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REGULARITY OF POWERS OF QUADRATIC SEQUENCES WITH APPLICATIONS TO BINOMIAL IDEALS

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ABSTRACT. In this article, we obtain an upper bound for the Castelnuovo-Mumford regularity of powers of an ideal generated by a homogeneous quadratic sequence in a polynomial ring in terms of the regularity of its related ideals and degrees of its generators. As a consequence, we compute upper bounds for the regularity of powers of several binomial ideals. We generalize a result of Matsuda and Murai to show that the regularity of J_G^s is bounded below by $2s+\ell(G)-1$ for all $s \geq 1$, where J_G denotes the binomial edge ideal of a graph G and $\ell(G)$ is the length of a longest induced path in G. We compute the regularity of powers of binomial edge ideals of cycle graphs, star graphs, and balloon graphs explicitly. Also, we give sharp bounds for the regularity of powers of almost complete intersection binomial edge ideals and parity binomial edge ideals.

1. INTRODUCTION

Huneke introduced the notion of d-sequence, in [17], and proved that the symmetric algebra and Rees algebra of an ideal generated by a d-sequence in a Noetherian ring are isomorphic, [15] (see [38] for a simple proof). He used the theory of d-sequence to study the depth of powers of ideals in a Noetherian ring R, [17]. He generalized this notion to weak d-sequence and analyzed behavior of R/I^n, when I is generated by a weak d-sequence, [16]. Raghavan further generalized the notion of weak d-sequence to quadratic sequence and studied the depth of R/I^n when I is generated by a quadratic sequence, [37]. In this paper, we obtain an upper bound for the Castelnuovo-Mumford regularity of R/I^n in terms of the regularity of related ideals, and degrees of the generators, where R is a standard graded polynomial ring over a field $\mathbb{K}$ and I is an ideal generated by a homogeneous quadratic sequence, Theorem 2.8, Corollary 2.11. To illustrate our result, we compute the regularity of powers of the defining ideal of a class of projective monomial curves, which in fact, is a binomial ideal generated by a quadratic sequence.

Ever since Cutkosky, Herzog and Trung, in [6], and independently Kodiyalam, in [26], proved that if I is a homogeneous ideal in a polynomial ring R, then $\operatorname{reg}(I^s) = as + b$ for $s \gg 0$, for some non-negative integers a, b, it has been a constant effort from the researchers to compute a and b for several classes of homogeneous ideals. They showed that a is at most the maximum degree of a minimal homogeneous generator of I. It has remained a challenge to compute the constant term in the linear polynomial. During the past decade, there has been a lot of research activity in this direction. In particular, if $I(G)$ denotes the monomial edge ideal corresponding to a finite simple graph G, then researchers have obtained an upper bound for the constant term for all graphs and have computed the constant term for several subclasses of graphs (see [1, 23] and the references therein). While monomial ideals have

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received a lot of attention in this direction, there are not many such results for binomial ideals. Recently, Raicu computed the linear polynomial corresponding to the asymptotic regularity function for $p \times p$ minors of an $m \times n$ matrix, $m \geq n$, [39]. As an application of our result, we get upper bounds for the regularity of powers of certain binomial ideals, namely, binomial edge ideals and parity binomial edge ideals.

Let G be a simple graph with the vertex set $[n] = \{1, \dots, n\}$ and the edge set $E(G)$. Let $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring where $\mathbb{K}$ is an arbitrary field. The binomial edge ideal corresponding to G, denoted by J_G, is the ideal generated by the set $\{x_iy_j - x_jy_i : i < j \text{ and } \{i,j\} \in E(G)\}$. The notion of binomial edge ideal was introduced by Herzog et al. in [13] and independently by Ohtani in [36]. Another class of binomial ideals associated with finite simple graphs are the parity binomial edge ideals. For a graph G on the vertex set $[n]$, the parity binomial edge ideal $\mathcal{I}_G$ is the ideal generated by the set $\{x_ix_j - y_iy_j : \{i,j\} \in E(G), i < j\} \subset S$. Kahle et al. introduced this notion, [24], and studied its various properties.

In the recent past, researchers have been trying to understand various properties of these ideals and their relationship with combinatorial properties of corresponding graphs. While there is some success in the case of binomial edge ideals, the parity binomial edge ideals are quite new and nothing much is known about them. One line of research is to estimate the regularity of these ideals using combinatorial invariants of corresponding graphs. In [32], Matsuda and Murai proved that $\ell(G) \leq \operatorname{reg}(S/J_G) \leq n-1$, where $\ell(G)$ denotes the length of a longest induced path in G. This bound, in general, is a weak one and there are improved bounds for several classes, (see for example [14, 21, 25, 29, 30, 31, 40]). For some classes of graphs, precise expressions for the regularity have also been computed, (see for example [10, 18, 42, 43]). The lower bound in the Matsuda-Murai bound for the regularity was a consequence of a more general result, namely, if H is an induced subgraph of G, then $\beta_{i,j}(S/J_H) \leq \beta_{i,j}(S/J_G)$ for all i,j. We generalize this result to all powers, that is, $\beta_{i,j}(S/J_H^s) \leq \beta_{i,j}(S/J_G^s)$ for all i,j and $s \geq 1$, whenever H is an induced subgraph of G, Proposition 3.3. As an immediate consequence, we obtain a general lower bound, namely, $2s + \ell(G) - 2 \leq \operatorname{reg}(S/J_G^s)$ for all $s \geq 1$, Corollary 3.4.

Computing the regularity of powers of (parity) binomial edge ideals of an arbitrary graph seems more challenging compared to the regularity of powers of monomial edge ideals. Even in the case of simple classes of graphs, the regularity of the powers of their binomial edge ideals is not known. So, naturally one restricts the attention to important subclasses. In [19], we studied the Rees algebra and first graded Betti numbers of binomial edge ideals which are almost complete intersections. We proved that almost complete intersection binomial edge ideals are generated by d-sequence. Cutkosky, Herzog and Trung proved that if I is an ideal generated by a d-sequence of n forms of the same degree r, then for all $s \geq n+1$, $\operatorname{reg}(I^s) = (s-n-1)r + \operatorname{reg}(I^{n+1})$, [6, Corollary 3.8]. This expression depends on the number of generators of I. Moreover, in our situation, computing the linear polynomial boils down to computing $\operatorname{reg}(I^{n+1})$, which itself is challenging when the graph has a large number of edges. Note that a d-sequence is a quadratic sequence. Moreover, in the case of ideals generated by d-sequence, the computation of the related ideals becomes much simpler. Using the upper bounds in Theorem 2.8 and Corollary 2.11, we compute the regularity of powers of binomial edge ideals of cycles, star graphs and balloon graphs. For other almost complete intersection binomial edge ideals, we obtain bounds for the regularity of their powers.

Theorem 1.1. *Let G be a finite simple graph and J_G denote its binomial edge ideal in the polynomial ring S.*

(1) *If $G = K_{1,n}$, then $\operatorname{reg}(S/J_G^s) = 2s$ for all $s \geq 1$.*
(2) *If $G = C_n$, then $\operatorname{reg}(S/J_G^s) = 2s + n - 4$ for all $s \geq 1$.*
(3) *If G is a tree such that J_G is an almost complete intersection ideal, then*
$$2s + \operatorname{iv}(G) - 2 \leq \operatorname{reg}(S/J_G^s) \leq 2s + \operatorname{iv}(G) - 1,$$
for all $s \geq 1$, where $\operatorname{iv}(G)$ denotes the number of internal vertices of G.
(4) *If G is a unicyclic graph on $[n]$ such that J_G is an almost complete intersection ideal, then*
$$2s + n - 5 \leq \operatorname{reg}(S/J_G^s) \leq 2s + n - 4$$
for all $s \geq 1$.

Bolognini et al. proved that if G is bipartite, then J_G and $\mathcal{I}_G$ are isomorphic, [3]. Therefore, to study parity binomial edge ideals, we consider graphs containing an odd-cycle. For parity binomial edge ideals, we prove:

Theorem 1.2. *For all $s \geq 1$,*

(1) *if $n \geq 3$ is an odd integer, then $\operatorname{reg}(S/\mathcal{I}_{C_n}) = 2s + n - 2$,*
(2) *if G is a graph on $[n]$ obtained by adding an edge between an odd cycle and an internal vertex of a path, then $2s + n - 5 \leq \operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 4$,*
(3) *if G is either a balloon graph on $[n]$ having odd girth or a graph obtained by adding a chord in an odd cycle C_n, then $2s + n - 4 \leq \operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 3$.*

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2. Regularity of powers of quadratic sequence

In this section, we study the regularity of powers of an ideal generated by quadratic sequence. First, we recall the definition of quadratic sequence from [37].

