REGULARITY OF SYMBOLIC POWERS OF EDGE IDEALS

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ABSTRACT. In this article, we prove that for several classes of graphs, the Castelnuovo-Mumford regularity of symbolic powers of their edge ideals coincide with that of their ordinary powers.

1. INTRODUCTION

This article is motivated by the results in the paper [5]. Gu et al. in [5] studied the properties and invariants associated with symbolic powers of edge ideals of unicyclic graphs. Let G be a finite simple graph on the vertex set $x_1, \ldots, x_n$ and $I(G)$ denote the ideal in the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ generated by $\{x_i x_j \mid \{x_i, x_j\}$ is an edge of $G\}$, where k is a field. There have been a lot of research on connection between algebraic properties of $I(G)^s$ with the combinatorial properties of G, see [2] and the references there in. In the geometrical context, the symbolic powers have more importance since it captures all polynomials that vanishes with a given multiplicity. Algebraically, the symbolic powers are harder to compute or handle. In our situation, we can observe that $I(G)^{(s)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} \mathfrak{p}^s$. It was proved by Simis, Vasconcelos and Villarreal that G is bipartite if and only if $I(G)^{(s)} = I(G)^s$ for all $s \geq 1$, [12]. It has been conjectured by N. C. Minh that if G is a finite simple graph, then $\mathrm{reg}(I(G)^{(s)}) = \mathrm{reg}(I(G)^s)$ for all $s \geq 1$, see [5]. Gu et al., in [5], proved this conjecture for odd cycles. Recently, the conjecture has been proved for the classes of unicyclic graphs, chordal graphs and Cameron-Walker graphs by Seyed Fakhari, [9, 10, 11]. In [7], Kumar and Selvaraja generalized a result of Seyed Fakhari to prove Minh's conjecture for a class of graphs obtained by attaching complete graphs to vertices of unicyclic graphs.

In this article, we extend some of the results in [5] to prove the equality of regularity of ordinary powers with that of symbolic powers for certain classes of graphs. Our main theorem is stated as follows:

Theorem 4.12. *Let G be a graph obtained by taking clique sum of a C_{2n+1} and some bipartite graphs. Let H be an induced subgraph of G on vertices $V \setminus \bigcup_{x \in V(C_{2n+1})} N_G(x)$. Assume that none of the vertices of H is part of any cycle in G. If $\nu(G) - \nu(H) \geq 3$, then* $\mathrm{reg}\left(I^{(s)}\right) = \mathrm{reg}\left(I^s\right)$.

As in [5], the approach is through understanding the symbolic power as a sum of product of ordinary powers of certain related ideals. We use this decomposition to study the regularity of symbolic powers of edge ideals of graphs whose each odd cycle is a dominant odd cycle.

Theorem 3.5. *Let G' be a clique sum of r cycles of size $2n + 1$, say $C_1, \ldots, C_r$, and G be a graph by taking the clique sum of G' and some bipartite graphs. If $N_G(C_i) = V(G)$ for any odd cycle C_i in G, then $\operatorname{reg}\left(I^{(s)}\right) = \operatorname{reg}\left(I^s\right)$ for all $s \geq 1$.*

The article is organized as follows. We collect the required terminologies and results in Section 2. In Section 3, we obtain the decomposition for symbolic powers in terms of ordinary powers and use it to prove Theorem 3.5. In the final section, we prove Theorem 4.12.

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2. PRELIMINARIES

Throughout this paper, all graphs considered are assumed to be finite and simple. For a graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$, S denotes the polynomial ring $\mathsf{k}[x_1, \ldots, x_n]$ and $\mathfrak{m}$ denotes the unique graded maximal ideal in S. In this section, we recall the definitions and results that are needed for the rest of the paper. We begin by recalling the some of the terminologies related to finite simple graphs.

Definition 2.1. Let G be a graph on the vertex set V. Then,

i) set $\alpha(G) := \min\{|C| : C \text{ is a vertex cover of } G\}$;

ii) the graph G is called *decomposable* if there exists a partition of $V = V_1 \sqcup \cdots \sqcup V_r$ such that $\sum \alpha(G_i) = \alpha(G)$, where G_i is induced subgraph of G on V_i. If G is not decomposable, then G is called *indecomposable*;

iii) for $T \subset V$, $G \setminus T$ denote the induced subgraph of G on the vertex set $V \setminus T$;

It was shown by Harary and Plummer, [6] that every indecomposable contains an odd cycle. We now recall the duplication and parallelization.

Definition 2.2. Let G be a graph on n vertices and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n$.

i) The *duplication* of a vertex x of G is the graph obtained from G by adding a vertex x' and all edges $\{x', y\}$ for all $y \in N_G(x)$.

ii) The *parallelization* of G with respect to $\mathbf{v}$, denoted by $G^{\mathbf{v}}$, is the graph obtained from G by deleting x_i if $v_i = 0$ and duplicating $v_i - 1$ times x_i if $v_i \geq 1$.

For an ideal I in a commutative ring A, let $\mathcal{R}_s(I) := \oplus_{n \geq 0} I^{(n)} t^n$ denote the symbolic Rees algebra of I. For a vector $\mathbf{v} \in \mathbb{N}^n$, let $\mathbf{x}^{\mathbf{v}}$ be the monomial $x_1^{v_1} \cdots x_n^{v_n} \in \mathsf{k}[x_1, \ldots, x_n]$. Martínez-Bernal et al. obtained the k-algebra generators for the symbolic Rees algebra:

Theorem 2.3. [8, Lemma 2.1] *Let G be a graph on V. Then*

$$\mathcal{R}_s(I) = \mathsf{k}[x^{\mathbf{v}}t^b : G^{\mathbf{v}} \text{ is an indecomposable graph}, \mathbf{v} \in \mathbb{N}^{|V|} \text{ and } b = \alpha(G^{\mathbf{v}})].$$

Here we recall the definition of implosive graphs.

Definition 2.4.

i) A graph G is called *implosive* if symbolic Rees algebra of I is generated by monomials of the form $x^{\mathbf{v}}t^b$, where $\mathbf{v} = \{0,1\}^{|V|}$.

ii) Let G_1 and G_2 be graphs. Suppose $G_1 \cap G_2 = K_r$ is a complete graph, where $G_1 \neq K_r$ and $G_2 \neq K_r$. Then $G_1 \cup G_2$ is called the *clique-sum* of G_1 and G_2.

