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Open-Loop and Feedback Nash Equilibria in Constrained Linear-Quadratic Dynamic Games Played over Event Trees ⋆

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Abstract

We characterize open-loop and feedback Nash equilibria for a class of constrained linear-quadratic multistage games having the following features: (a) The control variables are of two types, namely, control variables that enter the dynamics, but are not constrained, and control variables that are part of state-control constraints, but do not enter the dynamics; (b) The parameter values are uncertain, with the stochastic process being described by a event tree.

This paper takes stock on Reddy and Zaccour [1,2] where the same class of games were considered, but in a deterministic setting. Here, we follow the same approaches and rewrite the results in these two papers in a stochastic setting.

Key words: Multistage Games; Nash Equilibria; Games Played over Event Trees; Constrained Games; Complementarity Problem.

1 Introduction

Reddy and Zaccour in [1,2] analyzed a class of deterministic linear-quadratic finite-horizon multistage games where the control variables of each player are of two types, namely: (a) control variables that do not enter the state dynamics, but are part of joint state-control constraints; and (b) control variables that affect the state dynamics, but are not constrained. In [1], the authors provide conditions for the existence and uniqueness of an open-loop Nash equilibrium, and an approach to compute such an equilibrium. In [2], the same is achieved but for a feedback information structure and resulting feedback-Nash equilibrium. In this paper, we extend the approaches and results in [1,2] to the case where the games involve uncertain parameter values. The uncertainty is represented by an exogenously given event tree, that is, the players cannot influence the transition from one node to another.

An illustrative example of the class of games we are considering here is an oligopoly game à la Cournot with capacity constraints. In such model, there are a finite number of firms selling a homogeneous product, and where the market price is a decreasing function in the total quantity available on the market and (here) on some uncertain parameter values. Each player faces a capacity constraint and can invest to increase it, or disinvest to reduce it. In the parlance of dynamic games, the quantity produced and the investment are the control variables and the production capacity is the state variable. The quantity is an example of control variables of type (a) above, while the investment in capacity is of type (b). Indeed, the quantity does not enter the state dynamics, but is constrained by the capacity, while the investment is part of these dynamics, but is unconstrained.

The general model considered in this paper belongs to the class of dynamic games played over event trees (DGPET) introduced by Zaccour [3] and Haurie et al. [4], and further developed in Haurie and Zaccour [5]. Assuming noncooperative mode of play, examples of real (or realistic) implementations of this class of games in electricity markets include, e.g., Pineau and Murto [6], Genc et al. [7], Genc and Sen [8] and Pineau et al. [9], with the main

⋆ This paper heavily draws on Reddy and Zaccour [1,2]. To simplify the reader's task in appreciating the differences between a deterministic and a stochastic setting, we kept on purpose the same notation, structure and example in these papers. We would like to thank the three reviewers and the associate editor for their helpful comments. Research supported by NSERC Canada, grant RGPIN-2016-04975.

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objective of determining the equilibrium investments in different generation technologies in competitive electricity industries. Long-term cooperative agreements and their sustainability in DGPET were recently considered in a series of papers; see, e.g., Reddy et al. [10], Parilina and Zaccour [11,12,13]. For a comprehensive introduction to the class of DGPET, see Haurie et al. [14], and for a survey/tutorial on its applications in cooperative games, see [15].

The rest of the paper is organized as follows: In Section 2, we introduce the multistage stochastic game model. In Sections 3 and 4, we deal with S-adapted open-loop Nash equilibria, and S-adapted feedback Nash equilibria, respectively. In Section 5, we discuss the computation of an S-adapted Nash equilibrium. In Section 6, we provide a simple numerical illustration and we briefly conclude in Section 7.

1.1 Notations

We shall use the following notation. M' denotes the transpose of a matrix M. $\Pi_{i=1}^N A_i$ represents the Cartesian product of sets $A_1, A_2, \cdots, A_N$ and $\oplus_{i=1}^N M_i$ represents the block diagonal matrix obtained by taking the matrices $M_1, M_2, \cdots, M_N$ as diagonal elements in this sequence. $M \otimes N$ represents the Kronecker product of matrices M and N. The matrix with all entries as zeros is denoted by $\mathbf{0}$, and the identity matrix is represented by I. We call two vectors $x, y \in \mathbb{R}^n$ complementary if $x \geq 0, y \geq 0$ and $x'y = 0$, and $0 \leq x \perp y \geq 0$ denotes this condition. Let D be a $n \times n$ matrix and e be a $n \times 1$ vector. Let n be partitioned as $n = n_1 + n_2 + \cdots + n_K$. We represent $[D]_{ij}$ as the $n_i \times n_j$ submatrix associated with indices n_i (row) and n_j (column), and $[e]_i$ as the $n_i \times 1$ subvector associated with indices n_i. $\text{col}(e_i)_{i=1}^N$ ($\text{row}(e_i)_{i=1}^N$) represents single vector or matrix obtained by stacking the vectors or matrices $e_1, e_2, \cdots, e_N$ vertically (horizontally). The linear-complementarity problem [17] is defined as finding an $x \in \mathbb{R}^n$, given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, such satisfying the following conditions: $Mx + q \geq 0, x \geq 0, x'(Mx + q) = 0$. We denote this problem as $\text{LCP}(M, q)$ and we denote its solutions as SOL(LCP).