Let Λ be a finite poset. A subset $\Sigma \subseteq \Lambda$ is said to be a *poset ideal* if it satisfies the following property:
$$\text{if } \sigma \in \Sigma \text{ and } \lambda \in \Lambda \text{ with } \lambda \leq \sigma, \text{ then } \lambda \in \Sigma.$$
Let R be a commutative Noetherian ring with unity and $\{u_\lambda : \lambda \in \Lambda\}$ be a set of elements of R indexed by Λ. For $\Sigma \subseteq \Lambda$, let U_Σ denote the ideal of R generated by $\{u_\sigma : \sigma \in \Sigma\}$. Note that $U_\emptyset = (0)$. Let Σ be a poset ideal of Λ and $\lambda \in \Lambda$. We say that λ *lies just above* Σ if it satisfies the following:

i) $\lambda \notin \Sigma$ and
ii) $\sigma \in \Sigma$, whenever $\sigma \in \Lambda$ and $\sigma < \lambda$.

An element $\lambda \in \Lambda$ is said to *lie inside or just above* Σ if either $\lambda \in \Sigma$ or λ lies just above Σ.

Definition 2.1. ([37, Definition 3.3]) *Let Λ be a finite poset and $I \subset R$ be an ideal. Let "-" denote images in R/I. A set of elements $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ is said to be a quadratic sequence with respect to the ideal I if for every pair (Σ, λ), where Σ is a poset ideal of Λ and λ lies inside or just above Σ, there exists a poset ideal Θ of Λ such that*

(1) $(\bar{U}_\Sigma : \bar{u}_\lambda) \cap \bar{U}_\Lambda \subseteq \bar{U}_\Theta$,
(2) $u_\lambda U_\Theta \subseteq (U_\Sigma + I)U_\Lambda$.

A set of elements $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ is said to be a quadratic sequence if it is a quadratic sequence with respect to the zero ideal.

We now recall some basic properties of quadratic sequences from [37] which are required for our results.

Observation 2.2. ([37, Remark 3.4])

(1) Let I be an ideal of R. If $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ is a quadratic sequence with respect to I, then $\{\bar{u}_\lambda : \lambda \in \Lambda\} \subseteq R/I$ is also a quadratic sequence.
(2) If $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ is a quadratic sequence, then for any poset ideal $\Sigma \subset \Lambda$, $\{\bar{u}_\lambda : \lambda \in \Lambda \setminus \Sigma\} \subseteq R/U_\Sigma$ is a quadratic sequence.

Lemma 2.3. ([37, Corollaries 3.7 and 5.2]) *Let $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ be a quadratic sequence. Then*

(1) for every poset ideal Σ of Λ, $U_\Sigma \cap U_\Lambda^s = U_\Sigma U_\Lambda^{s-1}$ for any integer $s \geq 1$ and
(2) for any minimal element α of Λ, $\{\bar{u}_\lambda : \lambda \in \Lambda\} \subseteq R/(0 : u_\alpha)$ is a quadratic sequence.

Related ideals help to understand the structure of the ideals generated by quadratic sequence and their powers. We recall its definition here.

Definition 2.4. ([37, Definition 5.3]) *Let $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ be a quadratic sequence. An ideal $J \subseteq R$ is said to be a related ideal to the quadratic sequence if $J = U_\Lambda$ or $J = (U_\Sigma : u_\lambda) + U_\Lambda$ for some pair (Σ, λ), where Σ is a poset ideal of Λ and λ lies inside or just above Σ.*

In the following, we separate out a result from the proof of Theorem 5.4 of [37] which is required for the main theorem in this section.

Lemma 2.5. *Let $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ be a quadratic sequence and α be a minimal element of Λ. Let Σ be a poset ideal of Λ and $\lambda \in \Lambda$ lies inside or just above Σ. Then $((U_\Sigma + (0 : u_\alpha)) : u_\lambda) + U_\Lambda$ is a related ideal to the quadratic sequence $\{u_\lambda : \lambda \in \Lambda\}$.*

One of the important aspects in the study of powers of ideals generated by quadratic sequence is the existence of a filtration with some nice properties. We now prove a graded version of this result, [37, Theorem 5.4]. The proof is similar to that of the original result. We include it here for the sake of completeness.

Theorem 2.6. *Let $R = \oplus_{n \geq 0} R_n$ be a graded R_0-algebra, where R_0 is a Noetherian ring. Let Λ be a finite poset and $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ be a set of homogeneous elements of R with $\deg(u_\lambda) = d_\lambda > 0$. Set $d = \max\{d_\lambda : \lambda \in \Lambda\}$. If $\{u_\lambda : \lambda \in \Lambda\}$ is a quadratic sequence, then for every $s \geq 1$, there exists a graded filtration of R/U_Λ^s*

$$R/U_\Lambda^s = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = (0)$$

such that for every $0 \leq i \leq k-1$, there exists a related ideal V_i and $0 \leq d_i \leq d(s-1)$ with $M_i/M_{i+1} \simeq [R/V_i](-d_i)$.

Proof. We prove the assertion by induction on $|\Lambda|+s$. If $s=1$, then $(0)\subseteq R/U_\Lambda$ is the required filtration. Assume that $s\geq 2$. Let $|\Lambda|=1$. Set $\Lambda=\{\alpha\}$. Then, for any $s\geq 2$,

$$R/(u_\alpha^s)\supseteq (u_\alpha)/(u_\alpha^s)\supseteq\cdots\supseteq (u_\alpha^{s-1})/(u_\alpha^s)\supseteq (0) \tag{1}$$

is a filtration of $R/(u_\alpha^s)$. Take $\Sigma=\emptyset$, then α lies just above Σ. Hence, 2.1(1) and (2) translates to $(0:u_\alpha)\cap(u_\alpha)\subseteq U_\Theta$ and $u_\alpha U_\Theta=(0)$ for some poset ideal Θ. If $\Theta=\{\alpha\}$, then $u_\alpha^2=0$. Therefore, $u_\alpha\in(0:u_\alpha)$, and hence, $R/(0:u_\alpha)=R/((0:u_\alpha)+(u_\alpha))\cong(u_\alpha)$. Since both (u_α) and $(0:u_\alpha)+(u_\alpha)$ are related ideals, $R\supset(u_\alpha)\supset(0)$ is the required filtration. Now, suppose $\Theta=\emptyset$. Then, $(0:u_\alpha)\cap(u_\alpha)=(0)$. Let $k\geq 2$ and $au_\alpha^k=0$. Then, $au_\alpha^{k-1}\in(0:u_\alpha)\cap(u_\alpha)=0$. Hence, $a\in(0:u_\alpha^{k-1})$. Therefore, $(0:u_\alpha^k)=(0:u_\alpha^{k-1})$.

Now, we show that $\frac{(u_\alpha^k)}{(u_\alpha^{k+1})}\simeq\frac{(u_\alpha^{k-1})}{(u_\alpha^k)}(-d_\alpha)$ for $k\geq 2$. Consider $\mu_{u_\alpha}:(u_\alpha^{k-1})\to\frac{(u_\alpha^k)}{(u_\alpha^{k+1})}$, the multiplication by u_α. Let $a\in(u_\alpha^{k-1})$ be such that $au_\alpha\in(u_\alpha^{k+1})$. Write $a=fu_\alpha^{k-1}$ and $au_\alpha=gu_\alpha^{k+1}$ for some $f,g\in R$. Then, for $k\geq 2$, $(f-gu_\alpha)\in(0:u_\alpha^k)=(0:u_\alpha^{k-1})$ and so $fu_\alpha^{k-1}\in(u_\alpha^k)$. Therefore, for $k\geq 2$,

$$\frac{(u_\alpha^k)}{(u_\alpha^{k+1})}\simeq\frac{(u_\alpha^{k-1})}{(u_\alpha^k)}(-d_\alpha)\simeq\cdots\simeq\frac{(u_\alpha)}{(u_\alpha^2)}(-(k-1)d_\alpha)\simeq\frac{R}{((0:u_\alpha)+(u_\alpha))}(-kd_\alpha),$$

where the last isomorphism is obtained by proving that the kernel of the multiplication mapping from R to $(u_\alpha)/(u_\alpha^2)$ is $((0:u_\alpha)+(u_\alpha))$. Note that $(0:u_\alpha)+(u_\alpha)$ is a related ideal, and hence, the filtration (1) satisfies the required conditions. This completes the case $|\Lambda|=1$.