Remark 2.5. [4, Theorem 2.3, Theorem 2.5]

i) If G is a cycle, then G is implosive.

ii) The clique-sum of implosive graphs is implosive.

3. REGULARITY OF DOMINANT CYCLES

Gu et al. in [5] shows that if G is unicyclic graph with $C_{2n+1} = (x_1, \ldots, x_{2n+1})$, then $I^{(s)} = \sum_{i=0}^{k} I^{s-i(n+1)}(x_1 \cdots x_{2n+1})^i$, where $s = k(n+1) + r$ for some $k \in \mathbb{Z}$ and $0 \leq r \leq n$. In this section, we generalize some of the results in sections 3 and 5 of [5] and use it to compute the regularity of the symbolic powers, generalizing [5, Theorem 5.3].

Lemma 3.1. *Let G' be a clique sum of r cycles of size $2n + 1$, say $C_1, \ldots, C_r$, and G be a graph by taking the clique sum of G' and some bipartite graphs. Let $I = I(G)$ and $J = (u_{C_1}, \ldots, u_{C_r})$, where $u_{C_i} = \prod_{j=1}^{2n+1} x_{i_j}$, the product of variables corresponding to the vertices of the cycle C_i. Then $I^{(s)} = I^s$ for all $s \leq n$ and $I^{(s)} = \sum_{i=0}^{k} J^i I^{s-i(n+1)}$, where $s = k(n+1) + r$ for some $k \in \mathbb{Z}$ and $0 \leq r \leq n$.*

Proof. Since G is the clique sum of odd cycles and bipartite graphs, by [4, Theorems 2.3, 2.5], we get that G is implosive. By [6, Theorem 2], any indecomposable induced subgraph of G is contained in C_i for some i or an edge. Moreover, by [6, Corollary 1b], an indecomposable induced subgraph of C_i is either itself or an edge. Hence by Theorem 2.3, we get $\mathcal{R}_s(I) = S[It, Jt^{n+1}]$. Now comparing the graded components on both sides of the above equality, we get $I^{(s)} = I^s$ for all $s \leq n$ and $I^{(s)} = \sum_{i=0}^{k} J^i I^{s-i(n+1)}$. $\square$

To study the regularity of $I^{(s)}$, we need to understand the structure of $I^{(s)} \cap \mathfrak{m}^{2s}$. This is done by studying the intersection with each of the term appearing in the summation in the previous result.

Lemma 3.2. *Let G be a graph as in Lemma 3.1. Then*

$$I^{(s)} \cap \mathfrak{m}^{2s} = \sum_{i=0}^{k} J^i \mathfrak{m}^i I^{s-i(n+1)}.$$

Proof. By Lemma 3.1, it is enough to show that $J^i I^{s-i(n+1)} \cap \mathfrak{m}^{2s} = J^i \mathfrak{m}^i I^{s-i(n+1)}$. Since $J^i \mathfrak{m}^i I^{s-i(n+1)} \subset J^i I^{s-i(n+1)}$ and $J^i \mathfrak{m}^i I^{s-i(n+1)} \subset \mathfrak{m}^{2s}$, we get

$$J^i \mathfrak{m}^i I^{s-i(n+1)} \subset J^i I^{s-i(n+1)} \cap \mathfrak{m}^{2s}.$$

For the reverse containment, let $u \in J^i I^{s-i(n+1)} \cap \mathfrak{m}^{2s}$. Write $u = fgh$, where $f \in G(J^i)$, $g \in G(I^{s-i(n+1)})$. Note that $u \in \mathfrak{m}^{2s}$ implies that $\deg(u) \geq 2s$. Since $\deg(f) = i(2n+1)$ and $\deg(g) = 2s - 2i(n+1)$, we get that $\deg(h) \geq i$ which completes the proof. $\square$

As an immediate consequence, we obtain the intersection $I^{(s)} \cap \mathfrak{m}^{2s}$ for the class of graphs that we are considering.

Corollary 3.3. *Let G be a graph as in Lemma 3.1. If $N_G(C_i) = V$ for any odd cycle C_i in G, then $I^{(s)} \cap \mathfrak{m}^{2s} = I^s$.*

Proof. We show that $\mathfrak{m}J \subseteq I^{n+1}$. Let $x_i \in V(G)$ and $u_{C_i} = \prod_{j=1}^{2n+1} x_{i_j}$ be a minimal generator of J. Without loss of generality, let $x_{i_1} \in N_{C_i}(x_i)$. Then $x_i u_{C_i} = x_i x_{i_1} \cdot x_{i_2} x_{i_3} \cdots x_{i_{2n}} x_{i_{2n+1}} \in I^{n+1}$. Hence $\mathfrak{m}J \subseteq I^{n+1}$ so that $\mathfrak{m}^i J^i \subseteq I^{i(n+1)}$. $\square$

For a homogeneous ideal $I \subset S$, let $\alpha(I)$ denote the least degree of a minimal generator of I. The *Waldschmidt* constant of I is defined to be $\hat{\alpha}(I) := \lim_{s \to \infty} \frac{\alpha(I^{(s)})}{s}$. The real number $\rho(I) = \sup\{s/t \mid I^{(s)} \not\subset I^t\}$ is called *resurgence* number of I and $\rho_a(I) = \sup\{s/t \mid I^{(sr)} \not\subset I^{tr}$ for all $r \gg 0\}$ is called *asymptotic resurgence* number of I. We compute the Waldschmidt constant, resurgence and asymptotic resurgence number of the edge ideals of the graphs considered in Lemma 3.1.

Corollary 3.4. *Let G be as in Lemma 3.1. Then*

(1) $\alpha(I(G)^{(s)}) = 2s - \lfloor \frac{s}{n+1} \rfloor$ for all $s \in \mathbb{N}$;
(2) $\hat{\alpha}(I(G)) = \frac{2n+1}{n+1}$;
(3) $\alpha(I(G)^{(s)}) < \alpha(I^t)$ if and only if $I(G)^{(s)} \not\subset I^t$;
(4) $\rho(I(G)) = \rho_a(I(G)) = \frac{2n+2}{2n+1}$.

Proof. Since the proof is exactly same as the proof of [5, Theorem 3.6], we skip it here. $\square$

We now generalize [5, Theorem 5.3].

