REGULARIZATION OF LINEAR ILL-POSED PROBLEMS INVOLVING MULTIPLICATION OPERATORS

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ABSTRACT. We study regularization of ill-posed equations involving multiplication operators when the multiplier function is positive almost everywhere and zero is an accumulation point of the range of this function. Such equations naturally arise from equations based on non-compact self-adjoint operators in Hilbert space, after applying unitary transformations arising out of the spectral theorem. For classical regularization theory, when noisy observations are given and the noise is deterministic and bounded, then non-compactness of the ill-posed equations is a minor issue. However, for statistical ill-posed equations with non-compact operators less is known if the data are blurred by white noise. We develop a regularization theory with emphasis on this case. In this context, we highlight several aspects, in particular we discuss the intrinsic degree of ill-posedness in terms of rearrangements of the multiplier function. Moreover, we address the required modifications of classical regularization schemes in order to be used for non-compact statistical problems, and we also introduce the concept of the effective ill-posedness of the operator equation under white noise. This study is concluded with prototypical examples for such equations, as these are deconvolution equations and certain final value problems in evolution equations.

1. Introduction, background

This study is devoted to multiplication operators in the context of ill-posed linear operator equations

$$\mathrm{A}\, x = y \tag{1}$$

with bounded self-adjoint positive operators $\mathrm{A}\colon H \to H$ mapping in the (separable) Hilbert space H and possessing a non-closed range $\mathcal{R}(A)$.

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We fix a measure space (S, Σ, μ) on the set S with a *σ-finite* measure μ defined on the σ-algebra Σ, and consider instead of (1) the equation

$$b(s)f(s) = g(s), \quad s \in S \tag{2}$$

in the setting of the Hilbert space $L^2(S, \Sigma, \mu)$. In particular, the equation above is to hold μ-almost everywhere (μ-a.e.). This multiplication setup is prototypical based on the following version of the *Spectral Theorem.*

Fact (see e.g. [7] for a detailed discussion)**.** *For every bounded self-adjoint operator* $\mathrm{A}\colon H \to H$ *there is a σ-finite measure space* (S, Σ, μ)*, a real-valued essentially bounded function* $b \in L^\infty(S, \Sigma, \mu)$ *and an isometry* $U\colon H \to L^2(S, \Sigma, \mu)$ *such that* $U \mathrm{A} U^{-1} = M_b$*, where* M_b *is the multiplication operator, assigning* $f \in L^2(S, \Sigma, \mu) \mapsto b \cdot f \in L^2(S, \Sigma, \mu)$*.*

Since within this study the linear operator A is assumed to be bounded self-adjoint and positive, we have a constant $\overline{b} > 0$ such that the multiplier function b in (2) obeys the inequalities $0 < b(s) \leq \overline{b} < \infty$ for almost all $s \in S$. However, as a consequence of the ill-posedness of equation (1), which implies that zero is an accumulation point of the spectrum of A, the function b must have essential zeros, which means that $\operatorname{essinf}_{s\in S} b(s) = 0$.

So, the recovery of the element $x \in H$ in (1) from noisy data

$$y^\delta := \mathrm{A}\, x + \delta\eta \tag{3}$$

of the right-hand side y carries over to the reconstruction of the solution $f(s)$, $s \in S$, of equation (2) from noisy data

$$g^\delta := Uy^\delta = b \cdot (Ux) + \delta(U\eta) \tag{4}$$

of the right-hand side $g = U(Ax)$. The variable η turns to the noise $\xi := U\eta$, and further properties will be given in Definitions 4 and 5 below. The analysis will be different for bounded deterministic noise and for statistical white noise.

Thus, we consider the reconstruction of the function f in the Hilbert space $L^2(S, \Sigma, \mu)$, from the knowledge of the noisy data

$$g^\delta(s) := b(s)f(s) + \delta\xi(s), \quad s \in S, \tag{5}$$

where $b \in L^\infty(S, \Sigma, \mu)$ is given, and we assume that $\delta > 0$ denotes the noise level.

Statistical inverse problems under *white noise* and the reduction to multiplication problems as (5) were discussed in [2]. Multiplier equations as in (2) were studied in [8] from the regularization point of view for $S = (0, 1)$ and μ being the Lebesgue measure, which we throughout will denote by λ.

The outline of the remainder of this paper is as follows: In Section 2 we shall describe the framework for the analysis, and we will also provide several auxiliary results that might be of interest. Section 3 is devoted to the error analysis. The main results, presented in Propositions 3 & 4, yield estimations from above of the regularization error under bounded deterministic and white noise, respectively. Finally, Section 4 exhibits that the well-known (and typically non-compact) deconvolution and final value problems fit the considered framework after turning from operator equations to the current setup by means of the Fourier transform.

2. Notation and auxiliary results

In this section we shall first discuss the impact of properties of the multiplier function b on the intrinsic difficulty of the inverse problem. Then we turn to introducing the concept of solution smoothness. Finally, we introduce the concept of regularization schemes.

2.1. Degree of ill-posedness incurred by the multiplier function b.

As was mentioned in Section 1 ill-posedness of the multiplication problem (2) is a consequence of having zero as accumulation point of the essential range of b. For the character of ill-posedness, however, the location of the essential zeros of the function b should not be relevant. Therefore, a normalization of the function b in (2) is desirable. In this context, the increasing rearrangement of b was considered in the study [8]. For such setting we let b^* denote the increasing rearrangement

$$b^*(t) := \sup \{\tau : \mu(\{s : b(s) \le \tau\}) \le t\}, \quad t > 0.$$

However, this approach is limited to underlying S and μ with finite measure values $\mu(S)$.

Another normalization is the decreasing rearrangement b_* of the multiplier function b, which is based on the distribution function d_b, defined by $d_b(t) := \mu\{s \in S : b(s) > t\}$ for $t > 0$. Then we let the decreasing rearrangement of b be given as

$$b_*(t) := \inf \{\tau > 0 : d_b(\tau) \le t\}, \quad 0 < t < \mu(S).$$

Note that b_* is defined on $[0, \mu(S))$, equipped with the Lebesgue measure λ. In the context of ill-posed equations this normalization was first used in [6].

We also notice that for infinite measures, that is, for $\mu(S) = \infty$, the function b_* may have infinite value. Therefore we confine to the case of b satisfying the following assumption.

Assumption 1. *If $\mu(S) = \infty$ then the function b is assumed to* vanish at infinity*, in the sense that $\mu\{s \in S : b(s) > t\}$ is finite for every $t > 0$.*

Remark 1. Assumption 1 is important to guarantee the existence of a decreasing rearrangement b_* which is equimeasurable with b, meaning that it has the same distribution function as b, i.e.,

$$\lambda(\{\tau : b_*(\tau) > t\}) = \mu(\{s \in S : b(s) > t\}), \quad t > 0.$$

The characterization of cases when the function b_* is equimeasurable with b was first given by Day in [4], and for infinite measures $\mu(S) = \infty$ the Assumption 1 is known to be sufficient to guarantee this, see [3, Chapt. VII] for details. For calculus with decreasing rearrangements we also refer to [1, Chapt. 2]. Moreover, functions vanishing at infinity are important in Analysis, see [11, Chapt. 3].

