

Ready-to-use post-Newtonian gravitational waveforms for binary black holes with non-precessing spins: An update

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(Dated: January 22, 2016)

For black-hole binaries whose spins are (anti-) aligned with respect to the orbital angular momentum of the binary, we compute the frequency domain phasing coefficients including the quadratic-in-spin terms up to the third post-Newtonian (3PN) order, the cubic-in-spin terms at the leading order, 3.5PN, and the spin-orbit effects up to the 4PN order. In addition, we obtain the 2PN spin contributions to the amplitude of the frequency-domain gravitational waveforms for non-precessing binaries, using recently derived expressions for the time-domain polarization amplitudes of binaries with generic spins, complete at that accuracy level. These two results are updates to Refs. [1] for amplitude and [2] for phasing. They should be useful to construct banks of templates that model accurately non-precessing inspiraling binaries, for parameter estimation studies, and for constructing analytical template families that accounts for the inspiral-merger-ringdown phases of the binary.

PACS numbers: 04.25.Nx, 04.30.-w, 97.60.Jd, 97.60.Lf

I. INTRODUCTION

Recently there have been several improvements in modelling spinning binaries within the post-Newtonian formalism [3]. These developments include the computation of relative 2PN spin-orbit (SO) effects (corresponding to the 3.5PN order) in the equations of motion [4–6] as well as in the precession equations at the same relative accuracy level, and that of the near-zone metric at the 2PN order [7]. The work [7] also provided us with the energy function at 3.5PN order including spin-orbit (linear-in-spins) effects at the relative 2PN order, which is needed to compute the phase. Further, in Ref. [8], the 2PN SO contributions were incorporated to the gravitational-wave energy flux and (time-domain) phasing at the 3.5PN order. The tail-induced SO corrections to the two latter quantities were investigated in Ref. [9] at the order 4PN, where they are the only spin-orbit effects. On the other hand, the spin-spin (quadratic-in-spins, SS) interactions were recently included at the 3PN order [10], which means 1PN order beyond the leading SS terms presented in Ref. [1]. In addition, the leading cubic-in-spin terms entering the energy and the energy flux at 3.5PN were computed in [11]. The 2PN polarizations $h_{+, \times}$ accounting for both the spin-orbit and spin-spin effects were calculated explicitly in Ref. [12], extending the earlier works of Refs. [1, 13, 14]. Note that the tail-type spin-orbit corrections entering the 3PN amplitude are also available [15]. Hence, all spin contributions to the GW polarizations in the time-domain are known with 2PN accuracy, while the time-domain phasing is known to the 4PN, 3PN and 3.5PN orders, for the SO, SS and SSS effects, respectively.

Frequency domain amplitudes for non-precessing binaries, with spins (anti-)aligned to the orbital angular momentum vector, were first displayed to the 2PN order in Ref. [1]. Their expression complements that of the 3PN accurate polarizations for non-spinning binaries derived in [16, 17]. They model the spin-orbit effects at the leading (1.5PN) order and *partial* spin-spin effects at the 2PN order. More precisely, the spin-spin contributions to the GW amplitude presented in Ref. [1] are only those that arise due to couplings involving *both* spins, *i.e.* of the type (Spin(1)-Spin(2)), as at that time self-spin corrections (Spin(1)-Spin(1) and (Spin(2)-Spin(2))) were not available. In this work we make

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use of the above mentioned recent time-domain results for GW polarizations with all possible spin-dependent interactions to construct their frequency domain counterpart complete up to the 2PN order, by including the new 2PN SO and SS effects (besides those already present in [1]). Frequency-domain phasing with all SO contributions up to the 3.5PN order — except for those produced by the black-hole absorption at the 2.5PN order — and all SS contributions at the 2PN order, was provided in Ref. [2]. We extend that result by adding the tail-induced spin-orbit effects at the 4PN order, as well as the quadratic and cubic spin terms contributing to the phase at the 3PN and 3.5PN orders, respectively.

This paper is organized in the following manner. We begin Sec. II by showing the form of the Fourier domain signal and specifying our notations. The rest of the section is split into two parts. Section II A presents the phasing formula, which includes the spin-orbit contribution at the 4PN order, the quadratic spin terms at the 3PN order, and the cubic ones at the 3.5PN order. In Sec. II B we list our findings, complementing the outcomes of Ref. [1], related to the frequency domain amplitude of the waveform for non-precessing binaries in quasi-circular orbits. Finally, in Sec. IV we summarize our results and discuss their implications.