Theorem 3.5. *Let G be as in Lemma 3.1. If $N_G(C_i) = V$ for any odd cycle C_i in G, then $\operatorname{reg}\left(I^{(s)}\right) = \operatorname{reg}\left(I^s\right)$ for all $s \geq 1$.*

Proof. Suppose $\nu(G) = 1$. Then G is either C_5 or is the clique-sum of a C_3, say T, with several copies of C_3, say $G_1, \ldots, G_r$ along the edges of T and copies of P_2, say $G_{r+1}, \ldots, G_s$ along the vertices of T. If $G = C_5$, then the assertion is proved in [5]. If $G \neq C_5$, then G^c is the clique-sum of a K_s with $s - r$ copies of C_3 along the edges of K_s and with r many edges along the vertices of K_s. Hence G is a co-chordal graph. Therefore, S/I^s has linear resolution for all $s \geq 1$. Consider the short exact sequence

$$(1) \qquad 0 \longrightarrow \frac{S}{I^s} \longrightarrow \frac{S}{I^{(s)}} \oplus \frac{S}{\mathfrak{m}^{2s}} \longrightarrow \frac{S}{I^{(s)} + \mathfrak{m}^{2s}} \longrightarrow 0.$$

Note that since $I^{(s)}$ contains a minimal generator of degree $2s$, $\operatorname{reg}(S/I^{(s)}) \geq 2s - 1 = \operatorname{reg}(S/\mathfrak{m}^{2s})$. Also, $S/(I^{(s)} + \mathfrak{m}^{2s})$ is Artinian, $[S/(I^{(s)} + \mathfrak{m}^{2s})]_{2s-1} \neq 0$ and $[S/(I^{(s)} + \mathfrak{m}^{2s})]_{2s} = 0$ so that $\operatorname{reg}(S/I^{(s)} + \mathfrak{m}^{2s}) = 2s - 1$. Hence it follows from the exact sequence (1) that $\operatorname{reg}(S/I^{(s)}) \leq 2s - 1$. Therefore $\operatorname{reg}(S/I^{(s)}) = 2s - 1 = \operatorname{reg}(S/I^s)$.

Assume now that $\nu(G) \geq 2$. Since $S/(I^{(s)} + \mathfrak{m}^{2s})$ is Artinian, the regularity is given by the socle degree. Hence $\operatorname{reg}\left(\frac{S}{I^{(s)} + \mathfrak{m}^{2s}}\right) = 2s - 1 = \operatorname{reg}\left(\frac{S}{\mathfrak{m}^{2s}}\right)$ and by [5, Theorem 4.6] $\operatorname{reg}\left(\frac{S}{I^{(s)}}\right) \geq 2s + \nu(G) - 2$. Since $\nu(G) \geq 2$, this implies that $\operatorname{reg}\left(\frac{S}{I^{(s)}}\right) > \operatorname{reg}\left(\frac{S}{I^{(s)} + \mathfrak{m}^{2s}}\right)$. Hence it follows from the short exact sequence (1) that $\operatorname{reg}\left(I^{(s)}\right) = \operatorname{reg}\left(I^s\right)$. $\qquad\square$

Example 3.6. We would like to note here that the class of graphs that we have considered here is more general than unicyclic graphs with a dominating odd cycle which are considered in [5].

For example, the graphs given on the right are not unicyclic graphs but satisfy the hypotheses of Theorem 3.5. The first one is a clique sum of C_5 with some bipartite graphs which contain cycles. The second graph on the right is a clique sum of three C_3's.
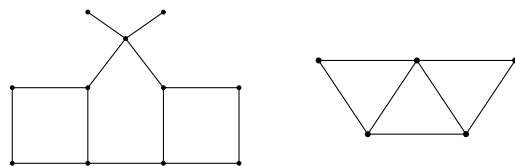
4. REGULARITY OF UNICYCLIC GRAPHS

In this section, we focus on graphs which has only one odd cycle. For the rest of the paper, let G be a graph obtained by taking clique-sum along the vertices or edges of an odd cycle C_{2n+1} and some bipartite graphs. Let $V(C_{2n+1}) = \{x_1, \ldots, x_{2n+1}\}$, $N_G(C_{2n+1}) \setminus V(C_{2n+1}) = \{y_1, \ldots, y_l\}$ and $V(G) \setminus N_G(C_{2n+1}) = \{z_1, \ldots, z_m\}$. Now we set $I = I(G)$, $\mu = x_1 \cdots x_{2n+1}$, $L = (x_1, \ldots, x_{2n+1}, y_1, \ldots, y_l)$, $K = (z_1, \ldots, z_m)$ and $\mathfrak{m}$ the homogeneous maximal ideal in $\mathsf{k}[L, K]$. For any monomial ideal J, let $G(J)$ denote the set of minimal monomial generators of J. We first give a refinement of the decomposition of $I^{(s)} \cap \mathfrak{m}^{2s}$.

Lemma 4.1. $I^{(s)} \cap \mathfrak{m}^{2s} = \sum_{i=0}^{k} \mu^i K^i I^{s-i(n+1)}.$

Proof. Using Lemma 3.2, we get $I^{(s)} \cap \mathfrak{m}^{2s} = \sum_{i=0}^{k} \mu^i \mathfrak{m}^i I^{s-i(n+1)}$. Since for any $a \in L$, we know that $a\mu \in I^{n+1}$, we get $L^i \mu^i \subset I^{i(n+1)}$. By above remark, we get

$$I^{(s)} \cap \mathfrak{m}^{2s} \subset \sum_{i=0}^{k} \sum_{t=0}^{i} \mu^i L^t K^{i-t} I^{s-i(n+1)} \subset \sum_{i=0}^{k} \sum_{t=0}^{i} \mu^{i-t} K^{i-t} I^{s-(i-t)(n+1)} = \sum_{i=0}^{k} \mu^i K^i I^{s-i(n+1)}.$$

Since each term of the summation on the right hand side is naturally contained in the left hand side, the reverse inclusion follows easily. $\square$

We now define an ordering, called *edgelex* ordering, among the monomial generators of $I(G)^s$ and $\mathfrak{m}^r I(G)^s$ following [1, Discussion 4.1]. This helps us in understand certain colon ideals which are crucial in the study of regularity of powers.

Definition 4.2. Let G be a graph with $E(G) = \{e_1, \dots, e_r\}$ and I be its edge ideal. For $A, B \in G(I^s)$, we say that $A >_{\text{edgelex}} B$ if there exists an expression $A = e_{i_1}^{a_1} \cdots e_{i_r}^{a_r}$ such that for all expressions $e_{i_1}^{b_1} \cdots e_{i_r}^{b_r} = B$, we have $(a_1, \dots, a_r) >_{\text{lex}} (b_1, \dots, b_r)$.