Now, assume that $|\Lambda|\geq 2$ and $s\geq 2$. Let $\alpha\in\Lambda$ be a minimal element. Consider the filtration $R/U_\Lambda^s\supseteq(u_\alpha,U_\Lambda^s)/U_\Lambda^s\supseteq(0)$. It follows from Lemma 2.3(1) that $\frac{(u_\alpha,U_\Lambda^s)}{(U_\Lambda^s)}\simeq\frac{(u_\alpha)}{(u_\alpha)\cap U_\Lambda^s}\simeq\frac{(u_\alpha)}{u_\alpha U_\Lambda^{s-1}}$. It is easy to see that the kernel of the multiplication map from R to $\frac{(u_\alpha)}{u_\alpha U_\Lambda^{s-1}}$ is $(0:u_\alpha)+U_\Lambda^{s-1}$. Therefore, $\frac{R}{(0:u_\alpha)+U_\Lambda^{s-1}}(-d_\alpha)\simeq\frac{(u_\alpha)}{u_\alpha U_\Lambda^{s-1}}$. Set $\bar{R}=R/(u_\alpha)$ and $\Lambda'=\Lambda\setminus\{\alpha\}$. Since $\{\bar{u}_\lambda:\lambda\in\Lambda'\}\subseteq R/(u_\alpha)$ is a quadratic sequence and $|\Lambda'|<|\Lambda|$, by induction $\bar{R}/\bar{U}_{\Lambda'}^s$ has a graded filtration

$$\bar{R}/\bar{U}_{\Lambda'}^s=N_0\supseteq N_1\supseteq\cdots\supseteq N_l=(0) \tag{2}$$

of $\bar{R}$-modules such that for each $0\leq j\leq l-1$, there exists $\bar{V}_j$, a related ideal to the quadratic sequence $\{\bar{u}_\lambda:\lambda\in\Lambda'\}$ and $0\leq d_j\leq d(s-1)$ such that $N_j/N_{j+1}\simeq[\bar{R}/\bar{V}_j](-d_j)$. If $\bar{V}_j=\bar{U}_{\Lambda'}$, then the pre-image of $\bar{V}_j$ in R is U_Λ, and therefore, $N_j/N_{j+1}\simeq[R/U_\Lambda](-d_j)$. So, assume that $\bar{V}_j=(\bar{U}_{\Sigma_j}:\bar{u}_{\lambda_j})+\bar{U}_{\Lambda'}$ for some poset ideal Σ_j of Λ' and $\lambda_j\in\Lambda'$ lies inside or just above Σ_j. The pre-image of $\bar{V}_j$ in R is $(U_{\Sigma_j\cup\{\alpha\}}:u_{\lambda_j})+U_\Lambda$ so that

$$N_j/N_{j+1}\simeq[R/((U_{\Sigma_j\cup\{\alpha\}}:u_{\lambda_j})+U_\Lambda)](-d_j).$$

Since α is a minimal element in Λ and Σ_j is a poset ideal of Λ', $\Sigma_j\cup\{\alpha\}$ is a poset ideal of Λ and λ_j lies inside or just above $\Sigma_j\cup\{\alpha\}$. Therefore, $(U_{\Sigma_j\cup\{\alpha\}}:u_{\lambda_j})+U_\Lambda$ is a related ideal to the quadratic sequence $\{u_\lambda:\lambda\in\Lambda\}$. Hence, $R/(u_\alpha,U_\Lambda^s)$ has the required graded filtration. Note that by Lemma 2.3(2), $\{u'_\lambda:\lambda\in\Lambda\}\subseteq R/(0:u_\alpha)$ is a quadratic sequence, where $'$ denotes the image modulo the ideal $(0:u_\alpha)$. Thus, by induction, there exists a graded filtration

$$R'/{U'}_\Lambda^{s-1}=L_0\supseteq L_1\supseteq\cdots\supseteq L_k=(0) \tag{3}$$

of R'-modules such that for each $0\leq i\leq k-1$, there exists a related ideal to $\{u'_\lambda:\lambda\in\Lambda\}$, $V'_i\subset R'$ and $0\leq d'_i\leq d(s-2)$ such that $L_i/L_{i+1}\simeq[R'/V'_i](-d'_i)$. Since V'_i's are related ideals to $\{u'_\lambda : \lambda \in \Lambda\}$, V'_i's are either U'_Λ or has the form $(U'_{\Sigma_i} : u'_{\lambda_i}) + U'_\Lambda$ for some poset ideal Σ_i of Λ and λ_i lying inside or just above Σ_i. Hence, $[R'/V'_i](-d'_i) \simeq [R/(((U_\Sigma + (0 : u_\alpha)) : u_\lambda) + U_\Lambda)](-d'_i)$ or $[R'/V'_i](-d'_i) \simeq [R/((0 : u_\alpha) + U_\Lambda)](-d'_i)$. By Lemma 2.5, $((U_\Sigma + (0 : u_\alpha)) : u_\lambda) + U_\Lambda$ is a related ideal to $\{u_\lambda : \lambda \in \Lambda\}$. Hence, we get a graded filtration for $[R'/{U'}^{s-1}_\Lambda](-d_\alpha) \simeq (u_\alpha, U^s_\Lambda)/U^s_\Lambda$. By combining the filtrations (2) and (3) we get the required filtration of R/U^s_Λ. □

The following result on the regularity is well-known and can be derived from the long exact sequence of Tor modules. We state it for the sake of convenience.

Lemma 2.7. *Let R be a standard graded ring and M, N, P be finitely generated graded R-modules. If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a short exact sequence with f, g graded homomorphisms of degree zero, then*

(1) $\operatorname{reg}(N) \le \max\{\operatorname{reg}(M), \operatorname{reg}(P)\}$,
(2) $\operatorname{reg}(M) \le \max\{\operatorname{reg}(N), \operatorname{reg}(P) + 1\}$,
(3) $\operatorname{reg}(P) \le \max\{\operatorname{reg}(M) - 1, \operatorname{reg}(N)\}$,
(4) $\operatorname{reg}(M) = \operatorname{reg}(P) + 1$ *if* $\operatorname{reg}(N) < \operatorname{reg}(M)$.

We now prove an upper bound for the regularity of powers of an ideal generated by a homogeneous quadratic sequence.

Theorem 2.8. *Let R be a standard graded polynomial ring over a field $\mathbb{K}$. Let Λ be a finite poset and $\{u_\lambda : \lambda \in \Lambda\} \subseteq R$ be a quadratic sequence. Then for $s \ge 1$*

$$\operatorname{reg}(R/U^s_\Lambda) \le d(s-1) + \max_{\Sigma,\lambda} \operatorname{reg}(R/((U_\Sigma : u_\lambda) + U_\Lambda)),$$

where Σ is a poset ideal of Λ and λ lies inside or just above Σ, and $d = \max\{\deg(u_\lambda) : \lambda \in \Lambda\}$.

Proof. By Theorem 2.6, there exists a graded filtration

$$R/U^s_\Lambda = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k = (0)$$

such that for every $0 \le i \le k-1$, there exists a related ideal V_i and $0 \le d_i \le d(s-1)$ with $M_i/M_{i+1} \simeq [R/V_i](-d_i)$. For $0 \le i \le k-1$, consider the following short exact sequence:

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_i/M_{i+1} \longrightarrow 0.$$

By applying Lemma 2.7 successively in above short exact sequences, we get

$$\begin{aligned} \operatorname{reg}(R/U^s_\Lambda) &\le \max\{\operatorname{reg}(M_i/M_{i+1}) : 1 \le i \le k-1\} \\ &= \max\{d_i + \operatorname{reg}(R/V_i) : 1 \le i \le k-1\} \\ &\le d(s-1) + \max_{\Sigma,\lambda} \operatorname{reg}(R/((U_\Sigma : u_\lambda) + U_\Lambda)), \end{aligned}$$

where Σ is a poset ideal of Λ and λ lies inside or just above Σ. □

We now illustrate the use of the above theorem with an example. This example also shows that the upper bound that we have obtained is sharp.

Example 2.9. Let $R = \mathbb{K}[x, y, z, w]$ be a polynomial ring and U denote the defining ideal for the projective monomial curve

$$(x : y : z : w) = (u^{b+c}, u^b v^c, u^c v^b, v^{b+c}),$$

with $\gcd(b,c)=1$ and $b>c$. Morales and Simis proved, [34, Proposition 2.2], that U is minimally generated by $b-c+2$ elements, which are

$$u_1 = xw - yz, u_2 = x^{b-c}z^c - y^b, \ldots, u_{b-c+1} = xz^{b-1} - w^{b-c-1}y^{c+1}, u_{b-c+2} = z^b - w^{b-c}y^c.$$

They proved that $u_1,\ldots,u_{b-c+2}$ is a quadratic sequence (even a weak d-sequence). Note that U is not generated by a d-sequence except when $b=3$ and $c=2$. Let $U_i = ((u_1,\ldots,u_{i-1}) : u_i)$. Then, from the proof of Proposition 2.2 of [34], we get $U_1=(0)$, $U_2=(xw-yz)$, $U_3 = \cdots = U_{b-c+2} = (x,y)$. Hence, the related ideals to the quadratic sequence $\{u_1,\ldots,u_{b-c+2}\}$ are either U or $U_i + U = (x,y,z^b)$ for $3 \le i \le b-c+2$.

