ELSEVIER

Bull. Sci. math. 135 (2011) 251–261

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

Restriction theorems for Higgs principal bundles

Indranil Biswas [a,*], Arijit Dey [b]

[a] *School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*

[b] *Hausdorff Institute of Mathematics, Poppelsdorfer Allee 45, Bonn 53115, Germany*

Received 29 March 2010

Available online 13 April 2010

Abstract

We prove analogues of Grauert–Mülich and Flenner's restriction theorems for semistable principal Higgs bundle over any smooth complex projective variety.

MSC: 14F05; 14L15

Keywords: Higgs bundles; Principal bundle; Semistability; Restriction theorems

1. Introduction

Let X be a smooth complex projective variety, with $\dim X \geqslant 2$. Fix a very ample line bundle $\mathcal{O}_X(1)$ on X. For any integer n, the line bundle $\mathcal{O}_X(1)^{\otimes n}$ will be denoted by $\mathcal{O}_X(n)$.

Let E be a semistable vector bundle over X. Then for a general smooth hypersurface D on X from the linear system $|\mathcal{O}_X(a)|$, the restriction of E to D is semistable if the integer a is sufficiently large. More generally, for any vector bundle V over X, the Harder–Narasimhan polygon of $V|_D$ can be estimated from the data of V (see [6, Ch. 3 and Ch. 7]).

We recall the Grauert–Mülich and Flenner restriction theorems.

* Corresponding author.
E-mail addresses: indranil@math.tifr.res.in (I. Biswas), arijitdey@gmail.com (A. Dey).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2010.04.003

Theorem 1.1 *(Grauert–Mülich theorem). Let E be a semistable torsionfree sheaf over X. Let*

$$D = \bigcap_{i=1}^{c} D_i$$

be a general complete intersection of hypersurfaces $D_i \in |\mathcal{O}_X(a_i)|$ such that $\dim D > 0$. *If the restriction $V|_D$ is not semistable, then consider the Harder–Narasimhan filtration*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{m-1} \subset V_m = V|_D.$$

Then for all $0 < j < m$,

$$0 < \mu(V_j/V_{j-1}) - \mu(V_{j+1}/V_j) \leqslant \max_{1 \leqslant i \leqslant c} \left(\prod_{i=1}^{c} a_i \right) \cdot \deg(X).$$

See [6, p. 59, Theorem 3.1.2] for a proof of the Grauert–Mülich theorem.

Theorem 1.2 *(Flenner's theorem). Let V be a semistable torsionfree sheaf on X of rank r. Take any integer a such that*

$$\frac{\binom{a+\dim X}{a} - ac - 1}{a} > \deg(X) \cdot \max\left\{ \frac{r^2 - 1}{4}, 1 \right\}.$$

Then the restriction of V to the general complete intersection $D = \bigcap_{i=1}^{c} D_i$ of positive dimension, where $D_i \in |\mathcal{O}_X(a)|$, is semistable.

See [6, p. 161, Theorem 7.1.1] for a proof of the above theorem.

In [3], the above theorems were generalized to principal G-bundles over X. Our aim here is to generalize them to Higgs G-bundles.

We prove the following two theorems (see Theorem 4.1 and Theorem 5.1):

Theorem 1.3. *Let (E, θ) be a semistable Higgs G-bundle on X. Then there is a nonempty open subset $S' \subset S$ such that for all $s \in S'$ the following holds: Let $(E, \theta)|_{Z_s}$ be the restriction of E to the complete intersection $Z_s := D_1 \cap \cdots \cap D_l$. If it is unstable, let (E_P, θ_P) be the Harder–Narasimhan reduction of $(E, \theta)|_{Z_s}$, and let (E_L, θ_L) be the Higgs L-bundle obtained by extending the structure group of the Higgs P-bundle (E_P, θ_P), where L is the Levi quotient of P. Then for any $\alpha \in \Pi'$ (see (2.2)),*

$$0 < \mu\big(E_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big) \leqslant \max\{a_i\} \left(\prod_{i=1}^{c} a_i \right) \deg(X),$$

where $(E_L(\mathfrak{g}^{\bar{\alpha}}), \theta(\mathfrak{g}^{\bar{\alpha}}))$ is the Higgs vector bundle associated to (E_L, θ_L) for the adjoint action of L on the Z_L-root space $\mathfrak{g}^{\bar{\alpha}}$.

Theorem 1.4. *Let $a \in \mathbb{N}$ be such that*

$$\frac{\binom{a+\dim(X)}{a} - l \cdot a - 1}{a} > \deg(X) \frac{\dim \mathfrak{g} - \dim \mathfrak{t}}{2}.$$

If (E, θ) is a semistable Higgs G-bundle, then the restriction $(E, \theta)|_{D_1 \cap \cdots \cap D_l}$ to a general complete intersection of positive dimension with $D_i \in |\mathcal{O}_X(a)|$ is Higgs semistable.

The notation used in Theorem 1.3 and Theorem 1.4 is explained in Section 4.