II. FREQUENCY DOMAIN WAVEFORMS FOR NON-PRECESSING BINARIES IN CIRCULAR ORBITS

Since we view this report as an extension of [1], we basically follow the definitions and notations provided in there. The reader must refer to that work for details. Nonetheless, we shall provide below some minimal compendium both to ensure a natural flow in the paper and to facilitate the reading. The frequency domain amplitude of a signal h_{strain} produced by a gravitational wave h_{ij} can be written, truncated at some accuracy level, in the following way (see Sec. VI B of Ref. [1] for a derivation), using geometrical units where $G = c = 1$:

$$\tilde{h}_{\text{strain}}(f) = \frac{M^2}{D_L} \sqrt{\frac{5\pi}{48}} \sum_{n=0}^4 \sum_{k=1}^6 V_k^{n-7/2} C_k^{(n)} e^{i(k \Psi_{\text{SPA}}(f/k) - \pi/4)}. \quad (1)$$

Here, $\tilde{h}_{\text{strain}}(f)$ denotes the waveform in the frequency domain¹ as observed by the detector while M and D_L stand for the total mass and the luminosity distance of the source, respectively. The index n indicates the PN order, whereas the index k keeps track of the different harmonics of the orbital phase. Hence, the above waveform is 2PN accurate and consists of 6 harmonics. For the k^{th} harmonic, the PN parameter $v \equiv v(t)$ entering the time domain waveform has been replaced by a function V_k of the GW frequency f , defined as $V_k(f) = (2\pi M f/k)^{1/3}$. The function $\Psi_{\text{SPA}}(f)$ represents essentially the phase of the first harmonic in the frequency domain as obtained under the Stationary Phase Approximation (SPA) [18, 19] (see Sec. VI B of [1] for details). Finally, the coefficients $C_k^{(n)}$'s depend on the intrinsic parameters of the binary, such as the masses and the spins, as well as the angular parameters specifying the binary's location and orientation.

The results of the present paper, along with those of Ref. [1], will allow one to write amplitude corrections completed up to 2PN order with all possible spin effects. As already stated, the waveform provided in [1] contains terms describing the spin-orbit effects at the leading order (1.5PN) and *part of* the spin-spin effects (corresponding to Spin(1)-Spin(2) interactions) at the 2PN order. The coefficients $C_k^{(n)}$ through which they appear are explicitly listed in Appendix D of Ref. [1]. Thus, for the brevity of presentation and the sake of avoiding repetition, we shall only show here those $C_k^{(n)}$'s that get modified due to inclusion of the spin-orbit and spin-spin effects at the 2PN order, as discussed in Sec. I. Below, we shall display our expression for the GW phase and amplitude in two separate subsections.

A. Corrections to the phasing formula

In order to determine the frequency domain phasing we follow the prescription of Ref. [20], which is based on an energy balance argument. In the case of quasi-circular non-precessing orbits, the two inputs needed for the phase derivation are the time domain center-of-mass energy E and the energy flux \mathcal{F} of the binary, both given in terms of the orbital frequency, the two relations are invariant for a large class of gauge transformations.

¹ For the Fourier transform, we adopt the convention that $\tilde{h}(f) = \int dt e^{2\pi i f t} h(t)$.

Schematically, we can write for the energy

$$E = -\frac{\eta m}{2} v^2 [E_{\text{NS}} + E_{\text{SO}} + E_{\text{SS}} + E_{\text{SSS}}], \quad (2)$$

where E_{NS} , E_{SO} , E_{SS} and E_{SSS} denote the non-spinning, the spin-orbit (linear-in-spins), the spin-spin (quadratic-in-spin), and the spin-spin-spin (cubic-in-spin) contributions to the energy, while $\eta = m_1 m_2 / M^2$ represents the symmetric mass ratio parameter, with m_1 and m_2 being the masses of the two companions. The non-spinning part of the energy is currently available to the 4PN accuracy beyond the Newtonian order [21]. However, for the present purpose, the 3PN expression of Ref. [22], completed with the results of [23], is sufficient since there cannot be any 3.5PN terms in the energy for quasi-circular orbits (see [3] for a discussion). The spin-orbit (linear-in-spin) corrections to the conservative part of the dynamics, starting from the 1.5PN order, are known with a relative 2PN accuracy, i.e., at the 3.5PN order beyond the Newtonian level [4, 6, 7]. The same relative accuracy has been achieved for the spin-spin (quadratic-in-spin) corrections [24–26], even though it corresponds now to the 4PN order, as the leading terms of that type arise at the 2PN approximation [1]. However, since the energy flux has not been determined yet with such precision, it will be sufficient for us to use the spin-spin part of the energy at the 3PN order. The explicit expressions of the 3.5PN spin-orbit and the 3PN spin-spin pieces of the energy can be found in the works [7] and [10], respectively. As for the cubic-in-spin pieces, which contribute at the 3.5PN order, they were only computed recently [11].

