let $n^a(m^a)$ be a new set of coordinates on the moduli space of ordinary Riemann surfaces. We obtain the patching function parametrized by n^a by means of the following identification:

$$\check{f}_{\alpha\beta}(z_\beta, \vec{n}) \;\equiv\; f_{\alpha\beta}(z_\beta, \vec{m}(\vec{n})) \quad . \tag{5.5}$$

The corresponding family of augmented Riemann surfaces is again given by the rule (5.2):

$$\begin{aligned} z_\alpha &= \check{f}_{\alpha\beta}(z_\beta, \widetilde{n}^a + \theta^+ \hat{n}^a) \\ &= \check{f}_{\alpha\beta}(z_\beta, \widetilde{n}^a) + \theta^+_\beta \partial_a \check{f}_{\alpha\beta}(z_\beta, \widetilde{n}^b)\hat{n}^a \\ \theta^+_\alpha &= \theta^+_\beta \quad . \end{aligned} \tag{5.6}$$

Comparing (5.4) and (5.6) using (5.5) shows that the two sets of coordinates on the moduli space of $TN = 0$ surfaces are related by the transition function

$$\begin{aligned} \widetilde{n}^a &= \widetilde{n}^a(\widetilde{m}^b) \quad , \\ \hat{n}^a &= \left.\frac{\partial n^a}{\partial m^b}\right|_{m=\widetilde{m}} \hat{m}^b \quad , \end{aligned}$$

which is not only of the form (5.3) but in fact split. Hence in particular the projection from augmented $N=0$ surfaces to ordinary ones is natural, and as we have already seen that this gives the desired projection from $TN=0$ surfaces to $N=0$.

For the case of $TN=1$, the situation is similar. Let $(\widetilde{m}^a,\ \hat{m}^a,\ \widetilde{\zeta}^\mu,\ \hat{\zeta}^\mu)$ be the coordinates of a point in the moduli space of augmented $N=1$ SRS and $(\widetilde{n}^a,\hat{n}^a,\widetilde{\phi}^\nu,\hat{\phi}^\nu)$ be the coordinates of the same point on another patch. Following similar arguments as for $TN=0$, we obtain

$$
\begin{aligned}
\widetilde{n}^a &= \widetilde{n}^a(\widetilde{m}^b,\widetilde{\zeta}^\nu) \quad ,\\
\hat{n}^a &= \frac{\partial n^a}{\partial m^b}\hat{m}^b + \frac{\partial n^a}{\partial \zeta^\nu}\hat{\zeta}^\nu \quad ,\\
\widetilde{\phi}^\mu &= \widetilde{\phi}^\mu(\widetilde{m}^b,\widetilde{\zeta}^\nu) \quad ,\\
\hat{\phi}^\mu &= \frac{\partial \phi^\mu}{\partial m^b}\hat{m}^b + \frac{\partial \phi^\mu}{\partial \zeta^\nu}\hat{\zeta}^\nu \quad ,
\end{aligned}
$$

which is again of the form (5.3) and hence the "forgetful" map is again natural. This can be seen to hold for the case of arbitrary TN since the only property which makes the transition function split is the global nature of θ^+. Thus, there exists a natural projection from the moduli space of TN-$SSRS$ to the moduli space of N-SRS. The significance of this result is that [1] it means we can use string-theory methods to get a measure on the big space, then integrate it over the fibers of this projection to get a measure on the smaller space, namely the moduli space of N-superconformal surfaces, which is where the observables of topological gravity should live.

6. Conclusion

In this paper, we have provided an intrinsic definition of N-SRS and TN-$SSRS$ which appeared naturally in (super)gravity and topological (super)gravity respectively. The intrinsic definitions are given in the context of G-structures. It is straightforward to define superconformal or semirigid G-structure from the coordinates given in the patch definition of SRS or $SSRS$. Much of our analysis was devoted to showing how one can recover the patch definition given a G-structure on a manifold. That is, we first obtained the necessary torsion constraints where needed and showed that the almost G-structure is integrable under such conditions.

Moreover, we have shown that there exists a natural projection from the moduli space of TN-$SSRS$ to that of N-SRS. Since a field theoretical realization of topological TN-gravity can yield an integration density on the moduli space of TN-$SSRS$, the natural projection allows us to integrate along the fibers of the projection and obtain an integration density on the moduli space of N-SRS. If there are non-trivial observables, then the field theory provides for us cohomology classes on the moduli space, thus probing its topology. This procedure has been used for the case $TN = 0$ in [3][4]; it would be interesting to see what topologies one can probe for $TN \geq 1$ cases.