Claim: For all $s \geq 1$, $\operatorname{reg}(R/U^s) = bs-1$.
Proof. First we prove that $\operatorname{reg}(R/U) = b-1$. In [34, Proposition 2.1], Morales and Simis computed a graded minimal R-presentation of U, which is as follows:

$$R(-(b+1))^{2(b-c)} \longrightarrow R(-2) \oplus R(-b)^{b-c+1} \longrightarrow U \longrightarrow 0.$$

This implies that $\beta_{1,1+j}(R/U) = 0 = \beta_{2,2+j}(R/U)$ for $j \geq b$. Note that (u_1,u_2) is a regular sequence and for $2 \le k \le b-c+1$, we consider the following short exact sequence:

$$0 \longrightarrow [R/(x,y)](-b) \xrightarrow{\cdot u_{k+1}} R/(u_1,\ldots,u_k) \longrightarrow R/(u_1,\ldots,u_{k+1}) \longrightarrow 0.$$

Since (u_1,u_2) is a regular sequence, $\operatorname{reg}(R/(u_1,u_2)) = b$. Applying Lemma 2.7 on the above short exact sequence, we get $\operatorname{reg}(R/(u_1,u_2,u_3)) \le b$, $\ldots$, $\operatorname{reg}(R/U) \le b$. It can be seen that $\beta_{2,2+b}([R/(x,y)](-b)) = 1$ and $\beta_{2,2+b}(R/(u_1,u_2)) = 1$ are the unique extremal Betti numbers of $[R/(x,y)](-b)$ and $R/(u_1,u_2)$ respectively. Since $\beta_{2,2+j+1}([R/(x,y)](-b)) = 0$ for $j \geq b$, we have the corresponding long exact sequence of Tor for $2 \le k \le b-c+1$:

$$0 \to \operatorname{Tor}_{3,3+j}\left(\frac{R}{(u_1,\ldots,u_k)}\right) \to \operatorname{Tor}_{3,3+j}\left(\frac{R}{(u_1,\ldots,u_{k+1})}\right) \to \operatorname{Tor}_{2,3+j}\left(\frac{R}{(x,y)}(-b)\right) \to \cdots$$

Thus, for $2 \le k \le b-c+1$ and $j \geq b$, $\beta_{3,3+j}(R/(u_1,\ldots,u_k)) = \beta_{3,3+j}(R/(u_1,\ldots,u_{k+1})) = 0$. Also we have for $2 \le k \le b-c+1$, $\beta_{i,i+j}(R/(u_1,\ldots,u_{k+1})) = 0$ for $i \geq 4$. Hence, $\operatorname{reg}(R/U) = b-1$.

Note that $U + U_i = U$ for $i = 1,2$ and for $3 \le i \le b+c-2$, $\operatorname{reg}(R/(U+U_i)) = \operatorname{reg}(R/(x,y,z^b)) = b-1$. Therefore, by Theorem 2.8, we have $\operatorname{reg}(R/U^s) \le bs-1$. Since there is an element of degree bs in U^s, $bs-1 \le \operatorname{reg}(R/U^s)$, and hence, $\operatorname{reg}(R/U^s) = bs-1$. $\square$

It follows from Theorem 2.8 that given an ideal U_Λ generated by a quadratic sequence, an upper bound for the regularity of all its powers can be obtained once we know the regularity of its related ideals. Given the ideal U_Λ and the poset structure of Λ, one can compute the related ideals and their regularity. But, in general, structure of the related ideals is not very well-understood. Here we study ideals generated by d-sequence for which the related ideals structure is much simpler.

Let $u_1,\ldots,u_n$ be homogeneous elements in R. Then, $u_1,\ldots,u_n$ is said to be a *homogeneous d-sequence* if

(1) u_i is not in the ideal generated by the rest of the u_j's and
(2) for all $k \geq i+1$ and all $i \geq 0$, $((u_1,\ldots,u_i) : u_{i+1}u_k) = ((u_1,\ldots,u_i) : u_k)$.

Costa, in [5], proved that $(u_1,\ldots,u_n)$ is a d-sequence if and only if for $1 \le i \le n$

$$((u_1,\ldots,u_{i-1}) : u_i) \cap (u_1,\ldots,u_n) = (u_1,\ldots,u_{i-1}).$$

It is clear that if $u_1, \dots, u_n$ is a d-sequence, then they form a quadratic sequence with respect to the poset $\{1 < \cdots < n\}$. Let $u_1, \dots, u_n$ be a homogeneous d-sequence in R such that $u_1, \dots, u_{n-1}$ is a regular sequence. Then, one can note that the related ideals to $\{u_1, \dots, u_n\}$ in R are of the form $(u_1, \dots, u_n)$ or $((u_1, \dots, u_{n-1}) : u_n) + (u_n)$. First we obtain an upper bound for the regularity of these related ideals.

Proposition 2.10. *Let $u_1, \dots, u_n$ be a homogeneous d-sequence with $\deg(u_i) = d_i$ in a standard graded polynomial ring R over a field $\mathbb{K}$ such that $u_1, \dots, u_{n-1}$ is a regular sequence. Then,*

$$\operatorname{reg}(R/(((u_1, \dots, u_{n-1}) : u_n), u_n)) \leq \max\{\operatorname{reg}(R/(u_1, \dots, u_n)), \sum_{i=1}^{n-1} d_i - n\}.$$

Proof. For convenience, let $U = (u_1, \dots, u_n)$ and $U' = (u_1, \dots, u_{n-1})$. Consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{(U' : u_n)}(-d_n) \xrightarrow{\cdot u_n} \frac{R}{U'} \longrightarrow \frac{R}{U} \longrightarrow 0.$$

Since $u_1, \dots, u_{n-1}$ is a regular sequence with $\deg(u_i) = d_i$,

$$\operatorname{reg}(R/U') = \sum_{i=1}^{n-1}(d_i - 1) = \sum_{i=1}^{n-1} d_i - (n-1).$$

Therefore, by Lemma 2.7,

$$\operatorname{reg}(R/(U' : u_n)) + d_n \leq \max\{\operatorname{reg}(R/U) + 1, \sum_{i=1}^{n-1} d_i - (n-1)\}.$$

Now, consider the following short exact sequence:

$$0 \longrightarrow \frac{R}{(U' : u_n^2)}(-d_n) \xrightarrow{\cdot u_n} \frac{R}{(U' : u_n)} \longrightarrow \frac{R}{((U' : u_n), u_n)} \longrightarrow 0.$$

Since $u_1, \dots, u_n$ is a d-sequence, $(U' : u_n^2) = (U' : u_n)$. Therefore, it follows from Lemma 2.7 that

$$\begin{aligned} \operatorname{reg}(R/((U' : u_n), u_n)) &= \operatorname{reg}(R/(U' : u_n)) + d_n - 1 \\ &\leq \max\left\{\operatorname{reg}(R/U), \sum_{i=1}^{n-1} d_i - n\right\} \end{aligned}$$

and this completes the proof. □

As an immediate consequence, we obtain an upper bound for the regularity of powers of this class of ideals.

Corollary 2.11. *Let R be a standard graded polynomial ring over a field $\mathbb{K}$ and $u_1, \dots, u_n$ be a homogeneous d-sequence with $\deg(u_i) = d_i$ in R such that $u_1, \dots, u_{n-1}$ is a regular sequence. Set $U = (u_1, \dots, u_n)$ and $d = \max\{d_i : 1 \leq i \leq n\}$. Then, for all $s \geq 1$,*

$$\operatorname{reg}(R/U^s) \leq d(s-1) + \max\{\operatorname{reg}(R/U), \sum_{i=1}^{n-1} d_i - n\}.$$

Proof. Immediately follows from Theorem 2.8 and Proposition 2.10. □

3. Regularity of powers of binomial edge ideals

While there is extensive research on the regularity of powers of monomial edge ideals corresponding to graphs, there is absolutely no such result in the case of binomial edge ideals. In this section, we obtain bounds as well as precise expressions for the regularity of powers of binomial edge ideals corresponding to some classes of graphs. We first recall the terminologies that are needed from graph theory for our purpose.