Let $J = I^s \mathfrak{m}^r$. Then for any $u, v \in G(J)$, we say that $u > v$ if there exists an expression $u = fu'$ such that for any expression of $v = gv'$ with $g \in G(I^s)$ and $v' \in \mathfrak{m}^r$, we have either $f >_{edgelex} g$ or $f = g$ and $u' >_{lex} v'$. Further, we say $u = fu'$ is a *maximal expression* of u if for any other expression $f_1 u_1 = u$ with $f_1 \in G(I^s)$ and $u_1 \in \mathfrak{m}^r$, we have $f >_{edgelex} f_1$.

We now recall the concept of edge-division given in [1, Definition 4.2]. Let G be a graph with $E(G) = \{e_1, \dots, e_r\}$ and I be its edge ideal of G. Let $u \in I^s$. Then for some j, we say that e_j *edge-divides* u if there exists $v \in I^{s-1}$ such that $u = e_j v$. We denote this by $e_j \mid^{edge} u$.

For example, if $G = C_5$ and $I(G) = (x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_1 x_5) \subset \mathsf{k}[x_1, \dots, x_5]$, then $(x_4 x_5)^2 >_{edgelex} (x_1 x_5)^2$. Note that with respect to the lex order, the inequality is reverse. Also, $x_1 x_2 \mid^{edge} x_1 x_2^2 x_3$ and $x_2 x_3 \mid x_1 x_2 x_3 x_4$. Note that the second one is a normal division, not an edge-division.

Most of the proofs that we do are by some type of induction. Understanding the behavior of the colon ideal is necessary to apply induction. We first generalize [1, Lemma 4.11].

Lemma 4.3. *Let G be a graph, I be its edge ideal and $\mathfrak{m}$ be the homogeneous maximal ideal in the appropriate polynomial ring. Let $J = I^s \mathfrak{m}^r$. Then there exists an ordering on minimal monomial generators of $J = (u_1, \dots, u_m)$ such that for $j < k$, either $(u_j : u_k) \subset I^{s+1} : u_k$ or there exists $i < k$ such that $(u_i : u_k)$ is an ideal generated by a variable and it contains $(u_j : u_k)$.*

Proof. Consider the ordering on $G(J)$ given in Definition 4.2. We prove the result by using induction on (s, r). For $j < k$, let $u_j = f_1 v_1$ and $u_k = f_2 v_2$ be maximal expressions, where $f_1, f_2 \in I^s$ and $v_1, v_2 \in \mathfrak{m}^r$.

If $r = 0$, then the assertion follows from [1, Lemma 4.11]. In particular, if $(s, r) = (1, 0)$, then the assertion holds true. Assume by induction that the assertion is true for all $(s_1, r_1) <_{lex} (s, r)$.

Let ab be the maximal edge such that $ab \mid^{edge} f_1$. If $ab \mid^{edge} f_2$, then write $f_1 = abf_1'$ and $f_2 = abf_2'$ for some $f_1', f_2' \in I^{s-1}$. Then $u_j' = u_j/ab = f_1'v_1$ and $u_k' = u_k/ab = f_2'v_2$ are in $I^{s-1}\mathfrak{m}^r$. Moreover, $(u_j' : u_k') = (u_j : u_k)$. By induction, either $(u_j' : u_k') \subseteq (I^s : u_k')$ or there exists and $i < k$ such that $(u_i' : u_k')$ is an ideal generated by variables and it contains $(u_j' : u_k')$. If $(u_j' : u_k') \subseteq (I^s : u_k')$, then clearly $(u_j : u_k) \subseteq (I^{s+1} : u_k)$. Suppose there exists an $i < k$ such that $(u_i' : u_k')$ is generated by a variable. Clearly $(abu_i' : u_k) = (u_i' : u_k')$. Hence it is enough to show that $abu_i' > u_k$ and set $u_i = abu_i'$. But this is obvious since $u_i' \in G(I^{s-1}\mathfrak{m}^r)$, ab is an edge and ab edge divides f_2.

Now we assume that $ab \nmid^{edge} f_2$. If $\gcd(ab, u_k) = 1$, then $(u_j : u_k) \subset (ab) \subset (I^{s+1} : u_k)$. Hence the assertion follows. Suppose $\gcd(ab, u_k) \neq 1$. Consider the case when $a \mid u_k$. If $a \mid v_2$, then we claim that $b \nmid u_k$. Suppose $b \mid u_k$. If $b \mid v_2$, then we can write $u_k = v_2'f_2'$, where $f_2' = \frac{ab}{e_j}f_2$ for some edge e_j with $e_j \mid^{edge} f_2$ and $v_2' = \frac{e_j}{ab}v_2$. Since $f_2' >_{edgelex} f_2$, the expression $u_k = v_2f_2$ is not maximal which is a contradiction. Hence $b \nmid v_2$ so that $b \mid f_2$. Let b' be such that $bb' \mid^{edge} f_2$. This implies that $ab \mid^{edge} \frac{af_2}{b'}$ and $\frac{af_2}{b'} \mid u_k$. Note that $\frac{af_2}{b'} >_{edgelex} f_2$, and hence f_2v_2 is not a maximal expression which is a contradiction to our assumption. This implies that $b \nmid u_k$ and $(u_j : u_k) \subset (b) \subset I^{s+1} : u_k$. If $a \mid f_2$, then, as in the earlier case, we get $b \nmid v_2$. Then there exists a vertex c such that $ac \mid^{edge} f_2$. Suppose $(u_j : u_k) \subset (b)$. Write $f_1 = abf_1'$ and $f_2 = acf_2'$ and take $u_i = abf_2'v_2$. Hence $u_i > u_k$ and $(u_i : u_k) = (b) \supset (u_j : u_k)$. Suppose $(u_j : u_k) \not\subset (b)$. Since $b \mid u_j$, $b \mid u_k$. Also, $b \nmid v_2$. Therefore, $b \mid f_2$ and there exists a vertex d such that $bd \mid^{edge} f_2$. If $(u_j : u_k) \subseteq (a)$, then by the symmetry of arguments, we get $(u_i : u_k) = (a) \supset (u_j : u_k)$, where $u_i = ab\frac{f_2}{bd}v_2$. Hence, for the rest of the proof we may assume that neither a nor b divides $(u_j : u_k)$.