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Appendix A. Solving the torsion constraints

We will show that under the constraints (4.16) to (4.19), we can find a unique Ω to represent b and y by $D_+\Omega$ and $D_3\Omega$ respectively and satisfying the constraint

$$D_+\Omega \; D_-\Omega = -\frac{1}{2}(D_3\Omega)^2. \tag{A.1}$$

Anticipating Appendix B we note that if $\Omega = \widetilde{\theta}^+$ then (A.1) is one of the superconformal conditions involving only $\widetilde{\theta}^+$ in (4.23) when F^{-1} is used.

We will now go through the constraints (4.16) to (4.19) and show how we get Ω. Eqn. (4.16) implies that $b = D_+\Omega$ where Ω is an odd superfield. To see that is possible, one way is to expand both b and Ω in components θ^i and constrain b by (4.16). Then it is straightforward that equating b and $D_+\Omega$ turns to algebraic equations between their components, thus solving for the components of Ω in terms of that of b. However there is a residual freedom $\Omega \to \Omega' = \Omega + \omega$ where $D_+\omega = 0$ leaves $b = D_+\Omega$ invariant. We will make use of this degree of freedom to make $y = D_3\Omega$. Substituting $b = D_+\Omega$ into (4.17), it implies that $y = D_3\Omega + B$, where $D_+B = 0$. Here we will use the freedom in choosing Ω' to cancel B, that is, $D_3\omega = -B$. This is possible because both ω and B are annihilated by D_+, and in components, it means solving two algebraic equations and two first order linear differential equations in the components of ω in terms of that of B. There is still a little freedom left in Ω', namely $\Omega'' = \Omega' + \phi$, where $D_+\phi = D_3\phi = 0$. In components, this means $\phi = \theta^-\phi_-$, and ϕ_- is a constant. Again, this constant will be used later on.

Dropping the primes, we now have $b = D_+\Omega$ and $y = D_3\Omega$; substituting both into (4.18) and (4.19), we obtain

$$D_3\Omega\ D_+D_3\Omega = -D_+\Omega\ D_+D_-\Omega \tag{A.2}$$

and

$$-\frac{1}{2}(D_3\Omega)^2\ D_3D_+\Omega + D_+\Omega\ D_3\Omega\ \partial_z\Omega + (D_+\Omega)^2\ D_3D_-\Omega = 0. \tag{A.3}$$

Eliminating $D_3\Omega\ D_+D_3\Omega$ in (A.3) by (A.2), we get

$$-\frac{1}{2}D_3\Omega\ D_-D_+\Omega + D_+\Omega\ D_-D_3\Omega = 0. \tag{A.4}$$

Equations (A.2), (A.3) and (A.4) can be rewritten as

$$D_i[D_-\Omega + \frac{1}{2}\frac{(D_3\Omega)^2}{D_+\Omega}] = 0, \tag{A.5}$$

where $i = +, 3, -$ respectively. This implies that whatever is inside the square bracket can at most be some arbitrary constant. This constant can be cancelled by the remaining free constant ϕ. By construction, D_3 and D_+ annihilate ϕ, and $D_-\phi = \phi_-$. Thus, ϕ_- will be chosen to cancel the arbitrary constant, and we are left with (A.1).

Appendix B. N=3 superconformal coordinate transformation

In this appendix, we will give the conditions for the $N = 3$ superconformal coordinate transformation. We will then show that there exists an $N = 3$ superconformal coordinate transformation when $\widetilde{\theta}^+$ is given subject to (A.1). This is needed in the proof of $N = 3$ semirigid integrability.

We expand the superconformal transformation in components. Let

$$\widetilde{z} = f + \theta^+\phi_+ + \theta^3\phi + \theta^-\phi_- + \theta^3\theta^+\widetilde{f}_+ + \theta^+\theta^-\widetilde{f} + \theta^-\theta^3\widetilde{f}_- + \theta^+\theta^-\theta^3\widetilde{\phi} \tag{B.1}$$

and

$$\widetilde{\theta}^i = \lambda^i + {(m^t)^i}_j\theta^j + {(\Gamma^t)^i}_j g^{jk}\tfrac{1}{2}\epsilon_{k\ell m}\theta^\ell\theta^m + \widetilde{\ell}^i\theta^+\theta^-\theta^3, \tag{B.2}$$

where $i = +, 3, -$ and

$${m_i}^j = \begin{pmatrix} t_+ & n_+ & s_+ \\ t & n & s \\ t_- & n_- & s_- \end{pmatrix} \quad , \quad {\Gamma_i}^j = \begin{pmatrix} \widetilde{\tau}_+ & \widetilde{\nu}_+ & \widetilde{\psi}_+ \\ \widetilde{\tau} & \widetilde{\nu} & \widetilde{\psi} \\ \widetilde{\tau}_- & \widetilde{\nu}_- & \widetilde{\psi}_- \end{pmatrix}, \tag{B.3}$$

$$\lambda^i = \begin{pmatrix} \tau \\ \nu \\ \psi \end{pmatrix} \quad , \quad \widetilde{\ell}^i = \begin{pmatrix} \widetilde{t} \\ \widetilde{n} \\ \widetilde{s} \end{pmatrix} \quad , \tag{B.4}$$

the metric $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and $\epsilon_{-3+} = 1$.