Let G be a simple graph with the vertex set $V(G) = [n]$ and edge set $E(G)$. A *complete graph* on $[n]$, denoted by K_n, is the graph with the edge set $E(G) = \{\{i, j\} : 1 \leq i < j \leq n\}$. For $A \subseteq V(G)$, $G[A]$ denotes the *induced subgraph* of G on the vertex set A, that is, for $i, j \in A$, $\{i, j\} \in E(G[A])$ if and only if $\{i, j\} \in E(G)$. For a vertex v, $G \setminus v$ denotes the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$. A subset U of $V(G)$ is said to be a *clique* if $G[U]$ is a complete graph. A vertex v of G is said to be a *simplicial vertex* if v is contained in only one maximal clique otherwise it is called an *internal vertex*. We denote the number of internal vertices of G by $\mathrm{iv}(G)$. For a vertex v, $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ denotes the *neighborhood* of v in G and G_v is the graph on the vertex set $V(G)$ and edge set $E(G_v) = E(G) \cup \{\{u, w\} : u, w \in N_G(v)\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is $|N_G(v)|$. A vertex v is said to be a *pendant vertex* if $\deg_G(v) = 1$. For an edge e in G, $G \setminus e$ is the graph on the vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. Let $u, v \in V(G)$ be such that $e = \{u, v\} \notin E(G)$, then we denote by G_e, the graph on the vertex set $V(G)$ and edge set $E(G_e) = E(G) \cup \{\{x, y\} : x,\ y \in N_G(u) \textit{ or } x,\ y \in N_G(v)\}$. A *cycle* is a connected graph G with $\deg_G(v) = 2$ for all $v \in V(G)$. A graph is said to be a *unicyclic* graph if it contains exactly one cycle. A graph is a *tree* if it does not contain a cycle. The length of a shortest induced cycle in G is called the *girth* of G. A *path graph* on n vertices, denoted by P_n, is a graph with the vertex set $[n]$ and the edge set $\{\{i, i+1\} : 1 \leq i \leq n-1\}$. A vertex v of G is said to be a *cut vertex* if $G \setminus v$ has more connected components than G. A *block* of a graph is a maximal nontrivial connected subgraph which has no cut vertex. If every block of a connected graph G is a complete graph, then G is called a *block graph*. Let G be a tree and $L(G) = \{v \in V(G) : \deg_G(v) = 1\}$. If $G[L(G)^c]$ is either empty or a path, then G is said to be a *caterpillar*. A collection of edges $\{e_1, \ldots, e_s\}$ is said to be a *matching* if $e_i \cap e_j = \emptyset$ for all $i \neq j$ and this is said to be an *induced matching* if the induced subgraph on the vertices of $\{e_1, \ldots, e_s\}$ has edge set $\{e_1, \ldots, e_s\}$. For a graph G, the induced matching number of G is the largest size of an induced matching, and it is denoted by $\nu(G)$.

Notation 3.1. Let G be a graph on $[n]$. We reserve the notation S for the polynomial ring $\mathbb{K}[x_i, y_i : i \in [n]]$. Also, if there is only one graph on $[n]$ in a given context, irrespective of the notation used for this graph, the polynomial ring associated with this graph would be denoted by S. If H is any other graph on $[k]$, then we set $S_H = \mathbb{K}[x_i, y_i : i \in [k]]$. If $k \leq n$, then S_H can be considered as a subring of S. Note that the graded Betti numbers of J_H considered as an ideal of S_H are the same as those when considered as an ideal of S. Therefore, for the convenience of notation, in such cases, we consider J_H as an ideal in S. For an edge $e = \{i, j\}$ with $i < j$, let $f_e = x_i y_j - x_j y_i$.

We first make some observations, which can be derived easily from the results existing in the literature.

Observation 3.2. (1) Let G be a disjoint union of k paths with $|V(G)| = n$. Then, it follows from [8, Corollary 1.2] that J_G is generated by a regular sequence in degree 2 of length $n-k$. Therefore, by virtue of [2, Lemma 4.4], $\mathrm{reg}(S/J_{P_n}^s) = 2s + n - k - 2$ for all $s \geq 1$.

(2) Let $G = K_n$ and $I = \mathrm{in}_{\mathrm{lex}}(J_G)$, where lex denotes the lexicographic order on S induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. By [4, Theorem 2.1], $I^s = \mathrm{in}_{\mathrm{lex}}(J_G^s)$. Hence, $\mathrm{reg}(S/J_G^s) \leq \mathrm{reg}(S/\mathrm{in}_{\mathrm{lex}}(J_G^s)) = \mathrm{reg}(S/I^s)$. Note that I is a quadratic squarefree monomial ideal. If H denotes the graph corresponding to the ideal I, then H is a weakly chordal bipartite graph, [10, Lemma 3.3]. Hence, by [22, Corollary 5.1], $\mathrm{reg}(S/I^s) = 2s + \nu(H) - 2$. It is easy to observe in this case that $\nu(H) = 1$. Hence, $\mathrm{reg}(S/I^s) = 2s - 1$ which implies that $\mathrm{reg}(S/J_G^s) \leq 2s - 1$. Since J_G^s is generated in degree $2s$, we get $\mathrm{reg}(S/J_G^s) = 2s - 1$. Therefore, J_G^s has linear resolution for all $s \geq 1$. For $s = 1$, this property has been proved by Saeedi Madani and Kiani in [41, Theorem 2.1].

Matsuda and Murai [32, Corollary 2.2] proved that if H is an induced subgraph of G, then $\beta_{i,j}(S/J_H) \leq \beta_{i,j}(S/J_G)$ for all i, j. We generalize this to all powers.

Proposition 3.3. *Let H be an induced subgraph of G. Then, for all $i, j \geq 0$ and $s \geq 1$ $\beta_{i,j}(S/J_H^s) \leq \beta_{i,j}(S/J_G^s)$.*

Proof. First we claim that $J_H^s = J_G^s \cap S_H$ for all $s \geq 1$, where J_H is the binomial edge ideal of H in S_H. Since generators of J_H^s are contained in J_G^s, $J_H^s \subseteq J_G^s \cap S_H$. Now, let $g \in J_G^s \cap S_H$. Let $g = \sum_{e_1,\ldots,e_s \in E(G)} h_{e_1,\ldots,e_s} f_{e_1} \cdots f_{e_s}$, where $h_{e_1,\ldots,e_s} \in S$. Now, consider the map $\pi : S \to S_H$ by setting $\pi(x_i) = 0 = \pi(y_i)$ if $i \notin V(H)$ and $\pi(x_i) = x_i, \pi(y_i) = y_i$ if $i \in V(H)$. If $e \in E(G) \setminus E(H)$, then $\pi(f_e) = 0$. If $e \in E(H)$, then $\pi(f_e) = f_e$. Since $g \in S_H$, $\pi(g) = g$. Therefore, we get

$$\begin{aligned} g &= \sum_{e_1,\ldots,e_s \in E(G)} \pi(h_{e_1,\ldots,e_s}) \pi(f_{e_1}) \cdots \pi(f_{e_s}) \\ &= \sum_{e_1,\ldots,e_s \in E(H)} \pi(h_{e_1,\ldots,e_s}) f_{e_1} \cdots f_{e_s}. \end{aligned}$$

Thus, $g \in J_H^s$. Hence, S_H/J_H^s is a $\mathbb{K}$-subalgebra of S/J_G^s. Consider, $S_H/J_H^s \overset{i}{\hookrightarrow} S/J_G^s \overset{\bar{\pi}}{\to} S_H/J_H^s$, where $\bar{\pi}$ is induced by the map π. Note that $\bar{\pi} \circ i$ is identity on S_H/J_H^s. Thus, S_H/J_H^s is an algebra retract of S/J_G^s. Now, the assertion follows from [35, Corollary 2.5]. ☐

Corollary 3.4. *Let G be a connected graph. Then, $\mathrm{reg}(S/J_G^s) \geq 2s + \ell(G) - 2$ for all $s \geq 1$, where $\ell(G)$ is the length of a longest induced path of G.*

Proof. Let H be a longest induced path of G. Then, H is an induced subgraph of G. By Observation 3.2, $\mathrm{reg}(S_H/J_H^s) = 2s + \ell(G) - 2$ for all $s \geq 1$. Hence, the assertion follows from Proposition 3.3. ☐

In [19], we had shown that $J_{K_{1,n}}$ is generated by a d-sequence. Schenzel and Zafar proved that $\mathrm{reg}(S/J_{K_{1,n}}) = 2$, [42]. We now compute the regularity of their powers.