For the subsequent analysis we shall first assume that our focus is on the Lebesgue measure $\mu = \lambda$, either on $[0, \infty)$ for the decreasing rearrangement, or on some bounded interval $[0, a]$ for the increasing rearrangement. In such case, Σ denotes the corresponding Borel σ-algebra. In fact, one may take $a := \|b\|_\infty$. The corresponding analysis extends to measures $\mu \ll \lambda$ with density $\frac{d\mu}{d\lambda}$ which obey $0 < \underline{c} \leq \frac{d\mu}{d\lambda} \leq \bar{C} < \infty$. If this is the case then it is easily seen that the increasing rearrangements b^*_μ and b^*_λ of b corresponding to μ and λ, respectively, satisfy

$$b^*_\mu(\underline{c}t) \leq b^*_\lambda(t) \leq b^*_\mu(\bar{C}t), \quad t > 0,$$

Similar argumants apply to the decreasing rearrangement. Thus, the asymptotic results as these will be established for the Lebesgue measure λ find their counterparts for other measures μ.

The first observation concerns the decreasing rearrangement. For $M > 0$ we assign the truncated function $\tilde{b}_M(t) := b(t)\chi_{(M,\infty)}(t)$, and the shifted (to zero) version $b_M(t) := \tilde{b}_M(t + M)$.

Proposition 1. *We have that*

$$\left(b_M\right)_*(t) = \left(\tilde{b}_M\right)_*(t) \leq b_*(t), \quad t > 0.$$

Proof. The equality is a result of the translation invariance of the Lebesgue measure, and the inequality follows from the fact that the decreasing rearrangement is order-preserving. □

Thus, the decreasing rearrangement does not take into account any zeros which are present on bounded domains. Only the behavior at infinity is reflected.

For the increasing rearrangement we shall follow a constructive approach. Here we shall assume that the function b is piece-wise continuous and has finitely many zeros. Specifically, we assume a representation

$$b(s) = \sum_{j \in A^+} b_j(s - s_j) + \sum_{j \in A^-} b_j(s_j - s), \quad 0 \le s \le a, \tag{6}$$

where

(i) the reals s_j $(j = 1, \dots, m)$ are the (distinct) locations of the zeros,
(ii) for each $j = 1, \dots, m$ we have that $b_j(0) = 0$, and there is a neighborhood of zero $[0, a_j)$ such that
   - $b_j \colon [0, a_j] \to \mathbb{R}^+,\ j = 1, \dots, m$ is *continuous and strictly increasing.*

   The function $\bar{b}$ satisfies essinf $\bar{b} > 0$.
(iii) The set $A = \{1, \dots, m\} = A^+ \sqcup A^-$ is decomposed into two disjoint subsets A^+ and A^-, possible empty, and
(iv) there is one function, say b_k such that its inverse b_k^{-1} dominates[1] all other functions b_j^{-1}, i.e., $b_j^{-1} \preceq b_k^{-1}$.

Thus, the function b is a superposition of a function bounded away from zero, and of increasing and decreasing parts. We also stress that domination as in item (iv) does not extend from functions f^{-1}, g^{-1} to the inverse functions f, g unless additional assumptions are made, we refer to [10] for a discussion.

Under the above assumptions we state the following result.

Proposition 2. *Let the function b be as in the equation (6), and let b_k be the function from item (iv) above. Then there is a constant $C \ge 1$ such that*

$$b_k(s) = (b_k)^*(s) \ge b^*(s) \ge (b_k)^* \left(\frac{s}{Cm} \right) = b_k \left(\frac{s}{Cm} \right),$$

for sufficiently small $s > 0$.

Proof. Clearly, the function b_k is increasing near zero, such that it coincides with its increasing rearrangement, which explains the outer equalities.

[1] We say that a (non-negative) function g dominates f, and write $f \preceq g$, if there are a neighborhood $[0, \varepsilon)$ and a constant $k > 0$ such that $f(t) \le k g(t),\ 0 \le t \le \varepsilon$.

To establish the inner inequalities we argue as follows. Recall that we need to control

$$\lambda(b \leq \tau) = \lambda\left(\bar{b}(s) + \sum_{j\in A^{+}} b_j(s - s_j) + \sum_{j\in A^{-}} b_j(s_j - s) \leq \tau\right).$$

If $\tau > 0$ is small enough then, by item (ii) the contribution of $\bar{b}$ is neglected, and the sub-level sets $\{b_j \leq \tau\}$ $(j = 1, \ldots, m)$ are disjoint intervals, which in turn yields

$$\begin{aligned}\lambda(b \leq \tau) &= \sum_{j\in A^{+}} \lambda(b_j(s - s_j) \leq \tau) + \sum_{j\in A^{-}} \lambda(b_j(s_j - s) \leq \tau)\\ &= \sum_{j\in A^{+}} b_j^{-1}(\tau) + \sum_{j\in A^{-}} b_j^{-1}(\tau)\\ &= \sum_{j\in A} b_j^{-1}(\tau).\end{aligned}$$

By the domination assumption from item (iv) we find a constant $C \geq 1$ such that

$$b_k^{-1}(\tau) \leq \sum_{j\in A} b_j^{-1}(\tau) \leq Cm b_k^{-1}(\tau). \tag{7}$$

Now, asking for the sup over all $\tau > 0$ such that $\lambda(b \leq \tau) \leq s$ we find that

$$b_k(s) \geq b^{*}(s) \geq b_k\left(\frac{s}{mC}\right),$$

for sufficiently small $s > 0$. This completes the proof. □

The above proposition asserts (heuristically) that the part in the decomposition of b in (6) which has the highest order zero determines the asymptotics of the increasing rearrangement.

2.2. **Solution smoothness.** In order to quantify the error bounds, we need to specify the way in which the solution smoothness will be expressed. This is given in terms of source conditions based on index functions. Here, and throughout, by an *index function*, we mean a strictly increasing continuous function $\varphi : (0, \infty) \to [0, \infty)$ such that $\lim_{t\to+0} \varphi(t) = 0$.

Definition 1 (source condition)**.** *A function $f \in L^2(S, \Sigma, \mu)$ obeys a source condition with respect to the index function φ and the multiplier function b if*

$$f(s) = [\varphi(b)](s)v(s) := \varphi(b(s))v(s), \quad s \in S, \quad \mu - a.e.$$

with $\|v\|_{L^2(S,\Sigma,\mu)} \leq 1$.