Let $(u_j : u_k) = (w)$. If $\gcd(f_1, w) = 1$, then $w \mid v_1$. Let $w = xw'$, where x is a variable, and take w_1 such that $w_1 \mid \frac{u_k}{\gcd(u_j, u_k)}$ with $\deg(w_1) = \deg(w')$. Set $u_i = \frac{u_jw_1}{w'}$. Since $f_1 >_{edgelex} f_2$, we have $u_i > u_k$, and $(u_i : u_k) = (x)$ which contains w.

Suppose $\gcd(f_1, w) \neq 1$. Let x be a vertex such that $x \mid w$ and $x \mid f_1$. Note that $x \neq a$. Since $x \mid f_1$, there exists y such that $xy \mid^{edge} f_1$. If y does not divide u_k, then $(u_j : u_k) \subset (xy)$. Since xy is an edge, this implies that $xyu_k \in I^{s+1}$, i.e., $xy \in I^{s+1} : u_k$. Hence $(u_j : u_k) \subset I^{s+1} : u_k$. Now assume that $y \mid u_k$. If $y \mid v_2$, then $xv_2f_2 \in I^{s+1}$, since xy is an edge and $f_2 \in I^s$. Therefore, $x \in I^{s+1} : u_k$. Hence $(u_j : u_k) \subset (x) \subseteq I^{s+1} : u_k$. If $y \mid f_2$, then there exists z such that $yz \mid^{edge} f_2$. Write $f_1 = abxyf_1''$ and $f_2 = acyzf_2''$ for some $f_1'', f_2'' \in I^{s-2}$. Since $(u_j : u_k) = (w)$, we get $u_j \mid wu_k$, and hence $abf_1''v_1 \mid w'zacf_2''v_2$. This implies that $(w'z) \subset (abf_1''v_1 : acf_2''v_2)$. Let $(abf_1''v_1 : acf_2''v_2) = (w_1')$. This gives us $abf_1''v_1 \mid w_1'acf_2''v_2$, and hence $u_j \mid w_1'xu_k$ which forces that $w'x \mid w_1'x$. This implies that $(abf_1''v_1 : acf_2''v_2)$ is equal either to (w') or to $(w'z)$. Note that $abf_1'' >_{edgelex} acf_2''$. Therefore by induction $(abf_1''v_1 : acf_2''v_2) \subset I^s : acf_2''v_2$ or there exists $u' \in G(I^{s-1}\mathfrak{m}^r)$ such that $(u' : acf_2''v_2)$ is generated by a variable and it contains $(abf_1''v_1 : acf_2''v_2)$.

Assume that $(abf_1''v_1 : acf_2''v_2) = (w') \supset (u_j : u_k)$. Suppose $(abf_1''v_1 : acf_2''v_2) \subset I^s : acf_2''v_2$. This implies that $(u_j : u_k) \subset (abf_1''v_1 : acf_2''v_2) \subset I^{s+1} : u_k$. Suppose there exists $u' \in G(I^{s-1}\mathfrak{m}^r)$ such that $(u' : acf_2''v_2) = (l)$ for some variable l which divides w'. Therefore, by taking $u_i = yzu'$, we get $(u_i : u_k) = (u' : acf_2''v_2) = (l)$ which divides w', and hence w.

Suppose $(abf_1''v_1 : acf_2''v_2) = (w'z)$. If $(abf_1''v_1 : acf_2''v_2) \subset I^s : acf_2''v_2$, i.e., $w'zacf_2''v_2 \in I^s$, then $w'zxyacf_2''v_2 \in I^{s+1}$, i.e., $wu_k \in I^{s+1}$. Suppose there exists $u' \in G(I^{s-1}\mathfrak{m}^r)$ such that $(u' : acf_2''v_2) = (l)$, where l is a variable and $l \mid w'z$. If $l = z$, then take $u_i = xyu'$. This gives us $(u_i : u_k) = (x)$. If $l \neq z$, then take $u_i = yzu'$. Then we get $(u_i : u_k) = (l)$. In both cases, $(u_i : u_k)$ is generated by a variable and it contains $(u_j : u_k)$ which completes the proof. $\square$

We now recall the definition of even-connection introduced by Banerjee in [1].

Definition 4.4. Let G be a graph and x and y be vertices of G. Then we say that x and y are *even connected* with respect to $u = e_1 \cdots e_s$ if there is a path $p_0p_1 \cdots p_{2k+1}$, $k \geq 1$ in G such that

i) $p_0 = x$ and $p_{2k+1} = y$.
ii) For all $1 \leq l \leq k$, we have $p_{2l-1}p_{2l} = e_i$ for some i.
iii) For all i, we have $|\{l \geq 0 : p_{2l-1}p_{2l} = e_i\}| \leq |\{j : e_j = e_i\}|$.

One of the most important property of the even connection is that it describes the generators of the colon ideal $I^s : u$.

Theorem 4.5. [1, Theorem 6.7] *Let G be a graph and I be its edge ideal. Let $u \in G(I^{s-1})$. Then $I^s : u = I + (xy : x$ is even connected to y with respect to $u)$.*

We further analyze the even-connected edges in this class of edge ideals and certain colon ideals which come up in the induction step.

Lemma 4.6. *Let G be a graph obtained by taking the clique-sum along the vertices or edges of an odd cycle C_{2n+1} and some bipartite graphs. Let $\{z_1, \ldots, z_m\} = V(G) \setminus N_G(C_{2n+1})$. Assume that z_i is not part of any cycle for all $i = 1, \ldots, m$. Then there exists an ordering on $G(I^s) = \{u_1, \ldots, u_r\}$ such that if z_i and z_j are even-connected with respect to u_t for some $1 \leq t \leq r$, then there exists $u_s > u_t$ such that $(u_s : u_t) = (z_k)$, where $k = \min\{i, j\}$.*

Proof. Since z_i is not part of any cycle, it follows that the induced subgraph on $V(G) \setminus N_G(C_{2n+1})$ is a forest. After a re-ordering of the vertices, assume that e_1 is a leaf in G having pendant vertex z_1 and e_i is a leaf in $G \setminus \{e_1, \ldots, e_{i-1}\}$ with pendant vertex z_i, for $i = 2, \ldots, m$. Set $z_1 > \cdots > z_m$, $e_1 > \cdots > e_m$ and on $E(G) \setminus \{e_1, \ldots, e_m\}$, set the lexicographic ordering with $y_1 > \cdots > y_l > x_1 > \cdots > x_{2n+1}$ and such that for any $e \in E(G) \setminus \{e_1, \ldots, e_m\}$, $e_m > e$. Now, take the edgelex ordering on I^s.