Theorem 3.5. *Let $G = K_{1,n}$ be a star graph for $n \geq 3$. Then, $\mathrm{reg}(S/J_G^s) = 2s$ for all $s \geq 1$.*

Proof. Let $G = K_{1,n}$ denote the star graph on the vertex set $[n+1]$ with the edge set $E(G) = \{\{i, n+1\} : 1 \leq i \leq n\}$. By [42, Theorems 4.1], $\mathrm{reg}(S/J_G) = 2$. It follows from [19, Proposition 4.8] that $J_G = (f_{1,n+1}, \ldots, f_{n,n+1})$ is generated by a quadratic sequence with respect to the poset $\Lambda = \{\{1, n+1\} < \cdots < \{n, n+1\}\}$. Let V be a related ideal to J_G. Then V is either J_G or of the form $((f_{1,n+1}, \ldots, f_{i,n+1}) : f_{i+1,n+1}) + J_G$ for some $i \geq 2$. If $V = ((f_{1,n+1}, \ldots, f_{i,n+1}) : f_{i+1,n+1}) + J_G$ for some $i \geq 2$, then by [33, Theorem 3.7], $V = J_H$, where H is the graph obtained from the complete graph on vertex set $\{1, \ldots, i, n+1\}$ by adding edges $\{j, n+1\}$ for $i+1 \leq j \leq n$ at the vertex $n+1$. Thus, by using [14, Theorem 8], $\operatorname{reg}(S/V) = \operatorname{reg}(S/J_H) \leq 2$. Therefore, by Theorem 2.8, we have $\operatorname{reg}(S/J_G^s) \leq 2(s-1)+2 = 2s$ for all $s \geq 1$. Since $\ell(G) = 2$, by Corollary 3.4, $\operatorname{reg}(S/J_G^s) \geq 2s$ for all $s \geq 1$. Hence, for all $s \geq 1$, $\operatorname{reg}(S/J_G^s) = 2s$. □

Now, we obtain the regularity of powers of binomial edge ideals of cycle graphs.

Theorem 3.6. *Let $n \geq 3$. Then, for all $s \geq 1$, $\operatorname{reg}(S/J_{C_n}^s) = 2s + n - 4$.*

Proof. Let $G = C_n$ be the cycle graph on $[n]$. By [43, Corollary 16], we have $\operatorname{reg}(S/J_G) = n-2$. Moreover, $f_{1,2}, \dots, f_{n-1,n}$ is a regular sequence and by [19, Theorem 4.9], $f_{1,2}, \dots, f_{n-1,n}, f_{1,n}$ is a d-sequence. Hence, it follows from Corollary 2.11 that $\operatorname{reg}(S/J_G^s) \leq 2s + n - 4$ for all $s \geq 1$. Since C_n contains an induced path of length $n-2$, it follows from Corollary 3.4 that $2s + n - 4 \leq \operatorname{reg}(S/J_G^s)$ for all $s \geq 1$. Hence, $\operatorname{reg}(S/J_G^s) = 2s + n - 4$ for all $s \geq 1$. □

In [20], it was proved that if G is a tree, then $\operatorname{iv}(G) + 1 \leq \operatorname{reg}(S/J_G)$. In the next result, we obtain a lower bound for all powers of almost complete intersection binomial edge ideals of trees. We also give an upper bound for this class.

Theorem 3.7. *If G is a tree such that J_G is an almost complete intersection ideal, then for all $s \geq 1$, $2s + \operatorname{iv}(G) - 2 \leq \operatorname{reg}(S/J_G^s) \leq 2s + \operatorname{iv}(G) - 1$.*

Proof. Let G be a tree such that J_G is an almost complete intersection ideal. Then, by [19, Theorem 4.1], G is obtained by adding an edge between two paths, either by adding an edge between two internal vertices or by adding an edge between an internal vertex and a pendant vertex. If G is obtained by adding an edge between an internal vertex of a path and a pendant vertex of another path, then we say that G is type T and if G is obtained by adding an edge between two internal vertices of two distinct paths, then we say that G is type H. Note that if G is T-type, then $\operatorname{iv}(G) = n-3$ and if G is H-type, then $\operatorname{iv}(G) = n-4$.

Let G be T-type. Then, by [20, Theorems 4.1, 4.2], G contains no Jewel graph as an induced subgraph, and hence, $\operatorname{reg}(S/J_G) = n-2$. Therefore, it follows from Corollary 2.11 that $\operatorname{reg}(S/J_G^s) \leq 2s + n - 4 = 2s + \operatorname{iv}(G) - 1$ for all $s \geq 1$. Let v denote a neighbor of the unique vertex of degree 3 in G. Then, $J_{G\setminus v}$ is generated by a regular sequence of length at least $n-3$. Hence, for all $s \geq 1$, $2s + n - 5 \leq \operatorname{reg}(S/(J_{G\setminus v})^s) \leq \operatorname{reg}(S/J_G^s)$, where the first inequality follows from [2, Lemma 4.4] and the second inequality follows from Proposition 3.3. Therefore, $2s + \operatorname{iv}(G) - 2 \leq \operatorname{reg}(S/J_G^s)$ for all $s \geq 1$.

Now, let G be H-type and $e = \{u, v\}$ be the edge such that $G \setminus e$ is a disjoint union of two paths. Then, the related ideal to J_G is either J_G or of the form $(J_{G\setminus e} : f_e) + J_G$. By [20, Theorems 4.1, 4.2], $\operatorname{reg}(S/J_G) = n-3$. It follows from [33, Theorem 3.7] that $(J_{G\setminus e} : f_e) + J_G = J_{(G\setminus e)_e \cup \{e\}}$. Since $(G \setminus e)_e \cup \{e\}$ is a block graph with no vertex contained in more than two maximal cliques, it follows from [20, Corollary 3.1] that $\operatorname{reg}(S/J_{(G\setminus e)_e \cup \{e\}}) = n - 3 = \operatorname{iv}(G) + 1$. Hence, by Theorem 2.8, $\operatorname{reg}(S/J_G^s) \leq 2s + n - 5 = 2s + \operatorname{iv}(G) - 1$. As in the previous case, it can be seen that $J_{G\setminus u}$ is generated by a regular sequence of length $n-4$. Hence, by [2, Lemma 4.4] and Proposition 3.3, we get $2s + \operatorname{iv}(G) - 2 = 2s + n - 6 = \operatorname{reg}(S/J_{G\setminus u}^s) \leq \operatorname{reg}(S/J_G^s)$ for all $s \geq 1$. □

Corollary 3.8. *If G is a caterpillar tree such that J_G is an almost complete intersection, then $\operatorname{reg}(S/J_G^s) = 2s + \operatorname{iv}(G) - 1$.*

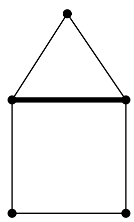
Proof. If G is a caterpillar tree on $[n]$, then G has a longest induced path, say P, such that $\operatorname{iv}(G) = \operatorname{iv}(P) = \ell(P) - 1$. Hence, by Corollary 3.4, $2s + \operatorname{iv}(G) - 1 \leq \operatorname{reg}(S/J_G^s)$. The upper bound follows from Theorem 3.7. □

We are unable to prove, but believe that the answer to the following question is affirmative:

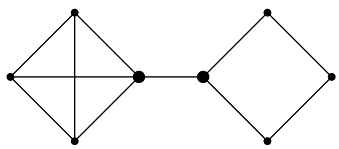
Question 3.9. *If G is a tree, then is $2s + \operatorname{iv}(G) - 2 \leq \operatorname{reg}(S/J_G^s)$ for all $s \geq 1$?*

Now, we deal with unicyclic graphs, other than cycles, whose binomial edge ideals are almost complete intersection. We first develop the tools required for that.

Definition 3.10. *Let G_1 and G_2 be two subgraphs of a graph G. If $G_1 \cap G_2 = K_m$, the complete graph on m vertices with $G_1 \neq K_m$ and $G_2 \neq K_m$, then G is called the clique sum of G_1 and G_2 along the complete graph K_m, denoted by $G_1 \cup_{K_m} G_2$. If $m = 1$, the clique sum of G_1 and G_2 along a vertex is denoted by $G_1 \cup G_2$. If $m = 2$ and $K_2 = e$, then the clique sum of G_1 and G_2 along e is denoted by $G_1 \cup_e G_2$.*



H_1



H_2

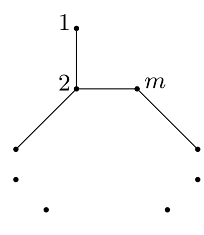
The first graph H_1 on the left is the clique sum of a K_3 and C_4 along an edge. The second graph, H_2, is a clique sum of a K_4, an edge and a C_4 along two vertices.

To understand the regularity of powers of binomial edge ideals of unicyclic graphs which are almost complete intersections, we first prove an auxiliary result.