Remark 2. Here, we briefly discuss the meaning of source conditions as given in Definition 1. Let the operator A and the multiplication operator M_b be related as described in the introduction. Then it is clear from the relation between the noisy data representations (3) and (4), that a source condition $f = Ux = \varphi(b)v,\ \|v\|_{L^2(S,\Sigma,\mu)} \le 1$ yields the representation $x = \varphi(A)w$ with $w := U^{-1}v \in H$ and $\|w\|_H = \|v\|_{L^2(S,\Sigma,\mu)} \le 1$. This is the standard form of general smoothness in terms of source conditions, given with respect to the forward operator A from (3), see [13, 14].

It is interesting to relate this to 'classical' smoothness in the sense of Hilbertian Sobolev spaces $H^p(S,\mu)$. Precisely, for smoothness parameter $p > 0$ we let $H^p(S,\mu)$ be the Hilbert space of all functions $f\colon S \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int_S |f(s)|^2 \left(1 + |s|^2\right)^p \, d\mu(s) \right)^{1/2} < \infty.$$

The question is, under which conditions this type of smoothness can be expressed in terms of source conditions as in Definition 1. We start with the following technical result.

Lemma 1. *Suppose that $\mu(S) = \infty$ and that the function b vanishes at infinity (cf. Assumption 1). Moreover, let there exist positive constants $M < \infty$ and $c > 0$ such that*

$$b(s) \ge c, \quad \text{for} \quad |s| \le M$$

and

$$\mu\left(\{x,\ b(x) > b(s)\}\right) \asymp |s|, \quad \text{for } |s| > M. \tag{8}$$

Then the function φ_*, given for sufficiently small $t > 0$ as

$$\varphi_*(t) := \frac{1}{\mu\left(\{x,\ b(x) > t\}\right)}, \tag{9}$$

constitutes an index function. Moreover, we have the asymptotics

$$\varphi_*(b(s)) \asymp \frac{1}{|s|} \quad \text{as} \quad |s| \to \infty. \tag{10}$$

Example 1 (power-type decay on $[0,\infty)$)**.** For $\kappa > 0$ we consider the functions $f \in L^2([0,\infty),\mu)$ with Lebesgue measure μ defined as

$$b(s) := \frac{1}{1 + s^{1/\kappa}}, \qquad 0 \le s < \infty.$$

Then the assumptions of Lemma 1 are fulfilled, and for sufficiently small $t > 0$, we have in this case

$$\mu\left(\{x,\ b(x) > t\}\right) = \left(\frac{1-t}{t}\right)^{\kappa}$$

and hence $\varphi_*(t) \asymp t^{\kappa}$ as $t \to +0$.

Corollary 1. *Under the assumptions of Lemma 1 consider the function φ_* as in (9). The function f belongs to $H^p(S,\mu)$ if and only if it obeys a source condition with respect to (a multiple of) the function $\varphi_*^p(t),\ t > 0$.*

Proof. Under the assumptions of Lemma 1 for the function b there are constants $0 < c < C < \infty$ such that

$$c \leq \inf_{s\in S}\left(1+|s|^2\right)\varphi_*^2\left(b(s)\right) \leq \sup_{s\in S}\left(1+|s|^2\right)\varphi_*^2\left(b(s)\right) \leq C. \tag{11}$$

This is easily seen for $|s| \leq M$, as given in Lemma 1. For $|s| > M$ we see that

$$\left(1+|s|^2\right)\varphi_*^2\left(b(s)\right) \asymp \left(1+|s|^2\right)|s|^{-2}.$$

But for $|s| \geq M$ we have that $1 \leq \left(1+|s|^2\right)|s|^{-2} \leq \left(1+M^2\right)M^{-2}$, where the right hand side bound follows from the monotonicity of $x \mapsto (1+x)/x,\ x > 0$, and this proves (11). Now, suppose that $f \in H^p(S,\mu)$. Consider the element $w(s) := \frac{f(s)}{\varphi_*^p(b(s))}$, where φ_* is as above. It is enough to show that $w \in L^2(S,\mu)$, i.e., that it serves as a source element. We have that

$$\begin{aligned}\int_S |w(s)|^2\ \mu(s) &\leq \int |f(s)|^2\left(1+|s|^2\right)^p\ d\mu(s)\sup_{s\in S}\frac{1}{\left(1+|s|^2\right)^p|\varphi_*^p(b(s))|^2}\\ &= \|f\|_p^2\sup_{s\in S}\frac{1}{\left[\left(1+|s|^2\right)|\varphi_*^2(b(s))|\right]^p},\end{aligned}$$

and the latter is finite by (11). On the other hand, under a source condition for f we can bound

$$\int_S |f(s)|^2\left(1+|s|^2\right)^p\ d\mu(s) \leq \|w\|_{L^2(S,\Sigma,\mu)}\sup_{s\in S}\left[\left(1+|s|^2\right)\left|\varphi_*^2(b(s))\right|\right]^p,$$

where the supremum is again bounded by (11). The proof is complete. □

2.3. **Regularization.** For reconstruction of $f(s)$, $s \in S$, we shall use *regularization schemes* $\Phi_\alpha\colon [0,\infty) \to \mathbb{R}^+$, parametrized by $\alpha > 0$, see e.g. [13].

Definition 2 (regularization scheme)**.** *A family* (Φ_α) *of real valued Borel-measurable functions* $\Phi_\alpha(t),\ t \geq 0, \alpha > 0,$ *is called a regularization if there are constants* $C_{-1} > 0$ *and* $C_0 \geq 1$ *such that*

(I) *for each* $t > 0$ *we have* $t\Phi_\alpha(t) \to 1$ *as* $\alpha \to 0,$
(II) $|\Phi_\alpha(t)| \leq \frac{C_{-1}}{\alpha}$ *for* $\alpha > 0,$ *and*
(III) *the function* $R_\alpha(t) := 1 - t\Phi_\alpha(t),$ *which is called a* residual function*, satisfies* $|R_\alpha(t)| \leq C_0$ *for all* $t \geq 0$ *and* $\alpha > 0.$

For the case of statistical noise, additional assumptions have to be made. These will be introduced and discussed later.

We apply a regularization (Φ_α) to a function b in the way

$$[\Phi_\alpha(b)](s) := [\Phi_\alpha \circ b](s) = \Phi_\alpha(b(s)), \quad s \in S.$$

Having chosen a regularization Φ_α, and given data g^δ we consider the function

$$f_\alpha^\delta(s) := [\Phi_\alpha(b)](s) g^\delta(s), \quad s \in S, \tag{12}$$

or in short $f_\alpha^\delta := \Phi_\alpha(b) g^\delta$, as a candidate for the approximate solution.

For the subsequent error analysis the following property of a regularization proves important, again we refer to [13, 16].