Suppose z_i and z_j are even connected with respect to $u_l = e_{i_1} \cdots e_{i_s}$. Without loss of generality, we may assume that $i < j$. Hence $z_j < z_i$ and $e_j < e_i$. Let $z_i p_1 \cdots p_{2k} z_j$, $k \geq 1$ be an even-connection in G.

We claim that $z_i p_1 > p_1 p_2$. If $z_i p_1 < p_1 p_2$, then $z_i < p_2$. This implies that $p_2 = z_{i_1}$ for some $i_1 < i$. Since z_{i_1} is obtained as a pendant vertex after removing $z_1, \ldots, z_{i_1-1}$ and both z_i and p_1 are less than z_{i_1}, $p_3 = z_{i_2}$ for some $i_2 < i_1$. Continuing like this, we obtain that $p_{2k+1} = z_j > z_i$ which is a contradiction to our assumption that $i < j$. Hence $z_i p_1 > p_1 p_2$. Set $u_s = z_i p_1 \dfrac{u_l}{p_1 p_2}$. Then $u_s > u_t$ and $(u_s : u_t) = z_i$. $\qquad\square$

Remark 4.7. Let $f = \mu^i g u \in G(\mu^i K^i I^{s-i(n+1)})$ and $M = \operatorname{supp}(g)$. Then we have the following:

i) Let $l \in M$ and $l' \in N_G(l)$. This implies $\dfrac{\mu}{x_j} ll'u \in I^{s-(i-1)(n+1)}$ for any j. Hence $l'\mu^i g u = \dfrac{g}{l}ll'\mu^i u \in \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)}$ which shows that $N_G(M) \subset \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f$.

ii) Let $l \in M \cup V(C_n)$ and $l' \in V(G)$ such that l and l' is an even connection with respect to $\dfrac{\mu u}{x_j}$ for some j. Hence $\dfrac{\mu}{x_j}ll'u \in I^{s-(i-1)(n+1)}$ for some j. Hence $l'\mu^i g u \in \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)}$ which shows that $l' \in \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f$.

To understand the colon with symbolic power, we study the colon with ideals in the decomposition of the symbolic power.

Lemma 4.8. *Let G be as in Lemma 4.6 and $f = \mu^i g u \in G(\mu^i K^i I^{s-i(n+1)})$ for $1 \leq i \leq k$ with $f \notin \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)}$, where $u \in I^{s-i(n+1)}$. Then*

$$\mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f = I + L',$$

where L' is an ideal containing L and generated by a set of variables.

Proof. Note that for any $a \in L$, we know that $a\mu \in I^{n+1}$, and hence we get

$$L \subset \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f.$$

We first claim that $I^{s-(i-1)(n+1)} : \mu u$ is generated in degree at most 2. Since $\dfrac{\mu u}{x_i} \in I^{s-(i-1)(n+1)-1}$, the ideal $I^{s-(i-1)(n+1)} : \dfrac{\mu u}{x_i}$ is of the form $I^{t+1} : e_1 \cdots e_t$ which is generated in degree 2. Let $v \in I^{s-(i-1)(n+1)} : \mu u$. This implies that $vx_i \in I^{s-(i-1)(n+1)} : \dfrac{\mu u}{x_i}$. Thus there exists a monomial v' of degree 2 such that $v' \mid vx_i$. If $x_i \nmid v'$, then $v' \mid v$. If $x_i \mid v'$, then $v'/x_i \in I^{s-(i-1)(n+1)} : \mu u$. Hence $I^{s-(i-1)(n+1)} : \mu u$ is generated in at most degree 2. Suppose $v \in G(I^{s-(i-1)(n+1)} : \mu u)$ such that $v \mid g$. Suppose $\deg(v) = 1$. Then $f = \mu^{i-1}\dfrac{g}{v}v\mu u \in \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)}$ which is a contradiction to our assumption that f is not in that ideal. Hence $\deg(v) = 2$. Then one can see as above that $f \in \mu^{i-1}K^{i-2}I^{s-(i-1)(n+1)}$. Hence $K \subseteq \mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f$ so that $\mu^{i-1}K^{i-1}I^{s-(i-1)(n+1)} : f = \mathfrak{m} = L' = I + L'$. For the rest of the proof, we may assume that if $v \in G(I^{s-(i-1)(n+1)} : \mu u)$, then $v \nmid g$.

Now, let $K^{i-1} = (g_1, \ldots, g_k, \ldots, g_r)$ with $g_j \mid g$ for $j = 1, \ldots, k$. Suppose $g = l_j g_j$ for $j = 1, \ldots, k$. Note that

$$\mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)} : f = \sum_{j=1}^{k} I^{s-(i-1)(n+1)} : \mu l_j u + \sum_{j=k+1}^{r} g_j I^{s-(i-1)(n+1)} : \mu g u.$$

We claim that for $k + 1 \leq j \leq r$, $g_j I^{s-(i-1)(n+1)} : \mu g u \subset \sum_{j=1}^{k} I^{s-(i-1)(n+1)} : \mu l_j u$. Suppose that $\gcd(g_j, g) = h_j$. Write $g_j = h_j g'_j$ and $g = h_j g'$. Now, let a be a monomial such that $a \mu g' u \in I^{s-(i-1)(n+1)}$. Hence $a g' \in I^{s-(i-1)(n+1)} : \mu u$ and this colon ideal is generated in at most degree 2, where the degree 2 generators are either edges or even connections. Then there exists a monomial generator v of $I^{s-(i-1)(n+1)} : \mu u$ dividing $a g'$. If $a \mu u \in I^{s-(i-1)(n+1)}$, then we are through. Assume that $a \mu u \notin I^{s-(i-1)(n+1)}$. If $v \mid a$, then $v \mu u$ and hence $a \mu u$ belongs to $I^{s-(i-1)(n+1)}$ which is a contradiction to our assumption. Hence $v \nmid a$. Also, $v \nmid g'$ (since $v \nmid g$). Hence we may write $v = l_j v'$ such that $l_j \mid g'$ and $v' \mid a$. This implies that $v' \in I^{s-(i-1)(n+1)} : l_j \mu u$, and hence $a \in I^{s-(i-1)(n+1)} : l_j \mu u$. Therefore $a \in \sum_{j=1}^{k} I^{s-(i-1)(n+1)} : \mu l_j u$ which proves the claim.