The superconformal conditions can be compactly written as

$$mgm^t = g(mgm^t)_{33} \quad , \tag{B.5}$$

$$(mg\Gamma^t)_{ij} = g_{ij}(mg\Gamma^t)_{33} + \epsilon_{ijk}(g^{-1}mg\partial_z\lambda)^k, \tag{B.6}$$

$$(mg\widetilde{\ell})_i = (\Gamma g\partial_z\lambda)_i - \frac{1}{4}\epsilon_{ijk}[2\Gamma g\Gamma^t + (\partial_z m)gm^t - mg\partial_z m^t]_{jk} \quad , \tag{B.7}$$

$$\partial_z f = (mgm^t)_{33} + (\partial_z\lambda)^t g\lambda, \tag{B.8}$$

$$(\phi)_i = (mg\lambda)_i \quad , \tag{B.9}$$

$$(\widetilde{f})_i = (\Gamma g\lambda)_i \quad , \quad \text{and} \tag{B.10}$$

$$\widetilde{\phi} = \widetilde{\ell}^t g\lambda - (mg\Gamma^t)_{33} \quad . \tag{B.11}$$

There are two things that one notices from (B.5) to (B.11). One is that m belongs to $SO(3,C)\times C^\times$, thus only four matrix elements are independent. The rest can be expressed in terms of the four independent variables. The other observation is that the components of $\widetilde{z}$ of the transformation are expressed in terms of the components of $\widetilde{\theta}^i$ given by (B.8) to (B.11). Thus (B.5) to (B.7) are internal superconformal conditions that have to be satisfied by $\widetilde{\theta}^i$.

Our problem is that we are given the components of $\widetilde{\theta}^+$, and we wish to see that there exists a superconformal coordinate transformation with this $\widetilde{\theta}^+$ by choosing the components of $\widetilde{\theta}^{3,-}$ to satisfy the internal superconformal conditions. Let us work with the case when ${N_+}^+$ is invertible. This implies that $b = D_+\widetilde{\theta}^+$ is also invertible and hence so is t_+. From the lowest component of (A.1), when Ω is identified with $\widetilde{\theta}^+$, we have $t^2 = -2t_+t_-$. Thus even though $\widetilde{\theta}^+$ is handed to us, we know that t is not independent of t_+ and t_-. We will take t_+ and t_- as two independent elements of m. There are two left, and we will choose them to be s_- and n. For now, the only constraint we put on s_- and n is that they are invertible. This gives m a invertible determinant. The rest of the five entries of m are expressed in terms of t_+, t_-, s_- and n by (B.5). Since $\widetilde{\theta}^+$ is given to us, we now have, in addition to s_- and n, the rest of the six elements of Γ, the lowest and highest components of $\widetilde{\theta}^3$ and $\widetilde{\theta}^-$ to choose to satisfy the eight conditions in (B.6) and three in (B.7). Since t_+, s_- and n are invertible, we invert them in (B.6) to solve for $\partial_z \nu$, $\partial_z \psi$, $\widetilde{\psi}_+$, $\widetilde{\nu}_{+,-}$ and $\widetilde{\nu}$, thus satisfying six of the eight conditions of (B.6). The two variables $\widetilde{\psi}_-$ and $\widetilde{\psi}$ have coefficients t_-. t_- is given to us and it may vanish. If it does not, then we can invert it and choose $\widetilde{\psi}_-$ and $\widetilde{\psi}$ to satisfy the last two conditions. If t_- vanishes then by (B.5) and by (A.1), we conclude that $n_-^2 = -2s_- t_- = 0$, $t_- = t = \widetilde{t} = \widetilde{\tau}_- = 0$ and $\widetilde{\tau} = \partial_z \tau = k\widetilde{\tau}_+$, where k is some even function. Under these circumstances, the two conditions become vacuous. Similarly, we invert t_+ and n in (B.7) to solve for the highest components of $\widetilde{\theta}^3$ and $\widetilde{\theta}^-$, $\widetilde{n}$ and $\widetilde{s}$ respectively, thus leaving one condition to be satisfied. The problem is that we cannot choose $\widetilde{t}$ to satisfy this equation, but one can see that if t_- is invertible, then we can invert n_- and choose $\partial_z \widetilde{n}$ to satisfy this condition. If t_- vanishes, then this condition becomes vacuous. Thus, all superconformal conditions can be satisfied given $\widetilde{\theta}^+$ and we are able to complete the rest of the superconformal coordinate transformation.

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