Definition 3 (qualification)**.** *Let* φ *be any index function. A regularization* (Φ_α) *is said to have qualification* φ *if there is a constant* $C_\varphi > 0$ *such that*

$$\sup_{t\geq 0} |R_\alpha(t)|\, \varphi(t) \leq C_\varphi \varphi(\alpha), \quad \alpha > 0.$$

Example 2 (spectral cut-off)**.** Let the regularization be given as

$$\Phi_\alpha^{\text{c-o}}(t) := \begin{cases} \frac{1}{t}, & t > \alpha \\ 0, & else. \end{cases}$$

This obeys the requirements of regularization with $C_{-1} = 1$. It has arbitrary qualification. That is, for any index function, the requirement in Definition 3 will be satisfied. The corresponding residual function is $R_\alpha = \chi_{\{t:\ t\leq\alpha\}}$, so that for a function b on S, $R_\alpha(b) = \chi_{\{s:\ b(s)\leq\alpha\}}$.

Example 3 (Lavrent'ev regularization)**.** This method corresponds to the function

$$\Phi_\alpha(t) := \frac{1}{t+\alpha}, \quad t \geq 0,\, \alpha > 0.$$

Lavrent'ev regularization is known to have at most 'linear' qualification. More generally, index functions $\varphi(t) := t^\nu,\ t > 0$, are qualifications whenever the exponent ν satisfies $0 < \nu \leq 1$.

For infinite measures μ, and under white noise, it will be seen that it is important that the regularization (Φ_α) will vanish for small $0 \leq t \leq \alpha$ This is formalized in

Assumption 2. *For each $\alpha > 0$ the function Φ_α vanishes on the set $\{t \geq 0 : \; t \leq \alpha\}$.*

This assumption holds true for spectral cut-off, but it is not fulfilled for most other regularizations. However, we can modify any regularization to obey Assumption 2.

Lemma 2. *Let (Φ_α) be any regularization with constants C_{-1} and C_0. Assign*

$$\tilde{\Phi}_\alpha(t) := \chi_{(\alpha,\infty)}(t)\Phi_\alpha(t), \quad t > 0.$$

Then $\left(\tilde{\Phi}_\alpha\right)$ is a regularization scheme with same constants C_{-1} and C_0. Moreover, an index function φ is a qualification of (Φ_α) if and only if it is a qualification of $\left(\tilde{\Phi}_\alpha\right)$ with constant $\tilde{C}_\varphi = \max\{C_\varphi, C_0\}$.

Proof. We verify the properties. For $t > \alpha$ the regularizations $\tilde{\Phi}_\alpha$ and Φ_α coincide, thus item (I) holds true. Also, $\left|\tilde{\Phi}_\alpha(t)\right| \leq |\Phi_\alpha(t)|$, such that we can let $\tilde{C}_0 := C_0$. Next, it is easy to check that $\tilde{R}_\alpha(t) = \chi_{(a,\infty)}(t)R_\alpha(t) + \chi_{(0,a]}(t)$, which allows us to prove the second assertion, after recalling that $C_0 \geq 1$.

Finally, we bound $\left|\tilde{R}_\alpha(t)\right| \varphi(t)$. Plainly, if $t \leq \alpha$ then

$$\left|\tilde{R}_\alpha(t)\right| \varphi(t) \leq C_0 \varphi(\alpha).$$

Otherwise, for $t > \alpha$ both functions $\tilde{R}_\alpha$ and R_α coincide, This completes the proof. □

Therefore, we may tacitly assume that the regularization of choice is accordingly modified to meet Assumption 2.

3. Error analysis

As stated in the introduction, we shall discuss error bounds, both for the classical setup of bounded deterministic noise, as well as for statistical white noise, to be defined now.

Definition 4 (deterministic noise)**.** *The noise term $\xi = \xi(s)$ is norm-bounded by one, i.e., $\|\xi\|_{L^2(S,\Sigma,\mu)} \leq 1$.*

We shall occasionally adopt the notation $\xi_s := \xi(s)$.

Definition 5 (white noise)**.** *There is some probability space* $(\Omega, \mathcal{F}, P)$ *such that the family* $\{\xi_s\}_{s\in S}$ *constitutes a centered stochastic process*[2] *with* $\mathbb{E}\xi_s = 0$ *for all* $s \in S$*, and* $\mathbb{E}\,|\xi_s|^2 = 1,\ s \in S$*.*

We return to the noisy equation (5). Writing (the unknown) $f_\alpha(s) := [\Phi_\alpha(b)](s)g(s), \quad s \in S$, by (5) and (12), we obtain

$$\begin{aligned} f - f_\alpha^\delta &= [f - \Phi_\alpha(b)(g)] - [\Phi_\alpha(b)(g^\delta) - \Phi_\alpha(b)(g)] \\ &= [f - \Phi_\alpha(b)(bf)] - [\Phi_\alpha(b)(g^\delta) - \Phi_\alpha(b)(g)] \\ &= [I - \Phi_\alpha(b)b]f - \delta\Phi_\alpha(b)\xi. \end{aligned}$$

Thus, we have the decomposition of the error of the reconstruction f_α^δ in a natural way, by using the residual function R_α, as

$$f - f_\alpha^\delta = R_\alpha(b)f - \delta\Phi_\alpha(b)\xi, \tag{13}$$

The term $R_\alpha(b)f$ is completely deterministic, the noise properties are inherent in $\Phi_\alpha(b)\xi$, only.

3.1. **Bounding the bias.** The (noise-free) term $R_\alpha(b)f$ in the decomposition (13) gives rise to the bias, defined as

$$b_f(\alpha) := \|R_\alpha(b)f\|_{L^2(S,\Sigma,\mu)},$$

and it is called the *profile function* in [9]. We shall assume that the solution admits a source condition as in Definition 1, and that the chosen regularization has this as a qualification so that

$$\begin{aligned} \|R_\alpha(b)f\|_{L^2(S,\Sigma,\mu)} &\le \|R_\alpha(b)\varphi(b(s))v(s)\|_{L^2(S,\Sigma,\mu)} \\ &\le \|R_\alpha(b)\varphi(b(s))\|_\infty \|v\|_{L^2(S,\Sigma,\mu)} \le C_\varphi\varphi(\alpha). \end{aligned} \tag{14}$$

We briefly highlight the case when μ is a finite measure. It is to be observed that if $f \in L^\infty(S,\mu)$, then

$$\|R_\alpha(b)f\|_{L^2(S,\Sigma,\mu)} \le \|R_\alpha(b)\|_{L^2(S,\Sigma,\mu)} \|f\|_{L^\infty(S,\mu)}. \tag{15}$$

Example 4. For Lavrent'ev regularization, spectral cut-off, and with function $b(s) := s^\kappa,\ s > 0$, with $\kappa > 0$, we see that

$$\|R_\alpha(b)\|^2_{L^2(S,\Sigma,\mu)} = \begin{cases} \alpha^2 \int (\alpha + s^\kappa)^{-2} d\mu(s), & \text{for Lavrent'ev regularization} \\ \mu\left(\{s :\ s^\kappa \le \alpha\}\right), & \text{for spectral cut-off.} \end{cases}$$

From this, we conclude that for Lavrent'ev regularization the bound in (15) is finite only if $\kappa > 1/2$, whereas for spectral cut-off this holds

[2]For each $s \in S$ we have a random variable $\xi_s \colon \Omega \to \mathbb{R}$.

for all $\kappa > 0$. If μ is the Lebesgue measure λ on $(0,1)$, then we find that

$$\|R_\alpha(b)\|_{L^2((0,1),\Sigma,\lambda)} \leq C\alpha^{1/(2\kappa)}$$

in either case.