2. Preliminaries

2.1. Higgs sheaf

Let X be an irreducible smooth projective variety over $\mathbb{C}$ of dimension n, with $n \geqslant 2$. The holomorphic cotangent bundle of X will be denoted by Ω_X^1.

A *Higgs sheaf* on X is a pair of the form (E, θ), where $E \longrightarrow X$ is a torsionfree sheaf, and

$$\theta : E \longrightarrow E \otimes \Omega_X^1$$

is an $\mathcal{O}_X$-linear homomorphism such that $\theta \wedge \theta = 0$ [9]. The homomorphism θ is called a *Higgs field* on E. A coherent subsheaf F of E is called θ*-invariant* if

$$\theta(F) \subset F \otimes \Omega_X^1.$$

A θ-invariant subsheaf will also be called a *Higgs subsheaf*.

Fix a very ample line bundle $H := \mathcal{O}_X(1)$ on X. The *degree* of a torsionfree coherent sheaf V on X is the degree of the restriction of V to the general complete intersection curve $D_1 \cap \cdots \cap D_{n-1}$, where $D_i \in |\mathcal{O}_X(1)|$. So,

$$\operatorname{degree}(V) = \big(c_1(V) \cup c_1(H)^{n-1}\big) \cap [X].$$

The quotient $\operatorname{degree}(V)/\operatorname{rank}(V) \in \mathbb{Q}$ is called the *slope* of V, and it is denoted by $\mu(V)$.

A Higgs sheaf (E, θ) is said to be *stable* (respectively, *semistable*) if for every Higgs subsheaf $F \subset E$ with $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$, the inequality

$$\mu(F) < \mu(E) \quad \big(\text{respectively, } \mu(F) \leqslant \mu(E)\big)$$

holds.

Given a Higgs sheaf (E, θ) over X, there is a unique strictly increasing filtration of Higgs subsheaves

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E \tag{2.1}$$

such that for each $i \in [1, k]$, the quotient E_i/E_{i-1} equipped with the Higgs field induced by θ is Higgs semistable, and furthermore,

$$\mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_j/E_{j-1}) > \cdots > \mu(E_k/E_{k-1}).$$

This filtration is known as the *Higgs Harder–Narasimhan filtration* of (E, θ) [10].

2.2. Higgs G-bundles

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. For a principal G-bundle $E \longrightarrow X$, let $E(\mathfrak{g})$ be the vector bundle associated to E for the adjoint action of G on $\mathfrak{g}$; it is called the adjoint vector bundle of E, and it is also denoted by $\operatorname{ad}(E)$.

A *Higgs G-bundle* on X is a pair of the form (E, θ), where $E \longrightarrow X$ is a principal G bundle

$$\theta : \mathcal{O}_X \longrightarrow E(\mathfrak{g}) \otimes \Omega_X^1$$

is an $\mathcal{O}_X$-linear homomorphism such that $\theta \wedge \theta = 0$; note that $\theta \bigwedge \theta$ is a section of $E(\mathfrak{g}) \otimes \Omega_X^2$ (it is defined using the Lie algebra structure of the fibers of $E(\mathfrak{g})$).

A Zariski open subset $U \subset X$ is said to be *big* if $\operatorname{codimension}(X \setminus U) \geqslant 2$.

Let E be a principal G-bundle on X. Let H be a closed algebraic subgroup of G. The quotient map $E \longrightarrow E/H$ will be denoted by q. A *reduction of structure group* of E to H over a big open subset U is a section

$$\sigma : U \longrightarrow E(G/H)$$

of the fiber bundle $E(G/H) = E/H \longrightarrow X$. Note that $q^{-1}(\sigma(U)) \longrightarrow U$ is a principal H-bundle.

If (H, σ) be a reduction of E to H over a big open subset U, and if

$$\theta_H \in \mathrm{H}^0\big(U, E_H(\mathfrak{h}) \otimes \Omega_U^1\big)$$

is a section such that the diagram

$$\begin{array}{ccc} \mathcal{O}_U & \xrightarrow{\ \theta\ } & E(\mathfrak{g}) \otimes \Omega_U^1 \\ & \searrow^{\theta_H} & \uparrow \\ & & E_H(\mathfrak{h}) \otimes \Omega_U^1 \end{array}$$

is commutative, then the quadruple (H, σ, θ_H, U) is called *Higgs reduction* of E to H. Sometime we will denote it by (E_H, θ_H) provided it does not cause any confusion.

Let $Z_G \subset G$ be the center. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. The Lie algebras of T and B will be denoted by $\mathfrak{t}$ and $\mathfrak{b}$ respectively. Let R_T be the set of roots of G with respect to T and $R_T^+ \subset R_T$ the set of positive roots. Let Δ be the set of simple roots of $\mathfrak{g}$. For $\alpha \in \mathfrak{t}^\vee - \{0\}$, let

$$\mathfrak{g}^\alpha = \big\{v \in \mathfrak{g}\colon\ [s, v] = \alpha(s)v,\ \text{for all } s \in \mathfrak{t}\big\}$$

be the root space.