Similarly, the energy flux has the following structure:

$$\mathcal{F} = \frac{32}{5} \eta^2 v^{10} [\mathcal{F}_{\text{NS}} + \mathcal{F}_{\text{SO}} + \mathcal{F}_{\text{SS}} + \mathcal{F}_{\text{SSS}}], \quad (3)$$

where \mathcal{F}_{NS} , \mathcal{F}_{SO} , \mathcal{F}_{SS} , and \mathcal{F}_{SSS} again denote the non-spinning, spin-orbit, spin-spin, and spin-spin-spin contributions to the energy flux. The non-spinning contributions up to the 3.5PN order beyond the leading quadrupolar flux are given in Refs. [23, 27]. For the spin-orbit terms, which first appear at the 1.5PN approximation, our current knowledge extends up to the 4PN order [9]. Let us point out that the 4PN spin-orbit piece of the energy flux comes from the so-called tail effect at the next-to-leading order (ignoring non spin-orbit terms). This non-linear effect can be understood as due to the back scattering of the wave on the spacetime curvature. It is hereditary in nature, which means that it depends on the past history of the binary evolution. Note that terms of this type (at the 3PN and 4PN order) are absent from the energy [9]. Spin-spin (or quadratic-in-spin) corrections, starting from the 2PN order, can be found up to the 3PN order in Refs. [1, 10]. Finally, the cubic-in-spin terms at the leading 3.5PN approximation were derived in [11].

With these time-domain expressions of the energy and the energy flux in hands, we are in the position to write the frequency domain phasing entailed by the SPA. Like the expressions above, it has the following general structure:

$$\Psi_{\text{SPA}}(f) = 2\pi f t_c - \phi_c + \left\{ \frac{3}{128\eta v^5} [\psi_{\text{NS}} + \psi_{\text{SO}} + \psi_{\text{SS}} + \psi_{\text{SSS}}] \right\}_{v=V_1(f)}, \quad (4)$$

where ϕ_c denote the orbital phase at the instant t_c of coalescence.

The complete 3.5PN accurate frequency domain phasing for non-spinning binaries is presented in Refs. [20, 28] while the spin-orbit terms up to the 3.5PN accuracy level and the spin-spin terms at the 2PN order are given in Refs. [1, 2]. The contributions to the phasing we add here include: (i) the tail-induced 4PN spin-orbits terms, (ii) the 3PN quadratic-in-spin terms, and (iii) the 3.5PN cubic-in-spin terms. Thus, the spin contributions to the phasing formula may be expressed as

$$\psi_{\text{Spin}} \equiv \psi_{\text{SO}} + \psi_{\text{SS}} + \psi_{\text{SSS}} = v^3 [\mathcal{P}_3 + \mathcal{P}_4 v + \mathcal{P}_5 v^2 + \mathcal{P}_6 v^3 + \mathcal{P}_7 v^4 + \mathcal{P}_8 v^5 + \dots]. \quad (5)$$

Refs. [1, 2] list the explicit expressions for \mathcal{P}_3 , \mathcal{P}_4 and \mathcal{P}_5 with the required accuracies. By contrast, the coefficients \mathcal{P}_6 and \mathcal{P}_7 there only include relative 1.5PN (leading linear-in-spin tail) and relative 2PN linear-in-spin contributions, respectively. In the present work, as discussed above, we add the relative 1PN quadratic-in-spin and the leading order cubic-in-spin corrections. In addition, we introduce a new coefficient \mathcal{P}_8 of order 4PN that corresponds to the tail-induced SO effect. The modified coefficients \mathcal{P}_6 , \mathcal{P}_7 , and the new coefficient \mathcal{P}_8 take the final following form:

$$\begin{aligned} \mathcal{P}_6 &= \pi \left[\frac{2270}{3} \delta \chi_a \cdot \hat{\mathbf{L}}_N + \left(\frac{2270}{3} - 520 \eta \right) \chi_s \cdot \hat{\mathbf{L}}_N \right] + \left(\frac{75515}{144} - \frac{8225}{18} \eta \right) \delta (\chi_a \cdot \hat{\mathbf{L}}_N) (\chi_s \cdot \hat{\mathbf{L}}_N) \\ &+ \left(\frac{75515}{288} - \frac{263245}{252} \eta - 480 \eta^2 \right) (\chi_a \cdot \hat{\mathbf{L}}_N)^2 + \left(\frac{75515}{288} - \frac{232415}{504} \eta + \frac{1255}{9} \eta^2 \right) (\chi_s \cdot \hat{\mathbf{L}}_N)^2, \end{aligned} \quad (6a)$$

$$\begin{aligned}
\mathcal{P}_7 &= \left(-\frac{25150083775}{3048192} + \frac{26804935}{6048}\eta - \frac{1985}{48}\eta^2 \right) \delta \boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \\
&+ \left(-\frac{25150083775}{3048192} + \frac{10566655595}{762048}\eta - \frac{1042165}{3024}\eta^2 + \frac{5345}{36}\eta^3 \right) \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \\
&+ \left(\frac{14585}{24} - 2380\eta \right) \delta (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N)^3 + \left(\frac{14585}{24} - \frac{475}{6}\eta + \frac{100}{3}\eta^2 \right) (\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N)^3 \\
&+ \left(\frac{14585}{8} - \frac{215}{2}\eta \right) \delta (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N)(\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N)^2 + \left(\frac{14585}{8} - 7270\eta + 80\eta^2 \right) (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N)^2 (\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N), \tag{6b}
\end{aligned}$$

$$\mathcal{P}_8 = \pi \left[\left(\frac{233915}{168} - \frac{99185}{252}\eta \right) \delta \boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N + \left(\frac{233915}{168} - \frac{3970375}{2268}\eta + \frac{19655}{189}\eta^2 \right) \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \right] (1 - 3 \ln v). \tag{6c}$$

In the above, $\boldsymbol{\chi}_s$ and $\boldsymbol{\chi}_a$ represent symmetric and anti-symmetric combinations of the spin vectors associated with the binary individual components $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$, namely

$$\begin{aligned}
\boldsymbol{\chi}_s &= \frac{1}{2}(\boldsymbol{\chi}_1 + \boldsymbol{\chi}_2), \\
\boldsymbol{\chi}_a &= \frac{1}{2}(\boldsymbol{\chi}_1 - \boldsymbol{\chi}_2). \tag{7}
\end{aligned}$$

The quantity $\hat{\mathbf{L}}_N$ is the unit vector pointing along the Newtonian orbital angular momentum. Coordinate frames and parameter conventions used here are identical to the ones employed in Ref. [1]; more details can be found in Sec II there; the parameter $\delta = (m_1 - m_2)/m$ represents the difference mass ratio. It should be emphasized that this result completes the SO phasing at the 4PN (relative 2.5PN) order, the SS phasing to the 3PN (relative 1PN) order, and the SSS phasing to the (leading) 3.5PN order in the frequency domain. In order to get the full 4PN phase, ignoring at this stage possible absorption effects for black holes, one still needs to add: (i) the 4PN non-spinning terms, which would require to know the energy flux at the same accuracy level, and (ii) the 3.5PN and 4PN SS terms, of tail and instantaneous types, respectively. The full phasing formula including the contributions listed in previous works [1, 2] is being provided in a separate file (**supl-mkaf16.m**), both for completeness and for convenient use, and is readable in MATHEMATICA.

B. Corrections to the Amplitude: 2PN spin-orbit and spin-spin effects

In this section, we present our findings concerning the amplitude of the signal from non-precessing binaries. The general structure of the waveform is given by Eq. (1). The frequency domain amplitudes in the absence of spins up to the 2.5PN order, the spin-orbit terms at the 1.5PN order, and partial spin-spin terms contributing at the 2PN order are listed in Ref. [1]. The related coefficients $\mathcal{C}_k^{(n)}$ entering Eq. (1) are defined in Eq.(6.13) and (6.14) of [1] and have been listed in Appendix D there. As discussed above, we shall only provide explicit expressions for those $\mathcal{C}_k^{(n)}$'s that get modified due to inclusion of 2PN spin-orbit and spin-spin terms computed in the time-domain by Ref. [12]. They read:

$$\begin{aligned}
\mathcal{C}_1^{(4)} &= s_i \left\{ F_+ \left[\delta \left[\frac{11i}{40} + \frac{5\pi}{8} + \frac{5i}{4} \log 2 + \left(\frac{7i}{40} + \frac{\pi}{8} + \frac{i}{4} \log 2 \right) c_i^2 \right] + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \left[-\frac{711}{448} + \frac{33}{16}\eta + \left(-\frac{65}{192} - \frac{23}{48}\eta \right) c_i^2 \right] \right. \\
&+ \boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \left[-\frac{711}{448} + \frac{173}{48}\eta + \left(-\frac{65}{192} + \frac{83}{48}\eta \right) c_i^2 \right] + i c_i F_\times \left[\delta \left[\frac{9i}{20} + \frac{3\pi}{4} + \frac{3i}{2} \log 2 \right] \right. \\
&\left. \left. + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \left[\left(-\frac{647}{336} + \frac{41}{24}\eta \right) - \frac{\eta}{8} c_i^2 \right] + \boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \left[\left(-\frac{647}{336} + \frac{125}{24}\eta \right) + \frac{\eta}{8} c_i^2 \right] \right] \right\} \Theta(F_{\text{cut}} - f), \tag{8a}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_2^{(4)} &= \frac{1}{\sqrt{2}} \left\{ F_+ \left[\frac{113419241}{40642560} + \frac{152987}{16128}\eta - \frac{11099}{1152}\eta^2 + \left(\frac{165194153}{40642560} - \frac{149}{1792}\eta + \frac{6709}{1152}\eta^2 \right) c_i^2 \right. \right. \\
&+ \left(\frac{1693}{2016} - \frac{5723}{2016}\eta + \frac{13}{12}\eta^2 \right) c_i^4 - \left(\frac{1}{24} - \frac{5}{24}\eta + \frac{5}{24}\eta^2 \right) c_i^6 + (1 + c_i^2) \left[\frac{49}{16} \delta (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N)(\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N) \right. \\
&\left. \left. + (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N)^2 \left(\frac{49}{32} - 6\eta \right) + (\boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N)^2 \left(\frac{49}{32} - \frac{\eta}{8} \right) \right] \right] + i c_i F_\times \left[\frac{114020009}{20321280} + \frac{133411}{8064}\eta - \frac{7499}{576}\eta^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\chi_a \cdot \hat{\mathbf{L}}_N \right)^2 \left(\frac{49}{16} - 12\eta \right) + \frac{49}{8} \delta (\chi_a \cdot \hat{\mathbf{L}}_N) (\chi_s \cdot \hat{\mathbf{L}}_N) + \left(\chi_s \cdot \hat{\mathbf{L}}_N \right)^2 \left(\frac{49}{16} - \frac{\eta}{4} \right) \\
& + \left(\frac{5777}{2520} - \frac{5555}{504} \eta + \frac{34}{3} \eta^2 \right) c_i^2 + \left(-\frac{1}{4} + \frac{5}{4} \eta - \frac{5}{4} \eta^2 \right) c_i^4 \left. \right\} \Theta(2F_{\text{cut}} - f), \tag{8b}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3^{(4)} & = \frac{s_i}{\sqrt{3}} \left\{ F_+ \left[\chi_a \cdot \hat{\mathbf{L}}_N \left[\frac{195}{64} - \frac{141}{16} \eta + \left(\frac{195}{64} - \frac{249}{16} \eta \right) c_i^2 \right] + \delta \chi_s \cdot \hat{\mathbf{L}}_N \left[\frac{195}{64} - \frac{39}{16} \eta + \left(\frac{195}{64} + \frac{69}{16} \eta \right) c_i^2 \right] \right. \right. \\
& + \left. \delta (1 + c_i^2) \left(-\frac{189i}{40} + \frac{9\pi}{8} + \frac{27}{4} i \log \left(\frac{3}{2} \right) \right) \right] + i c_i F_\times \left[\delta \left(-\frac{189i}{20} + \frac{9\pi}{4} + \frac{27}{2} i \log \left(\frac{3}{2} \right) \right) \right. \right. \\
& + \left. \left. \chi_a \cdot \hat{\mathbf{L}}_N \left[\left(\frac{195}{32} - 21\eta \right) - \frac{27}{8} \eta c_i^2 \right] + \delta \chi_s \cdot \hat{\mathbf{L}}_N \left[\left(\frac{195}{32} - \frac{3\eta}{2} \right) + \frac{27}{8} \eta c_i^2 \right] \right] \right\} \Theta(3F_{\text{cut}} - f). \tag{8c}
\end{aligned}$$