A similar bound, relying on Tikhonov regularization was first given in [8, Theorem 4.5].

3.2. **Error under deterministic noise.** Although the focus of this study is on statistical ill-posed problems, we briefly sketch the corresponding result for bounded deterministic noise as introduced in Definition 4. In this case we bound the error, starting from the decomposition (13), by using the triangle inequality, for $\alpha > 0$ as

$$\left\|f - f_\alpha^\delta\right\|_{L^2(S,\Sigma,\mu)} \leq \|R_\alpha(b)f\|_{L^2(S,\Sigma,\mu)} + \delta \left\|\Phi_\alpha(b)\xi\right\|_{L^2(S,\Sigma,\mu)}. \tag{16}$$

Now, using the item (2) in Definition 2, we obtain a bound for the noise term as

$$\delta \left\|\Phi_\alpha(b)\xi\right\|_{L^2(S,\Sigma,\mu)} \leq \delta \sup_{s\geq 0} |\Phi_\alpha(b(s))| \leq C_{-1}\frac{\delta}{\alpha}, \quad \alpha > 0.$$

This together with the estimate (14) for the noise free term gives the following

Proposition 3. *Suppose that the solution f satisfies the source condition as in Definition 1, and that a regularization (Φ_α) is chosen with qualification φ. Then*

$$\left\|f - f_\alpha^\delta\right\|_{L^2(S,\Sigma,\mu)} \leq C_\varphi \varphi(\alpha) + C_{-1}\frac{\delta}{\alpha}, \quad \alpha > 0.$$

The a priori parameter choice $\alpha_* = \alpha_*(\varphi,\delta)$ *from solving the equation*

$$\alpha\varphi(\alpha) = \delta \tag{17}$$

yields the error bound

$$\left\|f - f_\alpha^\delta\right\|_{L^2(S,\Sigma,\mu)} \leq 2\max\left\{C_\varphi, C_{-1}\right\}\varphi(\alpha_*), \tag{18}$$

uniformly for functions f which obey a source condition with respect to the index function φ.

3.3. **Error under white noise.** Here we assume that the underlying noise is as in Definition 5. Thus, since ξ is a random variable, it is a function of $\omega \in \Omega$ so that f_α^δ also a function of $\omega \in \Omega$. Hence, for each fixed $\omega \in \Omega$, from (13) we obtain

$$\begin{aligned} \left\|f - f_\alpha^\delta(\omega)\right\|_{L^2(S,\Sigma,\mu)}^2 &= \|R_\alpha(b)f\|_{L^2(S,\Sigma,\mu)}^2 + 2\delta\langle R_\alpha(b)f, \Phi_\alpha(b)\xi\rangle \\ &\quad + \delta^2 \int_S |\Phi_\alpha(b(s))|^2 \left|\xi_s(\omega)\right|^2 \, d\mu(s). \end{aligned} \tag{19}$$

The error of the regularization Φ_α under white noise is measured in RMS sense, that is, it is defined as

$$e(f, \Phi_\alpha, \delta)^2 := \mathbb{E} \left\| f - f_\alpha^\delta \right\|^2_{L^2(S,\Sigma,\mu)}, \tag{20}$$

where the expectation is with respect to the probability P governing the noise process. From the properties of the noise we deduce from (19) the *bias–variance decomposition*

(21)

$$\mathbb{E} \left\| f - f_\alpha^\delta \right\|^2_{L^2(S,\Sigma,\mu)} = \| R_\alpha(b) f \|^2_{L^2(S,\Sigma,\mu)} + \delta^2 \mathbb{E} \int_S |\Phi_\alpha(b(s))| \, |\xi_s(\omega)|^2 \; d\mu(s).$$

The first summand above, the squared bias, is treated as in § 3.1. It remains to bound the variance, that is, the second summand in (21). By interchanging expectation and integration we deduce that

$$\begin{aligned} \mathbb{E} \int_S |\Phi_\alpha(b(s))| \, |\xi_s(\omega)|^2 \; d\mu(s) &= \int_S |\Phi_\alpha(b(s))|^2 \, \mathbb{E} \, |\xi_s(\omega)|^2 \; d\mu(s) \\ &= \int_S |\Phi_\alpha(b(s))|^2 \; d\mu(s) \end{aligned} \tag{22}$$

For the above identity it is important to have the right hand side finite; that is, $\Phi_\alpha \circ b \in L^2(S, \Sigma, \mu)$.

In the subsequent analysis we shall distinguish the cases of finite measure μ, i.e., when $\mu(S) < \infty$ and the infinite case $\mu(S) = \infty$.

Plainly, if the measure μ is finite then we have from Definition 2 the uniform bound

$$\int_S |\Phi_\alpha(b(s))|^2 \; d\mu(s) \le \frac{C_{-1}^2}{\alpha^2} \mu(S), \quad \alpha > 0.$$

Otherwise, this needs not be the case as highlights the following

Example 5. Consider the multiplication operator

$$g := b \cdot f, \quad f \in L_2(\mathbb{R}, \lambda),$$

where

$$b(s) = \begin{cases} 0, & s < 0, \\ s, & 0 \le s \le 1, \\ 1, & s > 1, \end{cases}$$

with λ denoting the Lebesgue measure on $\mathbb{R}$.

Let (Φ_α) be an arbitrary regularization. From Definition 2 we know that $\Phi_\alpha(1) \to 1$ as $\alpha \to 0$, and hence there is $\alpha_0 > 0$ such that $\Phi_\alpha(1) \ge$

$1/2$ for $0 < \alpha \le \alpha_0$. Therefore, for each $s \ge 1$ and $0 < \alpha \le \alpha_0$ we have that $\Phi_\alpha(b(s)) = \Phi_\alpha(1) \ge 1/2$, and

$$\int_{\mathbb{R}} \left|\Phi_\alpha(b(s))\right|^2 \, d\lambda(s) \ge \int_1^\beta \left|\Phi_\alpha(b(s))\right|^2 \, d\lambda(s) \ge \frac{1}{4}(\beta - 1)$$

for every $\beta > 1$ so that the integral $\int_{\mathbb{R}} \left|\Phi_\alpha(b(s))\right|^2 \, d\lambda(s)$ is not finite. Consequently, the multiplication equation with the above $b(\cdot)$ cannot be solved with arbitrary accuracy (as $\delta \to 0$) under white noise by using any regularization.