For any parabolic subgroup P of G, there is a unique parabolic subgroup Q containing B such that Q is a conjugate of P.

Henceforth, by a parabolic subgroup P of G we will always mean that P contains B.

For any parabolic subgroup P containing B, there is a unique maximal connected T-invariant reductive $L(P)$ subgroup P. The composition

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism, where $R_u(P)$ is the unipotent radical of P. This subgroup $L(P)$ will be called the *Levi factor* of P. The Levi factor projects isomorphically to the quotient group $P/R_u(P)$. The group $P/R_u(P)$ is called the *Levi quotient* of P.

Let $\mathfrak{p}$ be the Lie algebra of the parabolic subgroup P. Let Π' be the set of simple roots $\alpha \in \Delta$ such that $-\alpha$ is not a root of $\mathfrak{p}$. Let

$$\Pi := \Delta \setminus \Pi' \tag{2.2}$$

be the complement.

The center of the Levi factor L of P, which is a torus, will be denoted by Z_L. Let

$$\mathfrak{z}_l := \mathrm{Lie}(Z_L)$$

be the Lie algebra of Z_L. For $\bar{\alpha} \in \mathfrak{z}_L$,

$$\mathfrak{g}^{\bar{\alpha}} = \big\{v \in \mathfrak{g}\colon\ [s, v] = \bar{\alpha}(s)v,\ \text{for all } \in \mathfrak{z}_l\big\}.$$

If $\alpha \in R_T \subset \mathfrak{t}^\vee$, the set $R_{Z_L} = \{\bar{\alpha}\colon\ \mathfrak{g}^{\bar{\alpha}} \neq 0\} \subset \mathfrak{z}_L^\vee$ of Z_L roots forms an abstract root system, but not necessarily reduced. If $\alpha \in R_T \subset \mathfrak{t}^\vee$ is a T-root, then the corresponding element in $R_{Z_L} \cup \{0\} \subset \mathfrak{z}_L$ will be denoted by $\bar{\alpha}$ (see §2 of [3] for more details). The spaces $\mathfrak{g}^{\bar{\alpha}}$ are not necessarily one-dimensional, in fact

$$\mathfrak{g}^{\bar{\alpha}} = \bigoplus_{\{\beta \in R_T\colon s.t. \bar{\beta} = \bar{\alpha}\}} \mathfrak{g}^{\beta},$$

where $\mathfrak{g}^{\beta}$ is the root space associated to the root $\beta \in R_T \subset \mathfrak{t}^\vee$. We have the following root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\bar{\alpha} \in R_{Z_L}} \mathfrak{g}^{\bar{\alpha}}.$$

Let E be a principal G-bundle on X. For any quasiprojective variety F on which G acts from the left, let $E(Y)$ be the associated fiber bundle. So $E(Y)$ is the quotient of $E \times Y$ where (e_1, y_1) and (e_2, y_2) of $E \times Y$ are identified if there is an element $g \in G$ such that $e_2 = e_1 g$ and $y_2 = g^{-1} \cdot y$.

Let $\rho : G \longrightarrow H$ be a homomorphism of connected reductive algebraic groups. For any principal G-bundle E_G, let $E_\rho(H)$ be the principal H-bundle obtained by extending the structure group of E_G using ρ. Note that we have a homomorphism

$$\rho' : \mathrm{ad}(E) \longrightarrow \mathrm{ad}\big(E_\rho(H)\big)$$

given by the homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{h}$ of H associated to ρ. Using ρ', a Higgs field on E produces a Higgs field on $E_\rho(H)$.

The Higgs structure θ_H on $E_\rho(H)$ $(= E \times_\rho H)$ is given by $\theta_H := (\rho' \otimes id) \circ \theta$ where $\rho' \otimes id : E(\mathfrak{g}) \otimes \Omega_X^1 \longrightarrow E(\mathfrak{h}) \otimes \Omega_X^1$ and $\theta : \mathcal{O}_X \longrightarrow E(\mathfrak{g}) \otimes \Omega_X^1$.

A character χ of a parabolic subgroup $P \subset G$ will be called *strictly anti-dominant* if χ is trivial on the connected component of Z_G, and the line bundle on G/P associated to the principal P-bundle $G \longrightarrow G/P$ for χ is ample.

A Higgs G-bundle (E, θ) is said to be *Higgs semistable* if for any Higgs reduction (E_P, θ_P) of (E, θ) to any proper parabolic subgroup $P \subset G$ over some big open subset U, and for any strictly anti-dominant character χ of P, the associated line bundle $L_\chi := E_P \times^\chi \mathbb{C}$ is of nonnegative degree.