2 The multistage stochastic game model

In this section, we first introduce the ingredients of an event tree, and next the details pertaining to the class of multistage games we are dealing with.

2.1 Modeling uncertainty with a scenario tree

Let $\mathbb{T} = \{0, 1, \ldots, T\}$ be the set of periods and denote by $\{\xi(t) : t \in \mathbb{T}\backslash\{T\}\}$ the exogenous stochastic process represented by an event tree. [1] This tree has a root node n_0 in period 0 and a finite set of nodes $\mathbf{n}_t$ in period $t \in \mathbb{T}$. Each node $n_t \in \mathbf{n}_t$ represents a possible sample value of the history of the $\xi(\cdot)$ process up to time t. Denote by $n_t^- \in \mathbf{n}_{t-1}$ the unique predecessor of node $n_t \in \mathbf{n}_t$, and by ν a successor of the node n_t. We denote by $n_t^+ \subset \mathbf{n}_{t+1}$ the set of all possible direct successors of node n_t, and by n_t^{++} the set of all nodes of the event tree having n_t as the root node. Having reached the node n_t at time t, n_t^{++} represents the residual uncertainty in the form of a subtree emanating from node n_t. Consequently, n_0^{++} describes the entire event tree. We illustrate a branch of the event tree in Figure 1. A path from the root node n_0 to a terminal node n_T is called a *scenario*, with the probabilities of all scenarios summing up to 1. We denote by $n_0 \rightsquigarrow n_t$ the set of all the nodes encountered along the unique sample path starting from n_0 to n_t. Let π_{n_t} be the probability of passing through node n_t, which corresponds to the sum of the probabilities of all scenarios that contain this node. In particular, $\pi_{n_0} = 1$ and π_{n_T} is equal to the probability of the single scenario that terminates in the node $n_T \in \mathbf{n}_T$. We denote by $\pi_{n_t}^\nu := \frac{\pi_\nu}{\pi_{n_t}}$ the transition probability from node n_t to a particular node $\nu \in n_t^+$, and by $\boldsymbol{\pi}_{n_t}^{n_t^+}$ the row vector of transition probabilities, that is,

$$\boldsymbol{\pi}_{n_t}^{n_t^+} = \left[\pi_{n_t}^{\nu^1} \ \pi_{n_t}^{\nu^2} \ \cdots \ \pi_{n_t}^{\nu^{|n_t^+|}}\right],$$

where $\nu^1, \nu^2, \cdots, \nu^{|n_t^+|}$ are the successors of node n_t. We enumerate the set of nodes at time t as $\mathbf{n}_t := \{n_t^1, n_t^2, \cdots, n_t^{|\mathbf{n}_t|}\}$. Let $z(\cdot)$ denotes the vector process (states and decisions) evolving on the scenario tree indexed by the nodes n_t, that is, at each node $n_t \in n_0^{++}$, we have $z(n_t) \in \mathbb{R}^n$. [2] We denote by z_{π,n_t^+} the conditional sum $z_{\pi,n_t^+} := \sum_{\nu \in n_t^+} \pi_{n_t}^\nu z_\nu$.

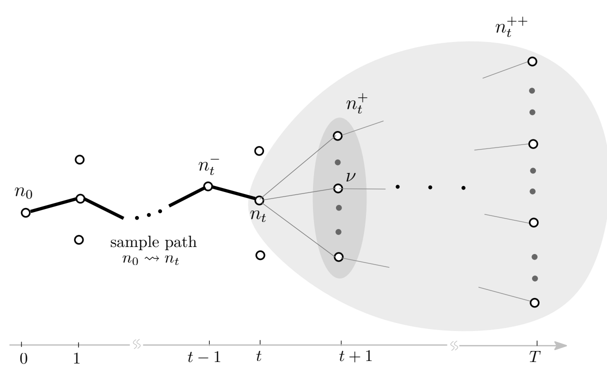

Fig. 1. Evolution of uncertainty described by an event tree

[1] Modeling uncertainty with scenario trees is a common practice in stochastic programming literature [16].

[2] We note that z_{n_t} takes values in a linear space.