Example 6 (Lavrent'ev regularization, continued)**.** Suppose that $\mu(S) = \infty$, and that b vanishes at infinity. For Lavrent'ev regularization we then see that

$$\begin{aligned}\int_S \left|\Phi_\alpha(b(s))\right|^2 \, d\mu(s) &\ge \int_{\{s,\ b(s) \le \alpha\}} \frac{1}{(\alpha + b(s))^2} \, d\mu(s) \\ &\ge \frac{1}{4\alpha^2} \mu\left(\{s,\ b(s) \le \alpha\}\right) = \infty.\end{aligned}$$

However, under Assumption 2 we have that

$$\int_S \left|\Phi_\alpha(b(s))\right|^2 \, d\mu(s) = \int_{\{b > \alpha\}} \left|\Phi_\alpha(b(s))\right|^2 \, d\mu(s),$$

and this will be finite for functions b vanishing at infinity.

We recall the decreasing rearrangement b_* of the multiplier function b. Since both b and b_* share the same distribution function we can use the transformation of measure formula to see for any (measurable) function $H\colon [0, \|b\|_\infty) \to \mathbb{R}$ that

$$\int_{[0,\mu(S))} \left|H(b_*(t))\right|^2 \, d\lambda(t) = \int_S \left|H(b(s))\right|^2 \, d\mu(s)$$

In particular this holds for spectral cut-off as in Example 2, used as $H(s) := \Phi_\alpha^{\text{c-o}}(b(s)\chi_{\{b(s) > \alpha\}}$, yielding

$$\int_{\{b_* > \alpha\}} \frac{1}{\left|b_*(t)\right|^2} \, d\lambda(t) = \int_{\{b > \alpha\}} \frac{1}{\left|(b(s))\right|^2} \, d\mu(s). \tag{23}$$

We now observe that from the definition of regularization functions, see Definition 2 we have for arbitrary regularization Φ_α that $\Phi_\alpha(t) \le \frac{C_0+1}{t}$, and hence that

$$\begin{aligned}\int_{\{b > \alpha\}} \left|\Phi_\alpha(b(s))\right|^2 \, d\mu(s) &\le (C_0 + 1)^2 \int_{\{b > \alpha\}} \frac{1}{\left|(b(s))\right|^2} \, d\mu(s) \\ &= (C_0 + 1)^2 \int_{\{b_* > \alpha\}} \frac{1}{\left|b_*(t)\right|^2} \, d\lambda(t).\end{aligned}$$

This gives rise to the following

Definition 6 (statistical effective ill-posedness)**.** *Suppose that we are given the function $b > 0$ on the measure space $(S, \mathcal{F}, \mu)$. For a function b that vanishes at infinity we call the function*

$$D(\alpha) := \left(\int_{\{b_* > \alpha\}} \frac{1}{|b_*(t)|^2} \, d\lambda(t) \right)^{1/2}, \quad \alpha > 0, \tag{24}$$

the statistical effective ill-posedness of the operator.

Example 7 (Spectral cut-off, counting measure)**.** Suppose that $S = \mathbb{N}$ and μ is the counting measure assigning $\mu(\{j\}) = 1,\ j \in \mathbb{N}$, and that the function $j \mapsto b(j)$ is non-increasing with $\lim_{j\to\infty} b(j) = 0$. Then it vanishes at infinity, and for each $\alpha > 0$ there will be a maximal finite number N_α with $b(N_\alpha) \geq \alpha > b(N_\alpha + 1)$.

In this case the statistical effective ill-posedness evaluates as

$$D(\alpha) = \left(\sum_{j=1}^{N_\alpha} \frac{1}{b_j^2} \right)^{1/2}, \quad \alpha > 0.$$

This corresponds to the 'degree of ill-posedness for statistical inverse problems' as given in [12]. The bias-variance decomposition from (19) is known to be order optimal, cf. [5].

The following bound simplifies the statistical effective ill-posedness, and can often be used.

Lemma 3. *Let Φ_α be any regularization. Under Assumptions 1 and 2 we have that*

$$D(\alpha) \leq \frac{1}{\alpha} \sqrt{\mu(\{s :\ b(s) > \alpha\})},\ \alpha > 0,$$

and

$$\int_S |\Phi_\alpha(b(s))|^2 \ d\mu(s) \leq \frac{C_{-1}^2}{\alpha^2} \mu\left(\{b > \alpha\}\right)$$

Proof. The result follows from Definition 2. □

We summarize the preceding discussion as follows. Suppose that the measure $\mu(S) = \infty$, and that assumptions 1 and 2 hold true. The error decomposition (19) then yields

$$\mathbb{E} \left\| f - f_\alpha^\delta \right\|_{L^2(S,\Sigma,\mu)}^2 \leq \|R_\alpha(b) f\|_{L^2(S,\Sigma,\mu)}^2 + \delta^2 (C_0 + 1)^2 D^2(\alpha), \quad \alpha > 0. \tag{25}$$

Using this we obtain the following analog of Proposition 3.

Proposition 4. *Suppose that the solution f satisfies the source condition as in Definition 1, and that a regularization Φ_α is chosen with qualification φ. Suppose, in addition, that Assumptions 1 and 2 hold. Then, for the case of white noise ξ,*

$$\mathbb{E}\left\|f - f_\alpha^\delta\right\|^2_{L^2(S,\Sigma,\mu)} \leq C_\varphi^2 \varphi(\alpha)^2 + \delta^2 (C_0+1)^2 D^2(\alpha), \quad \alpha > 0.$$

The a priori parameter choice $\alpha_* = \alpha_*(\varphi, D, \delta)$ *from solving the equation*

$$\varphi(\alpha) = \delta D(\alpha) \tag{26}$$

yields the error bound

$$\left(\mathbb{E}\left\|f - f_\alpha^\delta\right\|^2_{L^2(S,\Sigma,\mu)}\right)^{1/2} \leq \sqrt{2}\max\left\{C_\varphi, (C_0+1)\right\}\varphi(\alpha_*). \tag{27}$$

Proof. Follows from Definitions 3 and 6, (21), and (22). □

4. Operator equations in Hilbert space

As outlined in the introduction the setup of multiplication operators as analyzed here is prototypical for general bounded self-adjoint positive operators A: $H \to H$ mapping in the (separable) Hilbert space H due to the associated Spectral Theorem (cf. [7]), stated as **Fact**. It is an advantage of our focus on multiplication operators that we can include *compact* linear operators and *non-compact* ones as well.