We note that (E, θ) is Higgs semistable if and only if for any Higgs reduction $\sigma : U \longrightarrow E/P$ to any proper maximal parabolic subgroup $P \subset G$ over some big open subset U, the vector bundle $\sigma^* T_{\mathrm{rel}}$ is of nonnegative degree, where $T_{\mathrm{rel}} \longrightarrow E/P$ is the relative tangent bundle for the projection $E/P \longrightarrow X$ (see [8, p. 129, Definition 1.1] and [8, p. 131, Lemma 2.1]).

A Higgs reduction (E_P, θ_P) of (E, θ) over a big open subset of X is called a *Harder–Narasimhan reduction* if the following two conditions hold:

(1) The associated Higgs $L(P)$-bundle $(E_P(L(P)), \theta_{L(P)})$ is Higgs semistable, where $L(P) := P/R_u(P)$ is the Levi quotient of P.
(2) For each nontrivial character χ of P which is a nonnegative linear combination of simple roots with respect to B, the associated line bundle $E_P \times^\chi \mathbb{C}$ has positive degree.

For any Higgs G-bundle there is a unique Harder–Narasimhan reduction [4, Theorem 16].

3. Some useful results

In this section we will put down four lemmas which will be used in the proof of Theorem 4.1.

Lemma 3.1. *(See [1, Proposition 2.8].) Let E_1 and E_2 be two torsion free sheaves over X. Then*

(1) $\mathrm{Hom}(E_1, E_2) \neq 0$ *implies that* $\mu_{\min}(E_1) \leqslant \mu_{\max}(E_2)$.
(2) *If there exists a surjective homomorphism*

$$E_1 \longrightarrow E_2 \longrightarrow 0,$$

then $\mu_{\min}(E_2) \geqslant \mu_{\min}(E_1)$.

Lemma 3.2. *Let $V_1 \subset V_2 \subset \cdots \subset V_n = (V, \theta)$ be a filtration of Higgs shaves of a torsionfree Higgs sheaf (V, θ). Assume that V_1, with the induced Higgs field, is Higgs semistable. Also, assume that each successive quotient V_i/V_{i-1}, $2 \leqslant i \leqslant n$, with the induced Higgs field is Higgs semistable, and $\mu(V_i/V_{i-1}) < \mu(V_1)$. Then V_1 is the maximal destabilizing Higgs subsheaf of (V, θ) (meaning V_1 is the first term of the Harder–Narasimhan filtration of (V, θ)).*

Proof. The proof is exactly identical to the proof of Lemma 4.2 of [3]. □

Let $p : Z \longrightarrow X$ be a projective morphism with S integral, and let $O_Z(1)$ be a p-very ample line bundle on Z. The following lemma is the Higgs analogue of [6, p. 45, Theorem 2.3.2].

Lemma 3.3. *Let $(\mathcal{V}, \theta)$ be a Higgs torsionfree sheaf on Z which is flat over S. Then there exists an open subset $S' \subset S$ and a Higgs filtration*

$$0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k = \mathcal{V}|_{U'},$$

where $U' := p^{-1}(S')$, such that for all $s \in S'$, the restriction of the above filtration to the fiber Z_s is the Harder–Narasimhan filtration of the Higgs sheaf $(\mathcal{V}|_{Z_s}, \theta|_{Z_s})$ with respect to the polarization $\mathcal{O}_Z(1)|_{Z_s}$.

Proof. It can be proved by simply imitating the proof of Theorem 2.3.2 of [6]. □

We recall that Maruyama introduced the notion of a relative Harder–Narasimhan filtration for any family of torsionfree sheaves [7].

Lemma 3.4. *Let $(\mathcal{F}, \Theta)$ be a Higgs G-bundle over Z Then there exists an open dense set $S' \subset S$, an open set $U \subset Z' = p^{-1}(S')$ such that the codimension of the complement $Z_s \setminus U_s$ in Z_s is at least two for all $s \in S'$, where $U_s = U \cap Z_s$, and, furthermore, there is a Higgs parabolic reduction of structure group $(\mathcal{F}_P, \Theta_P)$ of $\mathcal{F}|_U$ to P, such that for all $s \in S'$, the restriction $(\mathcal{F}_P, \Theta_P)|_{U_s}$ is the Harder–Narasimhan reduction of the Higgs bundle $(\mathcal{F}, \Theta)|_{Z_s}$ with respect to polarization $\mathcal{O}_Z(1)|_{Z_s}$.*

Proof. Using Lemma 3.3 and [4, Proposition 12], this lemma is derived following the proof [1, Proposition 3.3]. □

4. Grauert–Mülich theorem

In this section we will prove Grauert–Mülich theorem for Higgs principal G-bundle. This theorem appeared first in [2] for vector bundles of rank 2 over projective spaces, and there it is attributed to Grauert and Mülich. Subsequently it was generalized for arbitrary rank by Spindler in [11], and to arbitrary projective varieties by Forster, Hirschowitz and Schneider in [5], and also by Maruyama [7]. In [3], this was extended to principal G-bundles [3, Theorem 4.1].