2.2 *Linear quadratic stochastic dynamic game with constraints*

Denote by $\mathbb{N} = \{1, 2, \ldots, N\}$ the set of players. Let $x_{n_t} \in \mathbb{R}^n$, where n is a given positive integer, be a state vector at node n_t. At each node $n_t \in \mathbf{n}_t$, player $i \in \mathbb{N}$ chooses two types of controls/actions, that is: (i) variables that enter the dynamics of the system, but not the constraints, which we denote by $u^i_{n_t} \in U^i_{n_t} \subset \mathbb{R}^{m_i}$; and (ii) variables that do not affect the dynamics, but constrain the decision making process, denoted by $v^i_{n_t} \in V^i_{n_t} \subset \mathbb{R}^{s_i}$. Here, $U^i_{n_t}$ and $V^i_{n_t}$ are the sets of admissible values for the two types of decision variables. The evolution of the state vector is described as follows:

$$x_{n_t} = A_{n_t^-} x_{n_t^-} + \sum_{i\in\mathbb{N}} B^i_{n_t^-} u^i_{n_t^-}, \qquad (1)$$
$$n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{0\},\ x_{n_0} = x_0,$$

where $A_{n_t} \in \mathbb{R}^{n\times n}$ and $B^i_{n_t} \in \mathbb{R}^{n\times m_i}$. According to the state equation (1), at each node $n_t \in \mathbf{n}_t$, the decisions of all players $\{u^i_{n_t},\ i \in \mathbb{N}\}$ will determine, in association with the current state x_{n_t}, the state x_ν for all the successor nodes $\nu \in n_t^+$. Note that the decisions are made by the players before the realization of uncertainty, which is reflected by the one-period lag in the dynamics. [3] Further, we assume that each player i must also select values of $v^i_{n_t} \in V^i_{n_t}$ at each node of the event tree, after the realization of uncertainty, which is similar to recourse variables in stochastic programming [16]. We suppose that the decision variables $v^i_{n_t}$ do not enter the dynamics directly but appear only in the form of inequality constraints, jointly with the state variable, at every node n_t as follows:

$$M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t} + r^i_{n_t} \geq 0,\ v^i_{n_t} \geq 0,\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}, \qquad (2)$$

where $M^i_{n_t} \in \mathbb{R}^{c_i\times n}$, $N^i_{n_t} \in \mathbb{R}^{c_i\times s_i}$ and $r^i_{n_t} \in \mathbb{R}^{c_i}$ for $i \in \mathbb{N}$. As an illustrative example, the u variables can be investments in production capacities to be engaged before knowing the realization of the random process at the next period, and the v variables are the quantities put on the market at current node, and are subject to the capacity constraints.

Remark 1 *It is easily seen that $x_{\nu_1} = x_{\nu_2}$, $\forall \nu_1, \nu_2 \in n_t^+$ such that $\nu_1 \neq \nu_2$, i.e., each state variable takes the same value at all successor nodes n_t^+ that emanate from node n_t. Further, in the above formalism, though the decision variables $\{u^i_{n_t},\ i \in \mathbb{N}\}$ do not enter explicitly the constraints (2), they affect indirectly these constraints through the state variables.*

[3] Here, we restrict our attention to one period and it is quite straight forward to incorporate longer delays. We will see in sections 3 and 4 that this one period lag assumption leads to recursive formulation in the computation of Nash equilibria.

We denote by $\mathbf{U}_{n_t} := \Pi_{i=1}^N U^i_{n_t}$ and $\mathbf{V}_{n_t} := \Pi_{i=1}^N V^i_{n_t}$ the joint decision sets of the players at a node $n_t \in \mathbf{n}_t$. The joint decisions of the players at a node $n_t \in \mathbf{n}_t$ are denoted by $\mathbf{u}_{n_t} := \mathrm{col}(u^i_{n_t})_{i=1}^N$ and $\mathbf{v}_{n_t} := \mathrm{col}(v^i_{n_t})_{i=1}^N$. We denote by $\tilde{\mathbf{u}} := \{\mathbf{u}^i_{n_\tau},\ n_\tau \in n_0^{++}\backslash\mathbf{n}_T\}$ and by $\tilde{\mathbf{v}} := \{\mathbf{v}^i_{n_\tau},\ n_\tau \in n_0^{++}\}$ a complete specification of actions defined for every node of the event tree. In our state equation formalism, player $i \in \mathbb{N}$ chooses, independently, the actions $u^i_{n_t}$ and $v^i_{n_t}$. The state variables are shared by all players and they enter the definition of players' objective functions. At each node $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}$, the cost incurred by player i is a function of the state variable x_{n_t} and the decision variables of all players, that is, $\mathbf{u}_{n_t}$ and $\mathbf{v}_{n_t}$, given by $g_i(n_t, x_{n_t}, \mathbf{u}_{n_t}, \mathbf{v}_{n_t})$. At a terminal node n_T, the cost incurred by player i is given by the function $g_{vi}(n_T, x_{n_T}, \mathbf{v}_{n_T})$. The state equations, state constraints and the cost functions define the following linear-quadratic dynamic game on the event tree (LQDGET):