Note that in deriving the 2PN terms in the SPA amplitude, we have taken into account all the spin contributions at the 2PN order instead of the partial ones that Ref. [1] used to be consistent with their spin inputs. To be more precise, we have resorted to the full expression of σ displayed in Eq. (6.24) when calculating the quantity \mathcal{S}_4 given by Eq. (6.11) of [1]. Similar to the phase we also provide a complete list of $\mathcal{C}_k^{(n)}$'s contributing at the 2PN order in the file (**supl-mkaf16.m**).

III. FOURIER TRANSFORM OF THE GW MODES

In this section, we provide the GW modes ($h_{\ell m}$) contributing to the waveform at the 2PN order. For this purpose, we must associate spherical coordinates (R, θ, ϕ) to the source in such a way that, following the conventions of [1], ϕ vanishes for an observer located on the earth while θ coincides with the inclination angle ι as measured by the same observer. As usual, the three vectors forming the standard orthogonal basis are referred to as e_r^i , e_θ^i and e_ϕ^i . The complex polarization $h \equiv h_+ - i h_\times \equiv -m^i m^j h_{ij}$, with $m^i = e_\theta^i - i e_\phi^i$, can be conveniently expanded in terms of the spherical harmonics with spin weight -2 , the ${}_{-2}Y_{\ell m}(\theta, \phi)$'s, whose precise definition is given by Eqs. (4.2)–(4.3) of Ref. [1]:

$$h(\theta, \phi) = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} {}_{-2}Y_{\ell m}(\theta, \phi). \tag{9}$$

The $h_{\ell m}$ modes of GW polarization have the following structure [1, 17]:

$$h_{\ell m} = \frac{2M\eta}{D_L} v^2 \sqrt{\frac{16\pi}{5}} \hat{h}_{\ell m} e^{-im\psi}. \tag{10}$$

Those for non-spinning binaries are listed in Eq. (9.4) of [17] whereas the $\tilde{h}_{\ell m}$'s for spinning binaries can be found in [1, 12]. Fourier transforms of these individual modes (as opposed to that of the full time-domain waveform) may be useful in many studies at the interface of analytical and numerical relativity. Hence we systematically provide them below. The procedure for computing those Fourier transforms is similar to the one used by Ref. [18, 19] which applied the stationary phase approximation to the individual harmonics. Following the same procedure, we obtain the Fourier transforms of the $\tilde{h}_{\ell m}$'s that are relevant for us. They have the form

$$\tilde{h}_{\ell m}(f) = \frac{M^2}{D_L} \pi \sqrt{\frac{2\eta}{3}} V_m^{-7/2} e^{-i(m \Psi_{\text{SPA}}(V_m) + \pi/4)} \hat{H}_{\ell m}(V_m). \tag{11}$$

Our results for $\hat{H}_{\ell m}(V_m) \equiv \hat{H}_{\ell m}$, consistently accounting for all spin effects (as well as for those in absence of spins) up to the 2PN order, read

$$\begin{aligned}
\hat{H}_{22} & = -1 + \left(\frac{323}{224} - \frac{451\eta}{168} \right) V_2^2 + \left[-\frac{27}{8} \delta \chi_a \cdot \hat{\mathbf{L}}_N + \chi_s \cdot \hat{\mathbf{L}}_N \left(-\frac{27}{8} + \frac{11}{6} \eta \right) \right] V_2^3 + \left[\frac{27312085}{8128512} + \frac{1975055}{338688} \eta \right. \\
& - \left. \frac{105271}{24192} \eta^2 + \left(\chi_a \cdot \hat{\mathbf{L}}_N \right)^2 \left(\frac{113}{32} - 14\eta \right) + \frac{113}{16} \delta (\chi_a \cdot \hat{\mathbf{L}}_N) (\chi_s \cdot \hat{\mathbf{L}}_N) + \left(\chi_s \cdot \hat{\mathbf{L}}_N \right)^2 \left(\frac{113}{32} - \frac{\eta}{8} \right) \right] V_2^4
\end{aligned}$$