For a positive integer m, let

$$S_m := \mathbb{P}\big(H^0\big(X, H^m\big)^*\big)$$

be the linear system of hypersurfaces of degree m and

$$Z_m := \big\{(x,s):\ s(x)=0,\ s \in S_m\big\}.$$

Then we have a diagram,

$$\begin{array}{ccc} Z_m & \xrightarrow{q_m} & S_m \\ \downarrow{\scriptstyle p_m} & & \\ X & & \end{array} \tag{4.1}$$

where p_m and q_m are the natural projections.

The fiber of q_m over $s \in S_m$ is embedded as hypersurface in X. So we always think of fibers of q_m as closed subschemes of X. Scheme theoretically, Z_m can be described in the following way. The evaluation map gives rise to a following exact sequence

$$0 \longrightarrow K_m \longrightarrow H^0\big(X, \mathcal{O}_X(m)\big) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(m) \longrightarrow 0. \tag{4.2}$$

Then $Z_m = \mathbb{P}(K_m^*)$ is a projective bundle over X with projection

$$p_m : Z_m \longrightarrow X. \tag{4.3}$$

Let $T_{Z_m/X} := T_{Z_m}/p_m^*(T_X)$ be the relative tangent sheaf for p_m in (4.3). We have the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_m} \longrightarrow q^*(K_m) \otimes p_m^*\big(\mathcal{O}_{Z_m}(1)\big) \longrightarrow T_{Z_m/X} \longrightarrow 0. \tag{4.4}$$

Let

$$Z := Z_{m_1} \times_X \cdots \times_X Z_{m_l} \quad \text{and} \quad S := \prod_{i=1}^{l} S_{m_i},$$

where all m_i are positive integers, and $l < \dim(X)$. We have natural projections

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow{\scriptstyle q} & \downarrow{\scriptstyle \pi_i} & \searrow{\scriptstyle p} & \\ S & & Z_{m_i} & & X \end{array} \tag{4.5}$$

induced from (4.1). The relative tangent sheaf $\mathcal{T}_{Z/X}$ is given by

$$\mathcal{T}_{Z/X} = \bigoplus_{i=1}^{l} \pi_i^*(\mathcal{T}_{Z_{m_i}/X}). \tag{4.6}$$

Since Z is a fiber product of projective bundles over X, we have following relations among Picard groups,

$$\operatorname{Pic}(Z) = q^*\big(\operatorname{Pic}(X)\big) \oplus p^*\big(\operatorname{Pic}(S)\big) = q^*\big(\operatorname{Pic}(X)\big) \oplus \mathbb{Z}^l. \tag{4.7}$$

Theorem 4.1. *Let (E,θ) be a semistable Higgs G-bundle on X. Then there is a nonempty open subset $S' \subset S$ such that for all $s \in S'$ the following holds: Let $(E,\theta)|_{Z_s}$ be the restriction of E to the complete intersection $Z_s := D_1 \cap \cdots \cap D_l$. If it is unstable, let (E_P,θ_P) be the Harder–Narasimhan reduction of $(E,\theta)|_{Z_s}$, and let (E_L,θ_L) be the Higgs L-bundle obtained by extending the structure group of the Higgs P-bundle (E_P,θ_P) to L, where L is the Levi quotient of P. Then for any $\alpha \in \Pi'$ (see (2.2)),*

$$0 < \mu\big(E_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big) \leqslant \max\{a_i\}\left(\prod_{i=1}^{c} a_i\right)\deg(X),$$

where $(E_L(\mathfrak{g}^{\bar{\alpha}}), \theta(\mathfrak{g}^{\bar{\alpha}}))$ is the Higgs vector bundle associated to (E_L,θ_L) for the adjoint action of L on the Z_L-root space $\mathfrak{g}^{\bar{\alpha}}$.

Proof. Consider $(\mathcal{F},\Theta) := q^*(E,\theta)$. It is a Higgs G-bundle on Z such that for each $s \in S$, the restriction $(\mathcal{F},\Theta)|_{Z_s}$ is a Higgs G-bundle isomorphic to $(E,\theta)|_{Z_s}$.

There is a dense open subset $S' \subset S$, an open subset $U \subset Z$, and a Harder–Narasimhan reduction $(\mathcal{F}_P,\Theta_P)$ of $(\mathcal{F},\Theta)|_U$ to a parabolic subgroup P of G, such that the following holds: For each point $s \in S'$, the induced reduction $(\mathcal{F}_P,\Theta_P)|_{U\cap Z_s}$ is the Harder–Narasimhan reduction of $(E,\theta)|_{Z_s}$ (this follows from Lemma 3.4).