$$\textbf{LQDGET}: \min_{\substack{\{u^i_{n_\tau},\ n_\tau\in n_0^{++}\backslash\mathbf{n}_T,\\ v^i_{n_\tau},\ n_\tau\in n_0^{++}\}}} J_i(x_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}), \qquad (3)$$
$$\text{subject to (1) and (2),}$$

where the total cost incurred by player i is given by

$$J_i(x_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) := \sum_{t=0}^{T-1} \sum_{n_t\in\mathbf{n}_t} \pi_{n_t} g_i(n_t, x_{n_t}, (u^i_{n_t}, u^{i-}_{n_t}), (v^i_{n_t}, v^{i-}_{n_t})) + \sum_{n_T\in\mathbf{n}_T} \pi_{n_T} gv_i(n_T, x_{n_T}, (v^i_{n_T}, v^{i-}_{n_T})),$$

where $g_i(n_t, x_{n_t}, (u^i_{n_t}, u^{i-}_{n_t}), (v^i_{n_t}, v^{i-}_{n_t})) = gu_i(n_t, x_{n_t}, (u^i_{n_t}, u^{i-}_{n_t})) + gv_i(n_t, x_{n_t}, (v^i_{n_t}, v^{i-}_{n_t}))$,

$$gu_i(n_t, x_{n_t}, (u^i_{n_t}, u^{i-}_{n_t})) = \sum_{j\in\mathbb{N}} \frac{1}{2} u^{j}_{n_t}{}' R^{ij}_{n_t} u^j_{n_t},$$

$$\begin{aligned} gv_i(n_t, x_{n_t}, (v^i_{n_t}, v^{i-}_{n_t})) &= \frac{1}{2} x'_{n_t} Q^i_{n_t} x_{n_t} + p^{i}_{n_t}{}' x_{n_t} + \\ &\frac{1}{2}\left[v^{i}_{n_t}{}'\ v^{i-}_{n_t}{}'\right] \begin{bmatrix} [D^i_{n_t}]_{i,i} & [D^i_{n_t}]_{i,i-} \\ [D^i_{n_t}]_{i-,i} & [D^i_{n_t}]_{i-,i-} \end{bmatrix} \left[v^{i}_{n_t}{}'\ v^{i-}_{n_t}{}'\right]' \\ &+ x'_{n_t}\left[[L^i_{n_t}]_i\ \ [L^i_{n_t}]_{i-}\right]\left[v^{i}_{n_t}{}'\ v^{i-}_{n_t}{}'\right]' \\ &+ \left[\left[d^{i}_{n_t}{}'\right]_i\ \ \left[d^{i}_{n_t}{}'\right]_{i-}\right]\left[v^{i}_{n_t}{}'\ v^{i-}_{n_t}{}'\right]', \end{aligned} \qquad (4)$$

where $Q^i_{n_t} \in \mathbb{R}^{n\times n}$, $p^i_{n_t} \in \mathbb{R}^n$, $D^i_{n_t} \in \mathbb{R}^{s\times s}$. $d^i_{n_t} \in \mathbb{R}^s$ and $L^i_{n_t} \in \mathbb{R}^{n\times s}$ for $i \in \mathbb{N}$ and $n_t \in n_0^{++}$. Further, we assume that for every $i \in \mathbb{N}$, $R^{il}_{n_t} \in \mathbb{R}^{m_i\times m_l}$, for all $l \in \mathbb{N}$ and $n_t \in n_0^{++}\backslash\mathbf{n}_T$. Notice that the optimization problem of player i depends on the actions of the other players denoted by i^-, and on the state dynamics in (1). We have the following assumptions.

Assumption 2

(a) *The action sets* $\{U^i_{n_t} \subset \mathbb{R}^{m_i},\ i \in \mathbb{N},\ n_t \in n_0^{++} \backslash \mathbf{n}_T\}$ *are such that the feasible action sets* $V^i_{n_t}(x_{n_t}) := \{v^i_{n_t} \in \mathbb{R}^{s_i} \,|M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t} + r^i_{n_t} \geq 0,\ v^i_{n_t} \geq 0\}$ *are non-empty, convex and bounded for all* $n_t \in n_0^{++}$ *and* $i \in \mathbb{N}$*;*
(b) *The matrices* $\{N^i_{n_t},\ i \in \mathbb{N},\ n_t \in n_0^{++}\}$ *have full rank;*
(c) *The matrices* $\{R^{ii}_{n_t},\ i \in \mathbb{N},\ n_t \in n_0^{++} \backslash \mathbf{n}_T\}$ *and* $\{[D^i_{n_t}]_{i,i},\ i \in \mathbb{N},\ n_t \in n_0^{++}\}$ *are positive definite.*

We highlight that the decision variables are indexed over the nodes of the event tree, which is equivalent to saying that the decisions are adapted to the history of the stochastic process $\xi(\cdot)$. Hence, the use of S-adapted information structure in the literature cited in the introduction, where S stands for sample.