Now we claim that $I^{s-(i-1)(n+1)} : \mu l_j u = I + I' + L_j$, where I' is the ideal generated by the even connections with respect to $\frac{\mu u}{x_a}$ for all a. Let $v \in I^{s-(i-1)(n+1)} : \mu l_j u$. Then $v x_a l_j \in I^{s-(i-1)(n+1)} : \frac{\mu u}{x_a}$. As $I^{s-(i-1)(n+1)} : \frac{\mu u}{x_a}$ is generated by edges and even connections with respect to $\frac{\mu u}{x_a}$, there exists $w = w_1 w_2$ which is an edge or an even connection with respect to $\frac{\mu u}{x_a}$ such that $w \mid v x_a l_j$. If $w \mid v$, then we are done. If $w \mid x_a l_j$, then this implies that $\mu l_j u \in I^{s-(i-1)(n+1)}$, and hence $f \in \mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)}$ which is a contradiction. Now, let $w \nmid v$ and $w \nmid x_a l_j$. We may assume that $w_1 \mid v$ and $w_2 \mid x_a l_j$. If $w_2 = x_a$, then $w_2 \in N_G(C_n) = L$. If $w_2 = l_j$, then $w_1 \in N_G(M) \subset L'$, by Remark 4.7(i). Hence, In either case, $w \in L'$. Hence we get that $\mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)} : f = I + L'$, where $L' = \sum_{j=1}^{k} L_j$. Now, Using Lemma 4.6, we know that $I' \subset L'$ which completes the proof. $\square$

In the process of understanding colon with symbolic power, in a step-by-step manner, we now study the colon with respect to the partial sums in the decomposition of symbolic powers.

Lemma 4.9. *Let G, μ, K, L be as defined in the beginning of the section. Assume that z_r is not part of any cycle for all $r = 1, \ldots, m$. Let $I = I(G)$ and for $1 \leq i \leq \lfloor \frac{s}{n+1} \rfloor + 1$, set $I_{i-1} = \sum_{t=0}^{i-1} \mu^t K^t I^{s-t(n+1)}$. Then there exists an ordering of $G \left(\mu^i K^i I^{s-i(n+1)} \right) = \{u_1, \ldots, u_r\}$ such that for all $j = 0, \ldots, r - 1$,*

$$(I_{i-1} + (u_1, \ldots, u_j)) : u_{j+1} = I + L'',$$

where L'' is an ideal containing L and generated by a subset of variables.

Proof. Let $u_{j+1} = \mu^i f u$, where $f \in G(K^i)$ and $u \in G(I^{s-i(n+1)})$. In order to prove the assertion we claim that for a fixed i and $t < i - 1$, if $\mu^i g u \notin \mu^t K^t I^{s-i(n+1)}$, then

$$\mu^t K^t I^{s-t(n+1)} : \mu^i g u \subset \mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)} : \mu^i g u.$$

We first consider the linear part of the left hand side colon ideal and show that it is contained in the right hand side. Note that

$$K^t I^{s-t(n+1)} : \mu^{i-t} g u = \sum_{j=1}^{k} I^{s-t(n+1)} : \mu^{i-t} \frac{g}{g_j} u + \sum_{j=k+1}^{r} \frac{g_j}{\gcd(g, g_j)} I^{s-t(n+1)} : \mu^{i-t} \frac{g}{\gcd(g, g_j)} u,$$

where $g_j \in K^t$ is a divisor of g for $1 \leq j \leq k$ and for $k + 1 \leq j \leq r$, $g_j \in K^t$ that does not divide g. As in the case of proof Lemma 4.8, it can be shown that the second term in the above summation is contained in the first. Hence, to prove the assertion, it is enough to consider the first summation.

First of all, note that $\mu^{i-t} u \in I^{s-t(n+1)-\left[\frac{i-t+1}{2}\right]}$. Again, as the proof of Lemma 4.8, one can see that this ideal is generated in degree at most 2, with the degree 2 part generated by some of the edges and even connections. Hence, if $l \in V(G)$ is such that $l \in K^t I^{s-t(n+1)-\left[\frac{i-t-1}{2}\right]} : \mu^{i-t} g u$, then there exists $l' \in V(C_{2n+1})$ or $l' \in \mathrm{supp}(g)$ such that $ll' \in E(G)$ or an even connection. By Remark 4.7, we get that $l \in \mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)} : \mu^i g u$. Now, note that $u \in I^{s-i(n+1)}$ and $\mu^{i-t} \in I^{(i-t)(n+1)-\left[\frac{i-t+1}{2}\right]}$. Hence by Lemma 4.3, we know that $K^t I^{s-t(n+1)-\left[\frac{i-t-1}{2}\right]} : \mu^{i-t} g u = I + L'_t$, where L'_t is generated by a set of variables and contains L. This implies that $\mu^t K^t I^{s-t(n+1)} : \mu^i g u \subset I + L' = \mu^{i-1} K^{i-1} I^{s-(i-1)(n+1)} : \mu^i g u$. This proves the claim.

Thus by Lemma 4.8, we have $I_{i-1} : u_{j+1} = (I^{s-i(n+1)+1} : u) + L'$. On $G(\mu^i K^i I^{s-i(n+1)})$, define an ordering induced by the ordering in Lemma 4.6 and Definition 4.2. By Lemma 4.3, there exists a largest ideal generated by a subset of variables, say L_2, such that $L_2 \subset (u_1, \ldots, u_j) : u_{j+1}$ and $(u_1, \ldots, u_j) : u_{j+1} \subset I^{s-i(n+1)+1} : u + L_2$. Take $L'' = L' + L_2$. Then it follows from Theorem 4.5 and Lemma 4.6 that $(I_{i-1} + (u_1, \ldots, u_j)) : u_{j+1} \subseteq I + L''$. Since $L_2 \subset (u_1, \ldots, u_j) : u_{j+1}$ and $I_{i-1} : u_{j+1} = (I^{s-i(n+1)+1} : u) + L' \supset I + L'$, we get the reverse containment as well. $\square$

Remark 4.10. Let H be the induced subgraph on $V(G) \setminus N_G(C_{2n+1})$. By Lemma 4.9, we know that

$$(I_{i-1} + (u_1, \ldots, u_j)) : u_{j+1} = I + L'',$$

where $L \subset L'' \subset \mathfrak{m}$. This implies that $I + L''$ corresponds to an induced subgraph of H. Therefore, we get

$$\mathrm{reg}\left(\frac{S}{(I_{i-1} + (u_1, \ldots, u_j)) : u_{j+1}}\right) \leq \mathrm{reg}\left(\frac{S}{I(H)}\right) = \nu(H).$$