Take any $\alpha \in \Pi' \subset \Delta_T$ (see (2.2)). Let Q be the maximal parabolic subgroup containing P associated to α. Let $(\mathcal{F}_Q,\Theta_Q)$ (respectively, (E_Q,θ_Q)) be the extension of structure group of $(\mathcal{F}_P,\Theta_P)$ (respectively, (E_P,θ_P)) to Q by the inclusion of P in Q. For each $s \in S'$, the restriction $(\mathcal{F}_Q,\Theta_Q)|_{U\cap Z_s}$ is a reduction of $(E,\theta)|_{Z_s}$ to the maximal parabolic Q. This reduction is given by a morphism

$$\sigma_Q : U \longrightarrow \mathcal{F}(G/Q).$$

Note that Z is projective bundle over X with fiber

$$\mathbb{P} := \prod_{i=1}^{l} \mathbb{P}(K_{a_i}),$$

and $E = \mathcal{F}/\mathbb{P}$. We have the following diagram

$$\begin{array}{ccccc} U & \xrightarrow{\sigma_Q} & \mathcal{F}(G/Q) & \xrightarrow{f} & E(G/Q)|_{X'} \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle g} & \\ & & X' & & \end{array} \tag{4.8}$$

where X' is the image $p(U)$ in X; this X' is a big open subset of X because p is a bundle map.

Let $\phi := f \circ \sigma_Q$, and $U_s = U \cap Z_s$. Consider the relative differential

$$D\phi : \mathcal{T}_{Z/X}|_U \longrightarrow \phi^*\mathcal{T}_{E(G/Q)/X}. \tag{4.9}$$

We will now show that $(D\phi)|_{U_s} \neq 0$ for a general $s \in Y'$.

If $D\phi = 0$ for a general s, then ϕ is constant on the fiber, hence ϕ factors through a morphism $\rho : X' \longrightarrow E(G/Q)|_{X'}$. This produces a reduction E_Q of $E|_{X'}$ to the maximal parabolic Q. It is a Higgs reduction because for each $s \in S'$,

$$\theta|_{U_s} : \mathcal{O}_{U_s} \longrightarrow \big(E(\mathfrak{g}) \otimes \Omega^1_X\big)\big|_{U_s}$$

factors through

$$\theta_Q|_{Z_s \cap U} : \mathcal{O}_{Z_s \cap U} \longrightarrow \big(E(\mathfrak{q})|_{Z_s \cap U} \otimes \Omega^1_{Z_s \cap U}\big),$$

where $\mathfrak{q}$ is the Lie algebra of Q. One can easily check that (E_Q, Θ_Q) contradicts the Higgs semistability of (E, θ). Hence for a general s, we have $D\phi(s) \neq 0$.

Now by Lemma 3.1(1),

$$\mu_{\min}(\mathcal{T}_{Z/X}|_{U_s}) \leqslant \mu_{\max}\big(\phi^* \mathcal{T}_{\mathcal{F}(G/Q)/X}|_{U_s}\big). \tag{4.10}$$

The theorem will be proved by analyzing the two sides of this inequality.

First we will calculate the left-hand side of the inequality in (4.10). For each $i \in [1, l]$, the Koszul complex associated to the evaluation map

$$e_i : H^0\big(X, \mathcal{O}_X(m_i)\big) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(m_i)$$

gives a surjection

$$\left(\bigwedge^2 H^0\big(X, \mathcal{O}_X(m_i)\big)\right) \otimes \mathcal{O}_X(-m_i) \longrightarrow \mathcal{K}_{m_i} = \operatorname{Ker}(e_i). \tag{4.11}$$

Composing (4.11) with the Euler exact sequence in (4.4) and restricting it to Z_s we get a surjective morphism

$$\bigoplus_{i=1}^{l} \left(\bigwedge^2 \big(H^0\big(\mathcal{O}_Z(m_i)\big) \otimes \mathcal{O}_Z(-m_i)\big)\right)\Bigg|_{Z_s} \longrightarrow \mathcal{T}_{Z/X}|_{Z_s}. \tag{4.12}$$

Note that,

$$\begin{aligned}\mu_{\min}\left(\left(\bigwedge^2 H^0(\mathcal{O}_{Z(m_i)}) \otimes \mathcal{O}_Z(-m_i)\right)\right)\Bigg|_{Z_s} &= \min_{1 \leqslant i \leqslant l}\{-m_i\} \cdot \deg(Z_s) \\ &= -\max_{1 \leqslant i \leqslant l}\{m_i\} \prod_{i=1}^{l} m_i \deg(X).\end{aligned}$$

Hence by Lemma (3.1)(2) we have

$$\mu_{\min}(T_{Z/X}|_{Z_s}) \geqslant -\max_{1 \leqslant i \leqslant l}\{m_i\} \prod_{i=1}^{l} m_i \deg(X). \tag{4.13}$$

Now we will calculate right-hand side of the inequality in (4.10). Denote the principal bundle $\mathcal{F}|_{U_s}$ on U_s by $\mathcal{F}^s$, denote the reduction $\mathcal{F}_P|_{U_s}$ by $\mathcal{F}^s_P$. Denote the extension of structure group of the principal P-bundle $\mathcal{F}^s_P$ to Q (respectively, L) by $\mathcal{F}^s_Q$ (respectively, $\mathcal{F}^s_L$). We have

$$\phi^* \mathcal{T}_{\mathcal{F}^s(G/Q)/X} \cong \mathcal{F}^s_Q(\mathfrak{g}/\mathfrak{q}) \cong \mathcal{F}^s_P(\mathfrak{g}/\mathfrak{q}).$$