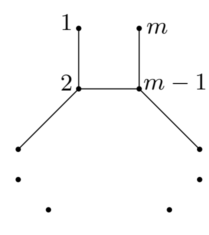
Proposition 3.11. *Let $n, m \geq 3$ be integers and G be the clique sum of a cycle C_n and a complete graph K_m along an edge e. Then, $\operatorname{reg}(S/J_G) = n - 1$.*

Proof. It is easy to notice that G contains an induced path of length $n-1$. Hence, by [32, Corollary 2.3], $n - 1 \leq \operatorname{reg}(S/J_G)$. Note that G is a graph on $n + m - 2$ vertices. Since K_m is the maximal clique of largest size in G, By [9, Theorem 2.1], $\operatorname{reg}(S/J_G) \leq (n + m - 2) - (m - 1) = n - 1$. □

Let G_1 and G_2 denote graphs on the vertex set $[m]$ with edge sets given by $E(G_1) = \{\{1, 2\}, \{2, 3\}, \ldots, \{m-1, m\}, \{2, m\}\}$ and $E(G_2) = \{\{1, 2\}, \{2, 3\}, \ldots, \{m-1, m\}, \{2, m-1\}\}$. Let $e = \{1, 2\}, e' = \{m-1, m\}$ denote edges in G_2.



G_1



G_2

Now, we proceed to study the regularity of powers of almost complete intersection binomial edge ideals of unicyclic graphs. In [19], we had proved that any such graph is obtained by attaching paths to the free vertices of G_1 and G_2. So, we first compute the regularity of powers of J_{G_1} and J_{G_2}.

Proposition 3.12. *Let $m \geq 4$ and G_1 be the graph as given above. Then,* $\operatorname{reg}(S/J_{G_1}) = m - 2$.

Proof. By [25, Theorem 3.2], $\operatorname{reg}(S/J_{G_1}) \leq m-2$. Note that $G_1 \backslash m$ is an induced path in G_1 of length $m-2$. So by [32, Corollary 2.3], $m-2 \leq \operatorname{reg}(S/J_{G_1})$. Hence, $\operatorname{reg}(S/J_{G_1}) = m-2$. □

Proposition 3.13. *Let G_2 be the graph as given above for $m \geq 6$. Then,* $\operatorname{reg}(S/J_{G_2}) = m-3$.

Proof. Note that $G_2 \setminus (m-1)$ is an induced path in G_2 of length $m-3$. So by [32, Corollary 2.3], $m-3 \le \operatorname{reg}(S/J_{G_2})$. Now, we prove that $\operatorname{reg}(S/J_{G_2}) \le m-3$. Consider the following short exact sequences:

$$0 \longrightarrow S/(J_{G_2\setminus e} : f_e)(-2) \xrightarrow{\cdot f_e} S/J_{G_2\setminus e} \longrightarrow S/J_{G_2} \longrightarrow 0 \tag{4}$$

$$0 \longrightarrow S/(J_{H\setminus e'} : f_{e'})(-2) \xrightarrow{\cdot f_{e'}} S/J_{H\setminus e'} \longrightarrow S/J_H \longrightarrow 0 \tag{5}$$

where $H = (G_2 \setminus e)_e$, $e = \{1, 2\}$ and $e' = \{m-1, m\}$. Since $G_2 \setminus e$ is a graph isomorphic to G_1 on $m-1$ vertices, we get $\operatorname{reg}(S/J_{G_2\setminus e}) = m-3$. By [33, Theorem 3.7], $J_{G_2\setminus e} : f_e = J_{(G_2\setminus e)_e}$ and $J_{H\setminus e'} : f_{e'} = J_{(H\setminus e')_{e'}}$. Note that $H \setminus e' = C_{m-3} \cup_{\{3,m-1\}} K_3$ and $(H \setminus e')_{e'} = C_{m-4} \cup_{\{3,m-2\}} K_4$. Thus, by using Proposition 3.11, $\operatorname{reg}(S/J_{H\setminus e'}) = m-4$ and $\operatorname{reg}(S/J_{(H\setminus e')_{e'}}) = m-5$. Now, applying Lemma 2.7 on short exact sequences (4) and (5), we get $\operatorname{reg}(S/J_H) \le m-4$, and hence, $\operatorname{reg}(S/J_{G_2}) \le m-3$. □

Now, we prove the bounds for the regularity of powers of binomial edge ideals of unicyclic graphs which are almost complete intersections.

Theorem 3.14. *Let G be a unicyclic graph on $[n]$ which is not a cycle such that J_G is an almost complete intersection ideal. Assume that $\mathbb{K}$ is an infinite field. Then, $2s+n-5 \le \operatorname{reg}(S/J_G^s) \le 2s+n-4$.*

Proof. If G is a unicyclic graph on $[n]$ such that J_G is an almost complete intersection ideal, then it was shown in [19] that G is obtained either by identifying a pendant vertex of a path with the vertex 1 in G_1 or by identifying a pendant vertex each of two paths with the vertices 1 and m in G_2 or attaching paths to every vertex of K_3. If u is a degree 3 vertex of G, then $G \setminus u$ is a disjoint union of two paths so that $J_{G\setminus u}$ is generated by a regular sequence of length $n-3$. Hence, by [2, Lemma 4.4] and Proposition 3.3, $2s+n-5 \le \operatorname{reg}(S/J_G^s)$. Since J_G is an almost complete intersection ideal, it follows from [7, Proposition 5.1] that there exists a set of homogeneous generators $\{F_1, \ldots, F_n\}$ of J_G such that $F_1, \ldots, F_{n-1}$ is a regular sequence in S. Since $J = (F_1, \ldots, F_{n-1})$ is an unmixed ideal, by [12, Theorem 4.7], $J : F_n = J : F_n^2$. Hence, J_G is generated by a homogeneous d-sequence $F_1, \ldots, F_n$. Since J_G is generated in degree 2, F_i has degree 2 for each i. Now, the upper bound follows directly from Corollaries 2.11 and [25, Theorem 3.2]. □

Remark 3.15. Suppose G is a balloon graph on $[n]$, i.e., G is obtained by identifying a pendant vertex of a path with the pendant vertex G_1. Let v denote a neighbor on the cycle of the degree 3 vertex in G. Then, $G \setminus v$ is a path of length $n-2$. Hence, by Proposition 3.3, $2s+n-4 \le \operatorname{reg}(S/J_G^s)$ for all $s \ge 1$. Therefore, by Theorem 3.14, $\operatorname{reg}(S/J_G^s) = 2s+n-4$.

Remark 3.16. If G is a unicyclic graph obtained by identifying a pendant vertex of two distinct paths to two pendant vertices of the graph G_2, then the regularity behaves in an unexpected manner. For example, if we take $G = G_2$ on 6 vertices, then G is a Cohen-Macaulay bipartite graph, [3]. Hence, $\operatorname{reg}(S/J_G) = 3$, [18]. Using any computational commutative algebra package, for example Macaulay 2 [11], it can be seen that $\operatorname{reg}(S/J_G^2) = 6 = \operatorname{reg}(S/J_G)+3$. This is different behavior in comparison with the regularity of powers of monomial edge ideals. It has been conjectured and is believed to be true, that $\operatorname{reg}(I(G)^s) \le 2s + \operatorname{reg}(I(G)) - 2$ for all $s \ge 1$, where $I(G)$ denotes the monomial edge ideal corresponding to a graph G, [1]. This example shows that such an inequality does not hold in the case of binomial edge ideals. Moreover, this gives a binomial edge ideal for which the stabilization index is bigger than 1.

4. Regularity of powers of parity binomial edge ideals

In this section, we study the regularity of powers of parity binomial edge ideals of some classes of graphs. It was proved by Bolognini et al., that for bipartite graphs, the parity binomial edge ideals are essentially the same as the binomial edge ideals, [3, Corollary 6.2]. This implies that the algebraic invariants associated to both these ideals are the same for bipartite graphs. Hence, we restrict our attention to graphs containing odd cycles. In [28], Kumar computed the regularity of parity binomial edge ideals of odd cycles. We compute the regularity of their powers.

Theorem 4.1. *Let $n \geq 3$ be an odd integer. Then, for all $s \geq 1$, $\operatorname{reg}(S/\mathcal{I}_{C_n}^s) = 2s + n - 2$.*

Proof. It follows from [27, Theorem 3.5] that $\mathcal{I}_{C_n}$ is generated by a regular sequence of length n in degree 2. Therefore, by virtue of [2, Lemma 4.4], $\operatorname{reg}(S/\mathcal{I}_{C_n}^s) = 2s+n-2$ for all $s \geq 1$. □

If H is an induced subgraph of G, then $\beta_{i,j}(S/\mathcal{I}_H) \leq \beta_{i,j}(S/\mathcal{I}_G)$ for all i, j, [28]. We generalize this to all powers. For an edge $\{i,j\}, i < j$, let $\bar{g}_e = x_i x_j - y_i y_j$.