We introduce the following additional notation: For any player $i \in \mathbb{N}$, the joint decisions are denoted by $\psi^i_{n_t} := (u^i_{n_t}, v^i_{n_t})$ for all the nodes $n_t \in n_0^{++} \backslash \mathbf{n}_T$ and by $\psi^i_{n_T} := v^i_{n_T}$ for the terminal nodes $n_T \in \mathbf{n}_T$. The joint decisions of all players at a node n_t and represented by $\boldsymbol{\psi}_{n_t} := (\psi^1_{n_t},\ \psi^2_{n_t},\ \cdots, \psi^N_{n_t})$, and we denote by $\boldsymbol{\Psi}_{n_t} := \Pi_{i=1}^N (U^i_{n_t} \times V^i_{n_t})$ and $\boldsymbol{\Psi}_{n_T} := \Pi_{i=1}^N V^i_{n_T}$ the joint action space for all nodes $n_t \in n_0^{++} \backslash \mathbf{n}_T$ and $n_T \in \mathbf{n}_\mathbf{T}$, respectively. The set of admissible actions taken by the players along a sample path $n_0 \rightsquigarrow n_t$ are then given by $\{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_t\}$; see Figure 1. The strategy or the collection of actions of player $i \in \mathbb{N}$ for all nodes is denoted by $\tilde{\psi}_i := \{\psi^i_{n_t},\ n_t \in n_0^{++}\}$. The strategy profile of all players is denoted by $\tilde{\boldsymbol{\psi}} := \{\boldsymbol{\psi}_{n_t},\ n_t \in n_0^{++}\}$.

Definition 3 (*S*-adapted strategy) *An admissible S-adapted strategy for player i is defined by* $\tilde{\psi}_i = \{\psi^i_{n_t},\ n_t \in n_0^{++}\}$*, that is, the plan of decisions adapted to the history of the random process* $\xi(\cdot)$ *represented by the event tree.*

Definition 4 (*S*-adapted Nash equilibrium) *An S-adapted Nash equilibrium is an admissible S-adapted strategy* $\tilde{\boldsymbol{\psi}}^* := (\tilde{\psi}^*_i, \tilde{\psi}^*_{i^-})$ *such that for every player i the following condition holds:*

$$J_i(x_0, \tilde{\boldsymbol{\psi}}^*) \leq J_i(x_0, (\tilde{\psi}_i, \tilde{\psi}^*_{i^-}),\ \forall\ \tilde{\psi}_i, \tag{5}$$

where $\tilde{\psi}^*_{i^-}$ *is defined as the S-adapted Nash equilibrium strategy vector of players* i^-*.*

In multistage games the interaction environment is dynamic, which is embedded in state variables and their evolution. It is well known [18,19] that the predictions of the game (e.g., Nash equilibrium) vary with the information used by the players during the (dynamic) decision making process. Therefore, in a dynamic game an information structure must be specified when players design their strategies. In the following, we introduce two widely used information structures in dynamic games.

2.3 S-adapted information structure

We consider two types of information structures, namely, S-adapted open-loop information structure and S-adapted feedback information structure. In the former, each player designs a rule that selects values of the control variables that depend on the current node of the event tree n_t and initial state x_0. In our setting, this can be translated into $u^i_{n_t} = \gamma^i_{\mathtt{o}n_t}(x_0) \in U^i_{n_t}$, for the unconstrained decisions, and $v^i_{n_t} \in V^i_{n_t}(x_{n_t})$, for the constrained variables, where $\gamma^i_{\mathtt{o}n_t}(\cdot)$ is a continuously differentiable mapping, and the set of all such mappings is denoted by $\Gamma^i_{\mathtt{o}}$. Note that the feasible action space for the variables that enter the constraints depends on the state of the system, and hence the notation $V^i_{n_t}(x_{n_t})$.

In an S-adapted feedback (or Markovian) information structure, each player designs a rule for choosing actions at each node n_t of the event tree that are based on the observation of the state variable x_{n_t}. The feedback rule is a mapping from the state space into the action set. In our setting, this can be translated into $u^i_{n_t} = \gamma^i_{\mathtt{f}n_t}(x_{n_t}) \in U^i_{n_t}$, for the unconstrained decisions, and $v^i_{n_t} \in V^i_{n_t}(x_{n_t})$, for the constrained variables, where $\gamma^i_{\mathtt{f}n_t}(\cdot)$ is a continuously differentiable mapping, and the set of all such mappings is denoted by $\Gamma^i_{\mathtt{f}}$. Note that the feasible action space for the variables that enter the constraints depends on the state of the system, and hence the notation $V^i_{n_t}(x_{n_t})$.