Since P acts on $\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q}$, there is a well-defined Higgs subbundle

$$\mathcal{F}^s_P\big((\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q})/\mathfrak{q}\big) \subset \mathcal{F}^s_P(\mathfrak{g}/\mathfrak{q}).$$

We will show that this is the maximal destabilizing Higgs subbundle of $\mathcal{F}^s_P(\mathfrak{g}/\mathfrak{q})$. Note that

$$\mathcal{F}^s_P\big(\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q}\big) \cong \mathcal{F}^s_L\big(\mathfrak{g}^{-\bar{\alpha}}\big).$$

Since Z_L acts on $\mathfrak{g}^{-\bar{\alpha}}$ by multiplication of scaler, $\mathcal{F}^s_L(\mathfrak{g}^{-\bar{\alpha}})$ is Higgs semistable (see [4, Lemma 13]).

Now we will prove that $\mathcal{F}^s_L(\mathfrak{g}^{-\bar{\alpha}})$ is the maximal Higgs subbundle of $\mathcal{F}^s_P(\mathfrak{g}/\mathfrak{q})$ with largest slope.

There is a maximal P invariant flag (see [3, p. 783, line 13])

$$\frac{\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q}}{\mathfrak{q}} \subset \frac{\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{g}^{-\bar{\beta}_2} \oplus \mathfrak{q}}{\mathfrak{q}} \subset \cdots \subset \frac{\mathfrak{g}}{\mathfrak{q}},$$

where $\bar{\alpha}$, $\bar{\beta}_i$'s are Z_L roots as described in [3, p. 783, line 6].

The above filtration induces a filtration of Higgs subbundles

$$\mathcal{F}^s_P\left(\frac{\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q}}{\mathfrak{q}}\right) \subset \mathcal{F}^s_P\left(\frac{\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{g}^{-\bar{\beta}_2} \oplus \mathfrak{q}}{\mathfrak{q}}\right) \subset \cdots \subset \mathcal{F}^s_P\left(\frac{\mathfrak{g}}{\mathfrak{q}}\right).$$

Each successive quotient in the above filtration is isomorphic to $\mathcal{F}^s_L(\mathfrak{g}^{\bar{\beta}_i})$ which is again Higgs semistable by [4, Lemma 13]. Since $\bar{\beta}_i - \bar{\beta}_1$ is a non-positive linear combination of simple roots, and $\mathcal{F}_L$ is Higgs semistable, we conclude that $\deg(\mathcal{F}^s_L(\chi_{a(\bar{\beta}_i - \bar{\beta}_1)})) < 0$ for some positive integer a. From [3, Lemma 4.3] we get that

$$\mu\big(\mathcal{F}^s_L\big(\mathfrak{g}^{\bar{\beta}_i}\big)\big) = \frac{\deg(\mathcal{F}_L(\chi_{a(\bar{\beta}_i - \bar{\beta}_1)}))}{a} + \mu\big(\mathcal{F}^s_L(\bar{\beta}_1)\big) < \mu\big(\mathcal{F}^s_L\big(\mathfrak{g}^{\bar{\beta}_1}\big)\big).$$

Hence using Lemma 3.2 we conclude that $\mathcal{F}^s_P(\mathfrak{g}^{-\bar{\alpha}} \oplus \mathfrak{q})$ is the maximal Higgs destabilizing subsheaf of $\mathcal{F}^s_L(\mathfrak{g}^{\bar{\alpha}})$. By [3, Lemma 4.3],

$$\mu_{\max}\big(\phi^*(\mathcal{T}_{\mathcal{F}^s(G/Q)/X})\big) = \mu_{\max}\big(\mathcal{F}^s_P(\mathfrak{g}/\mathfrak{q})\big) = \mu\big(\mathcal{F}^s_L\big(\mathfrak{g}^{-\bar{\alpha}}\big)\big) = -\mu\big(\mathcal{F}^s_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big). \tag{4.14}$$

Hence using (4.10), (4.13) and (4.14) we have,

$$\begin{aligned} 0 < \mu\big(\mathcal{F}^s_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big) &= -\mu_{\max}(\phi^*(\mathcal{T}_{\mathcal{F}^s(G/Q)/X}|_{U_s}) \leqslant -\mu_{\min}(\mathcal{T}_{Z/X}|_{U_s}) \\ &\leqslant \max\{a_i\} \prod_{i=1}^{l} a_i \deg(X). \end{aligned} \tag{4.15}$$

This completes the proof of the theorem. □

5. Flenner's theorem

Theorem 5.1. *Let $a \in \mathbb{N}$ be such that*

$$\frac{\binom{a+\dim(X)}{a} - l \cdot a - 1}{a} > \deg(X)\frac{\dim \mathfrak{g} - \dim \mathfrak{t}}{2}. \tag{5.1}$$

If (E, θ) is a semistable Higgs G-bundle, then the restriction $(E, \theta)|_{D_1 \cap \cdots \cap D_l}$ to a general complete intersection with $D_i \in |\mathcal{O}_X(a)|$ is Higgs semistable.