Proposition 4.2. *Let G be a graph and H be an induced subgraph of G. Then $\beta_{i,j}^{S_H}(S_H/\mathcal{I}_H^s) \leq \beta_{i,j}^{S}(S/\mathcal{I}_G^s)$, for all i, j, where $S_H = \mathbb{K}[x_k, y_k : k \in V(H)]$.*

Proof. The proof is essentially the same as the proof of Proposition 3.3. One only needs to replace f_{e_i} by $\bar{g}_{e_i}$. □

Let $\operatorname{oc}(G)$ denote the length of a longest induced odd cycle. If G has no induced odd cycle, then we assume that $\operatorname{oc}(G) = 0$. In [28], Kumar proved that $\operatorname{reg}(S/\mathcal{I}_G) \geq \max\{\ell(G), \operatorname{oc}(G)\}$. We generalize this to all powers.

Corollary 4.3. *Let G be a connected graph. Then, $\operatorname{reg}(S/\mathcal{I}_G^s) \geq 2s + \max\{\ell(G), \operatorname{oc}(G)\} - 2$ for all $s \geq 1$.*

Proof. Without loss of generality, we assume that G is a non-bipartite graph. Let H be a longest induced path of G. Then, H is an induced subgraph of G. Since, H is a bipartite graph, by [3, Corollary 6.2] and Observation 3.2, $\operatorname{reg}(S_H/\mathcal{I}_H^s) = 2s + \ell(G) - 2$ for all $s \geq 1$. Now, let H' be a longest induced odd cycle. By Theorem 4.1, $\operatorname{reg}(S_{H'}/\mathcal{I}_{H'}^s) = 2s + \operatorname{oc}(H') - 2 = 2s + \operatorname{oc}(G) - 2$ for all $s \geq 1$. Hence, the assertion follows from Proposition 4.2. □

For the rest of the section, we assume that $\operatorname{char}(\mathbb{K}) \neq 2$. In the next three results, we study the regularity of powers of parity binomial edge ideals which are generated by d-sequence.

Theorem 4.4. *Let G be a graph on $[n]$ obtained by adding an edge between an odd cycle and an internal vertex of a path. Then, $2s + n - 5 \leq \operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 4$, for all $s \geq 1$.*

Proof. Let G be obtained by adding edge $e = \{u, v\}$ between an odd cycle and a path, where u lies on an odd cycle and v is an internal vertex of a path. Note that $\mathcal{I}_G = \mathcal{I}_{G\setminus e} + (\bar{g}_e)$. By [27, Corollary 3.6], $\mathcal{I}_{G\setminus e}$ is a complete intersection ideal. It follows from the proof of [27, Theorem 3.8(1)] that $\mathcal{I}_G$ is generated by a d-sequence of length n such that first $n-1$ elements form a regular sequence. Thus, by Corollary 2.11,

$$\operatorname{reg}(S/\mathcal{I}_G^s) \leq 2(s-1) + \max\{\operatorname{reg}(S/\mathcal{I}_G), n-2\}, \text{ for all } s \geq 1.$$

Note that there exists an edge e' such that $G \setminus e'$ is a bipartite graph and $e' \cap \{u, v\} = \emptyset$. Consider the short exact sequence

$$0 \longrightarrow \frac{S}{\mathcal{I}_{G\setminus e'} : \bar{g}_{e'}}(-2) \xrightarrow{\cdot \bar{g}_{e'}} \frac{S}{\mathcal{I}_{G\setminus e'}} \longrightarrow \frac{S}{\mathcal{I}_G} \longrightarrow 0. \tag{6}$$

It follows from [27, Lemma 3.3] that $\mathcal{I}_{G\setminus e'} : \bar{g}_{e'} = \mathcal{I}_{G\setminus e'}$. Hence, by Lemma 2.7, $\operatorname{reg}(S/\mathcal{I}_G) = \operatorname{reg}(S/\mathcal{I}_{G\setminus e'}) + 1$. Note that $G \setminus e'$ is a tree of H-type. By [3, Corollary 6.2] and the proof of Theorem 3.7, $\operatorname{reg}(S/\mathcal{I}_{G\setminus e'}) = n-3$ so that $\operatorname{reg}(S/\mathcal{I}_G) = n-2$. Hence, $\operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s+n-4$, for all $s \geq 1$. Note that $G \setminus u$ is disjoint union of two paths on $n-1$ vertices. Therefore, $\operatorname{reg}(S/\mathcal{I}_{G\setminus u}^s) = 2s + n - 5$, for all $s \geq 1$. Thus, by Proposition 4.2, $2s + n - 5 \leq \operatorname{reg}(S/\mathcal{I}_G^s)$, for all $s \geq 1$. □

Theorem 4.5. *Let G be a balloon graph on $[n]$ having odd girth. Then, for all $s \geq 1$ $2s + n - 4 \leq \operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 3$.*

Proof. Since $\ell(G) = n - 1$, it follows from Corollary 4.3 that $2s + n - 4 \leq \operatorname{reg}(S/\mathcal{I}_G)$ for all $s \geq 1$. Let u be the vertex of degree three in G and v be a neighbor of u on the cycle. Set $e = \{u, v\}$. Then, $G \setminus e$ is a path graph. It follows from the proof of [27, Theorem 3.8(1)] that $\mathcal{I}_G$ is generated by a d-sequence of length n such that first $n-1$ elements form a regular sequence. Thus, by Corollary 2.11,

$$\operatorname{reg}(S/\mathcal{I}_G^s) \leq 2(s-1) + \max\{\operatorname{reg}(S/\mathcal{I}_G), n-2\}, \text{ for all } s \geq 1.$$

If one takes e' to be any edge on the cycle such that $e' \cap \{u\} = \emptyset$, then $G \setminus e'$ is a bipartite graph, and hence, it follows from [27, Lemma 3.3] that $\mathcal{I}_{G\setminus e'} : \bar{g}_{e'} = \mathcal{I}_{G\setminus e'}$. Thus, by Lemma 2.7, $\operatorname{reg}(S/\mathcal{I}_G) = \operatorname{reg}(S/\mathcal{I}_{G\setminus e'}) + 1$. Note that $G \setminus e'$ is a tree of T-type. Therefore, by [3, Corollary 6.2] and the proof of Theorem 3.7, $\operatorname{reg}(S/\mathcal{I}_{G\setminus e'}) = n-2$ so that $\operatorname{reg}(S/\mathcal{I}_G) = n-1$. Hence, $\operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 3$, for all $s \geq 1$. □

Theorem 4.6. *Let G be a graph obtained by adding a chord in an odd cycle C_n. Then, $2s + n - 4 \leq \operatorname{reg}(S/\mathcal{I}_G^s) \leq 2s + n - 3$, for all $s \geq 1$.*

Proof. Since $\ell(G) = n - 1$, it follows from Corollary 4.3 that $2s + n - 4 \leq \operatorname{reg}(S/\mathcal{I}_G)$ for all $s \geq 1$. Let $e = \{u, v\}$ be the chord in C_n. Then, $\mathcal{I}_{G\setminus e}$ is a complete intersection. Moreover, $\mathcal{I}_{G\setminus e} : \bar{g}_e^2 = \mathcal{I}_{G\setminus e} : \bar{g}_e$. Hence, $\mathcal{I}_G$ is generated by a d-sequence of length $n+1$. Therefore, by Corollary 2.11, $\operatorname{reg}(S/\mathcal{I}_G^s) \geq 2(s-1) + \max\{\operatorname{reg}(S/\mathcal{I}_G), n-1\}$. To complete the proof, it is enough to show that $\operatorname{reg}(S/\mathcal{I}_G) \leq n - 1$. The chord e splits the graph G into two induced cycles, an odd cycle and an even cycle, whose intersection is e. Let $e' = \{v, w\} \in E(G)$ be an edge of the induced odd cycle in G. Observe that $G \setminus e'$ is a balloon graph having even girth. Since $G \setminus e'$ is a bipartite graph and for a bipartite graph the parity binomial edge ideal is isomorphic to the binomial edge ideal, we conclude using [25, Theorem 3.2] that $\operatorname{reg}(S/\mathcal{I}_{G\setminus e'}) \leq n - 2$. Also, from [27, Lemma 3.3] we get $\mathcal{I}_{G\setminus e'} : \bar{g}_{e'} \simeq J_{(G\setminus e')_{e'}}$. Notice that $(G \setminus e')_{e'} = (G \setminus e')_v$ is not a path graph. Therefore, by [25], $\operatorname{reg}(S/J_{(G\setminus e')_v}) \leq n-2$. Hence, the assertion follows by applying Lemma 2.7, on the short exact sequence (6). □

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