Example 8 (Compact operator)**.** It was emphasized in [2] that the case of a compact positive self-adjoint operator A yields a multiplier version with $S = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$, and μ being the counting measure, i.e., $L^2(S,\Sigma,\mu) = \ell^2$, and multiplier $b := (b_j)_{j\in\mathbb{N}}$, where b_j denotes the jth eigenvalue taking into account (finite) multiplicities. White noise in ℓ^2 is given by a sequence of i.i.d. random variables $\xi_1, \xi_2, \dots$ with mean zero and variance one.

In the subsequent discussion we shall highlight the impact of the previous results, presented for equations with multiplication operator for specific operator equations with non-compact operator A.

4.1. Deconvolution.

Suppose that data y^δ are a real-valued function on $\mathbb{R}$ and given as

$$y^\delta(t) = (r * x)(t) + \delta\eta(t), \quad t \in \mathbb{R}. \tag{28}$$

In the above, $(r * x)(t) := \int_{\mathbb{R}} r(u-t)x(u)\,du$ for $t \in \mathbb{R}$. The noise η is assumed to be symmetric around zero and (normalized) weighted white noise $\eta(t) := w(t)dW_t,\ t \in \mathbb{R}$, with a square integrable weight normalized to $\|w\|_{L^2(\mathbb{R})} = 1$. The goal is to find approximately the function $x(t),\ t \in \mathbb{R}$, based on noisy data y^δ. This problem is usually called *deconvolution*.

4.1.1. *Turning to multiplication in frequency space.* In order to transfer the deconvolution task into the multiplication form (2) with noisy data (4) we use the Fourier transform to get

$$g^\delta(s) := \hat{y^\delta}(s) = \hat{r}(s)\hat{x}(s) + \delta\hat{\eta}(s), \quad s \in \mathbb{R}. \tag{29}$$

We make the following assumptions. First, we assume that the kernel function $r \in L^1(\mathbb{R})$ is non-negative, symmetric around zero, and that $u \mapsto r(u),\ u > 0$ is non-increasing. In this case its Fourier transform $b(s) := \hat{r}(s)$ is *non-negative and real valued.* Also, $b \in C_0(\mathbb{R})$, and zero is an accumulation point of the essential range of b. Thereby the corresponding multiplication operator does not have closed range. We denote $f(s) = \hat{x}(s),\ s \in \mathbb{R}$. Then it is easily checked that the Fourier transform $\xi(s) := \hat{\eta}(s)$ is centered Gaussian, and $\mathbb{E}\xi(s)\bar{\xi}(s') = 0$ whenever $s \neq s'$. By the properties of the noise, as described before, the variance is given as

$$\mathbb{E}\left|\xi(s)\right|^2 = \int_{\mathbb{R}} \left|w(u)\right|^2\ du = 1.$$

Thus we arrive at the multiplication equation (29) as in Section 1.

4.1.2. *Relation to reconstruction of stationary time series.* Historically, the deconvolution problem was first studied by Wiener in [17]. In that context the solution f in (5) is a stationary time series $f_s(\omega)$ with (constant) average signal strength $S_f := \mathbb{E}\left|f(s)\right|^2$. Then we may look for a (real valued) multiplier $h(s),\ s \in S$ such that $f^\delta(s) := h(s)g^\delta(s)$ is a MISE estimator, i.e., it minimizes (point-wise) the functional

$$\mathbb{E}_f\mathbb{E}_\xi\left|f^\delta(s) - f(s)\right|^2, \quad s \in S. \tag{30}$$

Assuming that the noise $\xi(s)$ is independent from the signal $f(s)$ the above minimization problem can be rewritten as

$$\mathbb{E}_f\mathbb{E}_\xi\left|f^\delta(s) - f(s)\right|^2 = \left|1 - h(s)b(s)\right|^2 S_f + \left|h(s)\right|^2 \mathbb{E}\left|\xi(s)\right|^2,\ s \in S.$$

The minimizing function $h(s)$ (in the general complex valued case, and with $\bar{b}$ denoting the complex conjugate to b) has the form

$$h(s) := \frac{\bar{b}(s)S_f}{\left|b(s)\right|^2 S_f + \delta^2} = \frac{\bar{b}(s)}{\left|b(s)\right|^2 + \frac{\delta^2}{s_f}}. \tag{31}$$

This approach results in the classical *Wiener Filter*, see [17]. Notice that the quotient $\sqrt{S_f}/\delta$ is the signal-to-noise ratio, a constant which is unknown, thus replacing δ^2/S_f by α we arrive at the reconstruction formula

$$f^\delta(s) := \frac{\bar{b}(s)}{\alpha + |b(s)|^2} g^\delta(s), \quad s \in \mathbb{R},$$

being the analog to Tikhonov regularization.

However, since here we assume b to be real and positive, one may propose the Lavrent'ev approach resulting in

$$f^\delta(s) := \frac{1}{\alpha + b(s)} g^\delta(s), \quad s \in \mathbb{R}, \tag{32}$$

and hence to the regularization scheme $\Phi_\alpha(t) := 1/(\alpha+t),\ \alpha > 0,\ t > 0$ as introduced in § 2. Other regularization schemes also apply.

4.2. **Final value problem.** In the final value problem (FVP), also known as backward heat conduction problem associated with the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Delta u(x,t), \quad x \in \Omega,\ 0 < t < \tau, \tag{33}$$

one would like to determine the initial temperature $f_0 := u(\cdot, 0)$, from the knowledge of the final temperature $f_\tau := u(\cdot, \tau)$. Here, the domain Ω is in $\mathbb{R}^d$. This problem is known to be ill-posed.

It can be considered as an operator equation with multiplication operator. A similar FVP was considered in the recent study [15].

4.2.1. $\Omega = \mathbb{R}^d$. In this case, on taking Fourier transform of the functions on both sides of equation (33) , we obtain

$$\frac{\partial}{\partial t} \hat{u}(s,t) = -c^2 |s|^2 \hat{u}(s,t), \quad s \in \mathbb{R}^d,\ 0 < t < \tau.$$

For each fixed $s \in \Omega$, the above equation is an ordinary differential equation, and hence the solution $\hat{u}(s,t)$ is given by

$$\hat{u}(s,t) = e^{-c^2 t |s|^2} \hat{f}_0(s), \quad s \in \mathbb{R}^d,$$

where $f_0(x) := u(x,0)$, $x \in \mathbb{R}^d$. In particular, with $t = \tau$, we have

$$\hat{u}(s,\tau) = e^{-c^2 \tau |s|^2} \hat{f}_0(s), \quad s \in \mathbb{R}^d.$$

Taking

$$f(s) := \hat{f}_0(s), \quad g(s) := \hat{f}_\tau(s), \quad b(s) := e^{-c^2 t |s|^2},$$

the above equation takes the form

$$b(s) f(s) = g(s), \quad s \in \mathbb{R}^d. \tag{34}$$

Here, one may assume that the actual data $g(\cdot)$ belongs to $\in L^2(\mathbb{R}^d)$. The problem is to determine the function $f(\cdot) \in L^2(\mathbb{R}^d)$ satisfying *multiplication* operator equation (34).