Proof. Let (E, θ) be a semistable Higgs G-bundle. Assume that the restriction of (E, θ) to a general complete intersection $Z_s = D_1 \cap \cdots \cap D_l$ is not Higgs semistable. Consider

$$(\mathcal{F}, \Theta) := q^*(E, \theta),$$

where q is the projection in (4.5). It is a Higgs G-bundle on the family Z such that for each $s \in S$, the restriction $(\mathcal{F}, \Theta)|_{Z_s}$ is a Higgs G-bundle on the complete intersection $Z_s \subset X$ which is isomorphic to $(E, \theta)|_{Z_s}$. By Lemma 3.4, there is an open subset $S' \subset S$ and an open subset $U \subset Z$ with a Higgs reduction $(\mathcal{F}_P, \Theta_P)$ of $(\mathcal{F}, \Theta)|_U$ to a parabolic subgroup $P \subset G$, such that for each $s \in S'$, the induced reduction $(\mathcal{F}_P, \Theta_P)|_{U \cap Z_s}$ is the Harder–Narasimhan reduction of $(E, \theta)|_{Z_s}$.

By (4.15),

$$-\mu_{\min}(\mathcal{T}_{Z/X}|_{Z_s}) \geqslant \mu\big(F_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big). \tag{5.2}$$

By (4.7) we have

$$\det\big(F_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big) = q^*(L_1) \otimes p^*(L_2)$$

with $L_1 \in \operatorname{Pic}(X)$ and $L_2 \in \operatorname{Pic}(S)$. This implies that $\deg(F_L^s(\mathfrak{g}^{\bar{\alpha}})) = a^c \deg(L_1)$. Since $(\mathcal{F}_P, \Theta_P)|_{U \cap Z_s}$ is the Harder–Narasimhan reduction of $(E, \theta)|_{Z_s}$, it follows that $\deg(L_1) \geqslant 1$. Hence we have,

$$\mu\big(F_L\big(\mathfrak{g}^{\bar{\alpha}}\big)\big) = \frac{\deg(F_L(\mathfrak{g}^{\bar{\alpha}}))}{\dim \mathfrak{g}^{\bar{\alpha}}} \geqslant \frac{a^c}{\dim(\mathfrak{g}^{\bar{\alpha}})} \geqslant \frac{2a^c}{\dim \mathfrak{g} - \dim \mathfrak{t}}. \tag{5.3}$$

In the proof of Flenner's theorem for vector bundles (see [6, Theorem 7.1.1, (7.1)]) it is shown that

$$\frac{a^{c+1}}{\binom{a+n}{a} - l \cdot a - 1} \deg(X) \geqslant -\mu_{\min}(\mathcal{T}_{Z/X}|_{Z_s}). \tag{5.4}$$

Combining (5.2), (5.3) and (5.4) we contradict (5.1). Hence the restriction $(E, \theta)|_{Z_s}$ is Higgs semistable for a general complete intersection subvariety $Z_s \subset X$. $\square$

References

[1] B. Anchouche, I. Biswas, Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, Amer. J. Math. 123 (2001) 207–228.

[2] W.P. Barth, Some properties of stable rank-2 vector bundles on $\mathbb{P}_n$, Math. Ann. 226 (1977) 125–150.

[3] I. Biswas, T.L. Gómez, Restriction theorems for principal bundles, Math. Ann. 327 (2003) 773–792.

[4] A. Dey, R. Parthasarathi, On Harder–Narasimhan reductions for Higgs principal bundles, Proc. Indian Acad. Sci. Math. Sci. 115 (2005) 127–146.

[5] O. Forster, A. Hirschowitz, M. Schneider, Type de scindage généralisé pour les fibrés stables, in: Vector Bundles and Differential Equations, Proc. Conf., Nice, 1979, in: Progr. Math., vol. 7, Birkhäuser, Boston, MA, 1980, pp. 65–81.

[6] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, Aspects Math., vol. E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[7] M. Maruyama, Boundedness of semistable sheaves of small ranks, Nagoya Math. J. 78 (1980) 65–94.

[8] A. Ramanathan, Stable principal bundles on a compact Riemann surface, Math. Ann. 213 (1975) 129–152.

[9] C.T. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. 75 (1992) 5–95.

[10] Carlos T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, Inst. Hautes Études Sci. Publ. Math. 79 (1994) 47–129.

[11] H. Spindler, Der Satz von Grauert–Mülich für beliebige semistabile holomorphe Vektorbündel über dem n-dimensionalen komplex-projektiven Raum, Math. Ann. 243 (1979) 131–141.