3 *S*-adapted open-loop Nash equilibrium

In this section, we provide necessary and sufficient conditions for the existence of an S-adapted open-loop Nash equilibrium for LQDGET. From definition 4 and from section 2.3, the S-adapted open-loop Nash equilibrium strategy of a player $i \in \mathbb{N}$ is obtained as the solution of a constrained optimal control problem defined on the event tree, where the strategies of players i^- are fixed at their equilibrium values $(\tilde{u}^*_{i^-}, \tilde{v}^*_{i^-})$. We introduce the Lagrangian associated with player i's minimization problem as follows:

$$\begin{aligned}\mathcal{L}_i =& J_i(x_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \\ &- \sum_{t=0}^{T} \sum_{n_t\in \mathbf{n}_t} \pi_{n_t} {\mu^i_{n_t}}' \left(M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t} + r^i_{n_t}\right) \\ &+ \Bigg(\sum_{t=1}^{T} \sum_{n_t\in \mathbf{n}_t} \pi_{n_t} {\lambda^i_{n_t}}' \big(A_{n_t^-} x_{n_t^-} + B^i_{n_t^-} u^i_{n_t^-} \\ &+ \sum_{j\in i^-} B^j_{n_t^-} {u^j_{n_t^-}}^* - x_{n_t}\big)\Bigg) + {\lambda^i_{n_0}}'(x_{n_0} - x_0), \quad (6)\end{aligned}$$

where $\{\lambda^i_{n_t},\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}\}$ are the co-state variables associated with the state transition over the event tree (1) and $\{\mu^i_{n_t},\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\}$ are the Lagrange multipliers associated with the constraints (2). We have the following theorem regarding the necessary conditions for S-adapted open-loop Nash equilibrium for LQGET.

Theorem 5 *Let assumption 2 holds true, then for an N-person LQGET, let $\{{u^i_{n_t}}^* \equiv \gamma^{i*}_{\circ n_t}(x_0),\ n_t \in n_0^{++}\backslash \mathbf{n}_T,\ {v^i_{n_t}}^*,\ n_t \in n_0^{++}\}$ be an S-adapted open-loop Nash equilibrium solution and $\{x^*_{n_t},\ n_t \in n_0^{++}\}$ be the corresponding equilibrium state trajectory. Then, there exists a finite sequence of co-state vectors $\{\lambda^i_{n_t},\ n_t \in n_0^{++}\}$ and multiplier vectors $\{\mu^i_{n_t},\ n_t \in n_0^{++}\}$ for each player $i \in \mathbb{N}$ such that the following conditions are satisfied:*

$$ {u^i_{n_t}}^* \equiv \gamma^{i*}_{\circ n_t}(x_0) = -(R^{ii}_{n_t})^{-1} {B^i_{n_t}}' \lambda^i_{\pi, n_t^+}, \quad (7)$$

$$ x^*_\nu = A_{n_t} x^*_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t} {u^l_{n_t}}^*, \quad (8)$$

$$ \nu \in n_t^+,\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\},\ x_{n_0} = x_0,$$

$$ \lambda^i_{n_t} = A'_{n_t} \lambda^i_{\pi, n_t^+} + Q^i_{n_t} x^*_{n_t} + p^i_{n_t} - {M^i_{n_t}}' {\mu^i_{n_t}}^* + L^i_{n_t} \mathbf{v}^*_{n_t}, \quad (9)$$

$$ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\},$$

$$ \lambda^i_{n_T} = Q^i_{n_T} x^*_{n_T} + p^i_{n_T} - {M^i_{n_T}}' {\mu^i_{n_T}}^* + L^i_{n_T} \mathbf{v}^*_{n_T},\ n_T \in \mathbf{n}_T, \quad (10)$$

$$ 0 \le \sum_{l\in\mathbb{N}} \left[D^i_{n_t}\right]_{il} {v^l_{n_t}}^* - {N^i_{n_t}}' {\mu^i_{n_t}}^* + \left[L^i_{n_t}\right]'_i x^*_{n_t} + \left[d^i_{n_t}\right]_i \perp {v^i_{n_t}}^* \ge 0,\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}, \quad (11)$$

$$ 0 \le M^i_{n_t} x^*_{n_t} + N^i_{n_t} {v^i_{n_t}}^* + r^i_{n_t} \perp {\mu^i_{n_t}}^* \ge 0, \quad (12)$$