$$+ \mathcal{O}(5), \quad (12a)$$

$$\begin{aligned} \hat{H}_{21} = & -\frac{\sqrt{2}}{3} \left\{ \delta V_1 - \frac{3}{2} (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N) V_1^2 + \delta \left(\frac{335}{672} + \frac{117}{56} \eta \right) V_1^3 + \left[\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \left(\frac{4771}{1344} - \frac{11941}{336} \eta \right) \right. \right. \\ & \left. \left. + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \left(\frac{4771}{1344} - \frac{2549}{336} \eta \right) + \delta \left(-\frac{i}{2} - \pi - 2i \log(2) \right) \right] V_1^4 \right\} + \mathcal{O}(5), \end{aligned} \quad (12b)$$

$$\begin{aligned} \hat{H}_{33} = & -\frac{3}{4} \sqrt{\frac{5}{7}} \left\{ \delta V_3 + \delta \left(-\frac{1945}{672} + \frac{27}{8} \eta \right) V_3^3 + \left[\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \left(\frac{161}{24} - \frac{85}{3} \eta \right) + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \left(\frac{161}{24} - \frac{17}{3} \eta \right) \right. \right. \\ & \left. \left. + \delta \left(-\frac{21i}{5} + \pi + 6i \log\left(\frac{3}{2}\right) \right) \right] V_3^4 \right\} + \mathcal{O}(5), \end{aligned} \quad (12c)$$

$$\hat{H}_{32} = -\frac{1}{3} \sqrt{\frac{5}{7}} \left\{ (1 - 3\eta) V_2^2 + 4\eta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N V_2^3 + \left(-\frac{10471}{10080} + \frac{12325}{2016} \eta - \frac{589}{72} \eta^2 \right) V_2^4 \right\} + \mathcal{O}(5), \quad (12d)$$

$$\begin{aligned} \hat{H}_{31} = & -\frac{1}{12\sqrt{7}} \left\{ \delta V_1 + \delta \left(-\frac{1049}{672} + \frac{17}{24} \eta \right) V_1^3 + \left[\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N \left(\frac{161}{24} - \frac{73}{3} \eta \right) + \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N \left(\frac{161}{24} - \frac{29}{3} \eta \right) \right. \right. \\ & \left. \left. + \delta \left(-\frac{7i}{5} - \pi - 2i \log(2) \right) \right] V_1^4 \right\} + \mathcal{O}(5), \end{aligned} \quad (12e)$$

$$\hat{H}_{44} = -\frac{4}{9} \sqrt{\frac{10}{7}} \left\{ (1 - 3\eta) V_4^2 + \left(-\frac{158383}{36960} + \frac{128221}{7392} \eta - \frac{1063}{88} \eta^2 \right) V_4^4 \right\} + \mathcal{O}(5), \quad (12f)$$

$$\hat{H}_{43} = -\frac{3}{4} \sqrt{\frac{3}{35}} \left\{ \delta (1 - 2\eta) V_3^3 + \frac{5}{2} \eta (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N - \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N) V_3^4 \right\} + \mathcal{O}(5), \quad (12g)$$

$$\hat{H}_{42} = -\frac{1}{63} \sqrt{5} \left\{ (1 - 3\eta) V_2^2 + \left(-\frac{105967}{36960} + \frac{75805}{7392} \eta - \frac{439}{88} \eta^2 \right) V_2^4 \right\} + \mathcal{O}(5), \quad (12h)$$

$$\hat{H}_{41} = -\frac{1}{84\sqrt{5}} \left\{ \delta (1 - 2\eta) V_1^3 + \frac{5}{2} \eta (\boldsymbol{\chi}_a \cdot \hat{\mathbf{L}}_N - \delta \boldsymbol{\chi}_s \cdot \hat{\mathbf{L}}_N) V_1^4 \right\} + \mathcal{O}(5), \quad (12i)$$

$$\hat{H}_{55} = -\frac{125}{96} \sqrt{\frac{5}{33}} \delta (1 - 2\eta) V_5^3 + \mathcal{O}(5), \quad (12j)$$

$$\hat{H}_{54} = -\frac{16}{9} \sqrt{\frac{2}{165}} (1 - 5\eta + 5\eta^2) V_4^4 + \mathcal{O}(5), \quad (12k)$$