Proposition 4.11. *Let the notation be as in Lemma 4.9 and H denote the induced subgraph on $V(G) \setminus N_G(C_{2n+1})$. If $\nu(G) - \nu(H) \geq 3$, then for $1 \leq i \leq \lfloor \frac{s}{n+1} \rfloor + 1$, $\mathrm{reg}(I^s) = \mathrm{reg}(I_{i-1})$.*

Proof. Result is true for $i = 1$. Assume that it is true for $i - 1$. Using Lemma 4.9, write $G(\mu^i K^i I^{s-i(n+1)}) = \{u_1, \ldots, u_r\}$ such that for all $j = 0, \ldots, r - 1$,

$$I_i = I_{i-1} + (u_1, \ldots, u_r) \text{ and } (I_{i-1} + (u_1, \ldots, u_j)) : u_{j+1} = I + L''.$$

For $j = 0$, consider the following exact sequence

$$0 \longrightarrow \frac{S}{I + L''}(-2s) \xrightarrow{\cdot u_1} \frac{S}{I_{i-1}} \longrightarrow \frac{S}{I_{i-1} + (u_1)} \longrightarrow 0.$$

From Remark 4.10, we know that

$$\operatorname{reg}\left(\frac{S}{I + L''}(-2s)\right) \leq 2s + \nu(H) \leq 2s + \nu(G) - 3 < \operatorname{reg}\left(\frac{S}{I^s}\right) = \operatorname{reg}\left(\frac{S}{I_{i-1}}\right),$$

where the third inequality follows from [3, Theorem 4.5]. Hence $\operatorname{reg}\left(\dfrac{S}{I_{i-1}}\right) = \operatorname{reg}\left(\dfrac{S}{I_{i-1} + (u_1)}\right)$. Assume by induction on j that $\operatorname{reg}(I_{i-1}+(u_1,\ldots,u_{j-1})) = \operatorname{reg}(I^s)$. Since $(I_{i-1}+(u_1,\ldots,u_j)) : u_{j+1} = I + L''$, we get the desired equality from the short exact sequence:

$$0 \longrightarrow \frac{S}{I + L''}(-2s) \xrightarrow{\cdot u_{j+1}} \frac{S}{I_{i-1} + (u_1, \ldots, u_j)} \longrightarrow \frac{S}{I_{i-1} + (u_1, \ldots, u_{j+1})} \longrightarrow 0.$$

$\square$

We are now ready to prove our second main theorem.

Theorem 4.12. *Let G be a graph obtained by taking clique sum of a C_{2n+1} and some bipartite graphs. Let H be an induced subgraph of G on vertices $V \setminus \bigcup_{x \in V(C_{2n+1})} N_G(x)$. Assume that none of the vertices of H is part of any cycle in G. If $\nu(G) - \nu(H) \geq 3$, then $\operatorname{reg}\left(I^{(s)}\right) = \operatorname{reg}\left(I^s\right)$.*

Proof. Let $s \geq 1$ and $k = \lfloor \frac{s}{n+1} \rfloor$. Consider the following exact sequence

$$0 \longrightarrow \frac{S}{I_k} \longrightarrow \frac{S}{I^{(s)}} \oplus \frac{S}{\mathfrak{m}^{2s}} \longrightarrow \frac{S}{I^{(s)} + \mathfrak{m}^{2s}} \longrightarrow 0,$$

where $I_k = \sum_{t=0}^k \mu^t K^t I^{s-t(n+1)}$. Since $\nu(G) \geq 2$, we have $\operatorname{reg}\left(\dfrac{S}{I^{(s)}}\right) > \operatorname{reg}\left(\dfrac{S}{\mathfrak{m}^{2s}}\right) = \operatorname{reg}\left(\dfrac{S}{I^{(s)} + \mathfrak{m}^{2s}}\right)$. Hence $\operatorname{reg}\left(\dfrac{S}{I_k}\right) = \operatorname{reg}\left(\dfrac{S}{I^{(s)}}\right)$. Since $\nu(G) - \nu(H) \geq 3$, by Proposition 4.11, we get that $\operatorname{reg}(I^s) = \operatorname{reg}(I^{(s)})$. $\square$

Remark 4.13. If the odd cycle in G is of length at least 9, then the condition $\nu(G) - \nu(H) \geq 3$ is always satisfied.

(1) If the unique odd cycle in G is of length 7, then the hypothesis of Theorem 4.12 is satisfied if a P_3 is attached to C_7.

(2) If the unique odd cycle in G is of length 5, then the hypothesis of Theorem 4.12 is satisfied if either two P_3's are attached to a single vertex or a P_3 and a P_2 are attached to adjacent vertices (see figure below).

(3) If the unique odd cycle in G is of length 3, then the hypothesis of Theorem 4.12 is satisfied if either two P_3's are attached to a single vertex or on each vertex of C_3 a P_3 is attached (see figure below).

(4) It may also be noted that the class of graphs considered in Theorem 4.12 is not a subset of unicyclic graphs. It also includes graphs which are obtained by taking clique sum of copies of C_4 along the edges of an odd cycle (see figure below).

We illustrate with pictures, some of the graphs for which the regularity of the symbolic powers of their edge ideals are same as that of their regular powers.
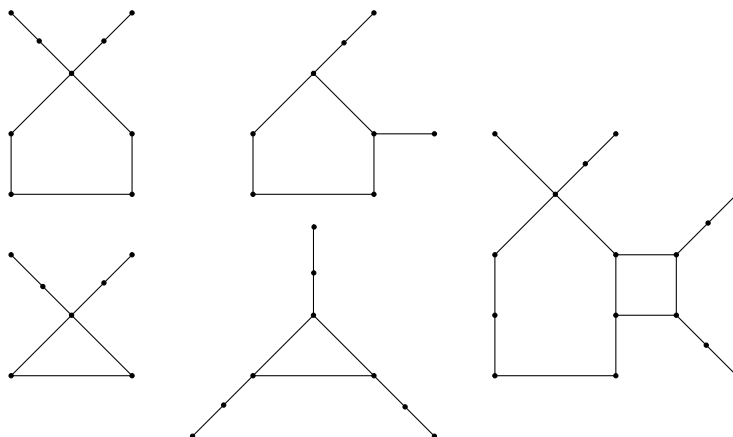
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