$$ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}.$$

The necessary conditions (7) follow directly as the Lagrangian (6) is strictly convex in $u^i_{n_t}$; $R^{ii}_{n_t}$ is a positive definite matrix. Further, the terminal conditions (10) are the transversality conditions associated with the constrained optimal-control problems. The conditions (11-12) are the Kraush-Khun-Tucker conditions at each time period $t \in \mathbb{T}$ represented in complementarity form. Note that (8-10) represent parametric two-point boundary value problems (TPBVPs), parametrized with optimal multipliers and joint actions $\{(\mathbf{v}^*_{n_t}, \boldsymbol{\mu}^*_{n_t}),\ n_t \in n_0^{++}\}$. Further, the complementarity conditions (11-12) represent parametric linear complementarity problems, parametrized with the optimal state trajectory $\{x^*_{n_t},\ n_t \in n_0^{++}\}$. Collecting (11-12) for all players $i \in \mathbb{N}$ we have at every node $n_t \in n_0^{++}$ the following parametric linear complementarity problem

$$\begin{aligned}\text{pLCP}(x^*_{n_t})\ : 0 \le& \begin{bmatrix} \mathbf{D}_{n_t} & -\mathbf{N}'_{n_t} \\ \mathbf{N}_{n_t} & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{v}^*_{n_t} \\ \boldsymbol{\mu}^*_{n_t}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{L}}'_{n_t} \\ \bar{\mathbf{M}}_{n_t}\end{bmatrix} x^*_{n_t} \\ &+ \begin{bmatrix}\mathbf{d}_{n_t} \\ \boldsymbol{r}_{n_t}\end{bmatrix} \perp \begin{bmatrix}\mathbf{v}^*_{n_t} \\ \boldsymbol{\mu}^*_{n_t}\end{bmatrix} \ge 0, \quad (13)\end{aligned}$$

where $[\mathbf{D}_{n_t}]_{ij} = \left[D^i_{n_t}\right]_{ij}$, $\mathbf{N}_{n_t} = \oplus_{i=1}^N N^i_{n_t}$, $\bar{\mathbf{L}}_{n_t} = \text{row}(\left[L^i_{n_t}\right]_i)_{i=1}^N$, $\bar{\mathbf{M}}_{n_t} = \text{col}(M^i_{n_t})_{i=1}^N$, $\mathbf{d}_{n_t} = \text{col}(\left[d^i_{n_t}\right]_i)_{i=1}^N$ and $\mathbf{r}_{n_t} = \text{col}(r^i_{n_t})_{i=1}^N$.

Consequently, the necessary conditions for the constrained Nash equilibrium lead to a coupled system of parametric two-point boundary value problem and parametric linear complementarity problems. Next, exploiting the linearity and separable cost structure (4) we show in the following theorem that the two-point boundary value problem can be solved by restricting the costate variables to be affine in the state variable. We have the following assumption.

Assumption 6 *The costate variable is affine in the state vector*

$$ \lambda^i_{n_t} := K^i_{n_t} x^*_{n_t} + \beta^i_{\circ n_t},\ n_t \in n_0^{++},\ i \in \mathbb{N}. \quad (14)$$

Theorem 7 *Let assumptions 2 and 6 hold true. Further, let the set of matrices $\{\Lambda^{n_t}_{n_t^+},\ n_t \in n_0^{++}\backslash \mathbf{n}_T\}$ defined in* (15) *be invertible, then the parametric equations (7-10) are uniquely solvable, and the matrices $\{K^i_{n_t}, \beta^i_{\circ n_t}\}$ in* (14) *are obtained recursively from (16-19).*

$$ \Lambda^{n_t}_{n_t^+} = I + \sum_{l\in\mathbb{N}} S^l_{n_t} K^l_{\pi, n_t^+},\ S^l_{n_t} = B^l_{n_t} (R^{ll}_{n_t})^{-1} {B^l_{n_t}}' \quad (15)$$

$$ K^i_{n_t} = Q^i_{n_t} + A'_{n_t} K^i_{\pi, n_t^+} \left(\Lambda^{n_t^+}_{n_t}\right)^{-1} A_{n_t} \quad (16)$$

$$ K^i_{n_T} = Q^i_{n_T},\ n_T \in \mathbf{n}_T \quad (17)$$

$$\begin{aligned}\beta^i_{\circ n_t} =& p^i_{n_t} - {M^i_{n_t}}' \mu^i_{n_t} + L^i_{n_t} \mathbf{v}^*_{n_t} \\ &+ A'_{n_t} \Big(\beta^i_{\circ\pi, n_t^+} - K^i_{\pi, n_t^+} \left(\Lambda^{n_t}_{n_t^+}\right)^{-1} \sum_{l\in\mathbb{N}} S^l_{n_t} \beta^l_{\circ\pi, n_t^+}\Big) \quad (18)\end{aligned}$$

$$ \beta^i_{\circ n_T} = p^i_{n_T} - {M^i_{n_T}}' \mu^i_{n_T} + L^i_{n_T} \mathbf{v}^*_{n_T},\ n_T \in \mathbf{n}_T \quad (19)$$