$$\hat{H}_{53} = -\frac{9}{32\sqrt{55}} \delta (1 - 2\eta) V_3^3 + \mathcal{O}(5), \quad (12l)$$

$$\hat{H}_{52} = -\frac{2}{27\sqrt{55}} (1 - 5\eta + 5\eta^2) V_2^4 + \mathcal{O}(5), \quad (12m)$$

$$\hat{H}_{51} = -\frac{1}{144\sqrt{770}} \delta (1 - 2\eta) V_1^3 + \mathcal{O}(5), \quad (12n)$$

$$\hat{H}_{66} = -\frac{18}{5} \sqrt{\frac{3}{143}} (1 - 5\eta + 5\eta^2) V_6^4 + \mathcal{O}(5), \quad (12o)$$

$$\hat{H}_{65} = \mathcal{O}(5), \quad (12p)$$

$$\hat{H}_{64} = -\frac{128}{495\sqrt{39}} (1 - 5\eta + 5\eta^2) V_4^4 + \mathcal{O}(5), \quad (12q)$$

$$\hat{H}_{63} = \mathcal{O}(5), \quad (12r)$$

$$\hat{H}_{62} = -\frac{2}{297\sqrt{65}} (1 - 5\eta + 5\eta^2) V_2^4 + \mathcal{O}(5), \quad (12s)$$

$$\hat{H}_{61} = \mathcal{O}(5). \quad (12t)$$

Let us emphasize that the source frame used to express the above polarizations (and hence the GW modes) is identical to the one of Refs. [1, 12] (with spin contributions) but differs from that of Ref. [17] (without spinning contributions). The former frame has been defined so that the azimuthal angle ϕ locating the observer vanishes there,

while the latter is such that $\phi = \pi/2$. From Eq. (9) and the property of the spin weighted spherical harmonics, we see that $h_{\ell m}^{(1)} = i^m h_{\ell m}^{(17)}$. Although we list all the modes contributing to the waveform at the 2PN level here, for the convenience of the user, we list these expressions in the file (`supl-mkaf16.m`) which we provide as a supplemental material to our paper.

IV. CONCLUSIONS

Based on the recent developments in modelling the spinning binaries [9–12], we have computed the tail-induced 4PN spin-orbit contribution, the 3PN quadratic-spin correction and the 3.5PN cubic-spin correction to the frequency domain phasing of the GW signal, as well as the *complete* spin contributions to the amplitude of the frequency domain waveform at the 2PN order. The 4PN phase presented here only accounts for tail-induced spin-orbit effects, which must be supplemented by non-spinning contributions at this order, but these contributions are currently out of reach due to lack of necessary inputs for the calculation. On the other hand, some of the higher-order spin effects are still missing beyond the 3PN order. Those are: (i) the instantaneous quadratic-in-spin contributions at the 4PN order (including those resulting from the interactions between the two spins on the one hand, and the effect of the spin-induced mass quadrupoles of the black holes on the other hand), (ii) a quadratic-in-spin piece of gravitational-wave tails at the 3.5PN order. Moreover, when at least one of the two companions is a spinning black hole, the imprint of the corresponding absorption has yet to be incorporated to the flux at the 2.5PN order [29–31] beyond the leading quadrupolar piece, with a 1.5PN relative accuracy [32]. This generates additional terms at the 2.5PN, 3.5PN and 4PN orders in the energy balance equation that is used to determine the orbital phase expression.

Our new frequency domain amplitude corrections involve spin-orbit as well as spin-spin terms at the 2PN order. The polarizations and the spherical harmonic modes of the waveform in the frequency domain are now complete at this approximation level.

These results will be useful for many purposes. One immediate application would be in the construction of high accuracy templates for the search of aligned spin binaries [33, 34]. The spin effects in the amplitude and phase of the waveform will also help in reducing the errors associated with the parameter estimation of the spinning binary signals [2, 35]. In addition, these waveforms could be useful to study the effect of spins for various tests of strong field gravity proposed in the literature [36–41]. Last but not least, these terms could play a crucial role in constructing analytical inspiral-merger-ringdown waveforms [42] including higher GW modes.

Acknowledgments

This work was initiated during the ICTS Program on Numerical relativity organized by the International Center for Theoretical Sciences, Bangalore, in June-July 2013. KGA was partly funded by a grant from the Infosys foundation. Useful conversations with P Ajith and Bala Iyer are gratefully acknowledged.

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