We may recall that the map $h \mapsto \hat{h}$ is a bijective linear isometry from $L^2(\mathbb{R}^d)$ into itself. Therefore, if f_τ^δ is a noisy data, then

$$\|f_\tau - f_\tau^\delta\|_{L^2(\mathbb{R}^d)} = \|g - g^\delta\|_{L^2(\mathbb{R}^d)},$$

where $g^\delta := \hat{f_\tau^\delta}$. Hence, if f^δ is an approximate solution corresponding to the noisy data g^δ, and if f_0^δ is the inverse Fourier transform of f^δ, then we have

$$\|f_0 - f_0^\delta\|_{L^2(\mathbb{R}^d)} = \|f - f^\delta\|_{L^2(\mathbb{R}^d)}.$$

Thus, in order to obtain the error estimates for the regularized solutions corresponding to noisy measurements f_τ^δ, it is enough to consider the noisy equation as in (5), that is,

$$g^\delta(s) = b(s)f(s) + \delta\xi(s), \quad s \in \mathbb{R}^d.$$

4.2.2. Ω *is a bounded domain in* $\mathbb{R}^d$. For the purpose of illustration, let us assume that the the temperature is kept at 0 at the boundary of Ω, that is,

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega.$$

Then the solution of the equation (33) along with the the initial condition

$$u(x,0) = f_0(x), \quad x \in \Omega,$$

is given by (see [14, § 4.1.2])

$$u(x,t) = \sum_{n=1}^{\infty} e^{-c^2\lambda_n^2 t}\langle f_0, v_n\rangle v_n(x).$$

Here (λ_n) is a non-decreasing sequence of non-negative real numbers such that $\lambda_n \to \infty$ as $n \to \infty$ and (v_n) is an orthonormal sequence of functions in $L^2(\Omega)$. In fact, each λ_n is an eigenvalue of the operator $(-\Delta)$ with corresponding eigenvector v_n. For $t = \tau$, taking $f_\tau := u(\cdot,\tau)$, we have

$$f_\tau(x) = \sum_{n=1}^{\infty} e^{-c^2\lambda_n^2 \tau}\langle f_0, v_n\rangle v_n(x).$$

Equivalently,

$$\langle f_\tau, v_n\rangle = e^{-c^2\lambda_n^2\tau}\langle f_0, v_n\rangle, \quad n \in \mathbb{N}. \tag{35}$$

Writing

$$g := (\langle f_\tau, v_n\rangle), \quad f := (\langle f_0, v_n\rangle), \quad b := (e^{-c^2\lambda_n^2\tau}),$$

the system of equations in (35) takes the form a multiplication operator equation

$$b(n)f(n) = g(n), \quad n \in \mathbb{N}, \tag{36}$$

where g and f are in $\ell^2(\mathbb{N})$ and b is in $c_0(\mathbb{N})$, the space all null sequences.

As in § 4.2.1, we have

$$\|f_\tau - f_\tau^\delta\|_{L^2(\Omega)} = \|g - g^\delta\|_{\ell^2(\mathbb{N})} \quad \text{and} \quad \|f_0 - f_0^\delta\|_{L^2(\Omega)} = \|f - f^\delta\|_{\ell^2(\mathbb{N})},$$

where $g^\delta \in \ell^2(\mathbb{N})$ and $f_0^\delta \in L^2(\Omega)$ are constructed from the bijective linear isometry $h \mapsto (\langle h, v_n\rangle)$ from $L^2(\Omega)$ onto $\ell^2(\mathbb{N})$, that is,

$$g^\delta(n) := \langle f_\tau^\delta, v_n\rangle \quad \text{and} \quad f_0^\delta := \sum_{n=1}^\infty \langle f^\delta, v_n\rangle v_n.$$

References

[1] C. Bennett and R. Sharpley. *Interpolation of Operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[2] L. Cavalier. Inverse problems with non-compact operators. *J. Statist. Plann. Inference*, 136(2):390–400, 2006.

[3] K. M. Chong and N. M. Rice. *Equimeasurable Rearrangements of Functions*. Queen's Papers in Pure and Applied Mathematics, No. 28. Queen's University, Kingston, Ont., 1971.

[4] P. W. Day. *Rearrangements of Measurable Functions*. Ph.D.-Thesis –California Institute of Technology. ProQuest LLC, Ann Arbor, MI, 1970.

[5] L. Ding and P. Mathé. Minimax rates for statistical inverse problems under general source conditions. *Comput. Methods Appl. Math.*, 18(4):603–608, 2018.

[6] H. W. Engl, B. Hofmann, and H. Zeisel. A decreasing rearrangement approach for a class of ill-posed nonlinear integral equations. *J. Integral Equations Appl.*, 5(4):443–463, 1993.

[7] P. R. Halmos. What does the spectral theorem say? *Amer. Math. Monthly*, 70:241–247, 1963.

[8] B. Hofmann and G. Fleischer. Stability rates for linear ill-posed problems with compact and non-compact operators. *Z. Anal. Anwendungen*, 18(2):267–286, 1999.

[9] B. Hofmann and P. Mathé. Analysis of profile functions for general linear regularization methods. *SIAM J. Numer. Anal.*, 45(3):1122–1141, 2007.

[10] R. Krämer and P. Mathé. Modulus of continuity of Nemytskiĭ operators with application to the problem of option pricing. *J. Inverse Ill-Posed Probl.*, 16(5):435–461, 2008.

[11] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[12] P. Mathé. Degree of ill-posedness of statistical inverse problems. Preprint 954, WIAS Berlin, August 2004.

[13] P. Mathé and S. V. Pereverzev. Geometry of linear ill-posed problems in variable Hilbert scales. *Inverse Problems*, 19(3):789–803, 2003.

[14] M. Thamban Nair. *Linear Operator Equations – Approximation and Regularization*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.

[15] M. Thamban Nair. On truncated spectral regularization for an ill-posed evolution equation. *arXiv e-prints*, page arXiv:1907.11076, July 2019.

[16] M. Thamban Nair, S. V. Pereverzev, and U. Tautenhahn. Regularization in Hilbert scales under general smoothing conditions. *Inverse Problems*, 21(6):1851–1869, 2005.

[17] N. Wiener. *Extrapolation, Interpolation, and Smoothing of Stationary Time Series. With Engineering Applications*. The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass; John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1949.

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