PROOF. As the terminal condition should hold for all $x^*_{n_T}$, we have $K^i_{n_T} = Q^i_{n_T}$, and $\beta^i_{\circ n_T} = p^i_{n_T} - {M^i_{n_T}}'\mu^i_{n_T} + L^i_{n_T}\mathbf{v}^*_{n_T}$ for $i \in \mathbb{N}$ and $n_T \in \mathbf{n}_T$. The state vector computed at the terminal nodes n_T then satisfies

$$
\begin{aligned}
x^*_{n_T} &= A_{n_T^-} x^*_{n_T^-} + \sum_{l\in\mathbb{N}} B^l_{n_T^-} {u^l_{n_T^-}}^* \\
&= A_{n_T^-} x^*_{n_T^-} - \sum_{l\in\mathbb{N}} B^l_{n_T^-} (R^{ll}_{n_T^-})^{-1} {B^l_{n_T^-}}' \lambda^l_{\pi,(n_T^-)^+}, \quad (20)
\end{aligned}
$$

where $(n_T^-)^+ \subset \mathbf{n}_T$ is the set of nodes with the node n_T^- as the common ancestor, and $\lambda^i_{\pi,(n_T^-)^+} = \sum_{\nu\in(n_T^-)^+} \pi^\nu_{n_T^-}\lambda^i_\nu$ is the conditional sum of λ^i_ν, which is the costate vector evaluated at the successor nodes $\nu \in (n_T^-)^+$. From remark 1 we have that the equilibrium state vector evaluated at the successor nodes must take the same value, i.e., $x^*_{\nu_1} = x^*_{\nu_2} = x^*_{n_T},\ \nu_1 \neq \nu_2,\ \nu_1,\nu_2 \in (n_T^-)^+$. Therefore, we have

$$
\begin{aligned}
\lambda^i_{\pi,(n_T^-)^+} &= \sum_{\nu\in(n_T^-)^+} \pi^\nu_{n_T^-} K^i_\nu x^*_\nu + \sum_{\nu\in(n_T^-)^+} \pi^\nu_{n_T^-} {\beta_\circ}^i_\nu \\
&= K^i_{\pi,(n_T^-)^+} x^*_{n_T} + {\beta_\circ}^i_{\pi,(n_T^-)^+}. \quad (21)
\end{aligned}
$$

Taking $S^l_{n_T^-} := {B^l}_{n_T^-}(R^{ll}_{n_T^-})^{-1}{B^{l\prime}}_{n_T^-}$ and substituting (21) in (20), we obtain

$$
\Big(I + \sum_{l\in\mathbb{N}} S^l_{n_T^-} K^l_{\pi,(n_T^-)^+}\Big) x^*_{n_T} = A_{n_T^-} x^*_{n_T^-} - \sum_{l\in\mathbb{N}} S^l_{n_T^-} {\beta_\circ}^l_{\pi,(n_T^-)^+}.
$$

Since $\Lambda^{(n_T^-)}_{(n_T^-)^+} := \Big(I + \sum_{l\in\mathbb{N}} S^l_{n_T^-} K^l_{\pi,(n_T^-)^+}\Big)$ is invertible, the equilibrium state vector evaluated at the node n_T is given by

$$
x^*_{n_T} = \big(\Lambda^{n_T^-}_{(n_T^-)^+}\big)^{-1} \left(A_{n_T^-} x^*_{n_T^-} - \sum_{l\in\mathbb{N}} S^l_{n_T^-} {\beta_\circ}^l_{\pi,(n_T^-)^+} \right),
$$

and the costate variable evaluated at the ancestor node n_T^- is then given (from (9)) by

$$
\begin{aligned}
\lambda^i_{n_T^-} =& A'_{n_T^-} \lambda^i_{\pi,(n_T^-)^+} + Q^i_{n_T^-} x^*_{n_T^-} + p^i_{n_T^-} - {M^i_{n_T^-}}' \mu^i_{n_T^-} + L^i_{n_T^-} \mathbf{v}^*_{n_T^-} \\
=& \Big(Q^i_{n_T^-} + A'_{n_T^-} K^i_{\pi,(n_T^-)^+} \big(\Lambda^{n_T^-}_{(n_T^-)^+}\big)^{-1} A_{n_T^-} \Big) x^*_{n_T^-} \\
&+ p^i_{n_T^-} - {M^i_{n_T^-}}' \mu^i_{n_T^-} + L^i_{n_T^-} \mathbf{v}^*_{n_T^-} + A'_{n_T^-} \bigg({\beta_\circ}^i_{\pi,(n_T^-)^+} \\
&- K^i_{\pi,(n_T^-)^+} \big(\Lambda^{n_T^-}_{(n_T^-)^+}\big)^{-1} \sum_{l\in\mathbb{N}} S^l_{n_T^-} {\beta_\circ}^l_{\pi,(n_T^-)^+} \bigg). \quad (22)
\end{aligned}
$$

Note that the left-hand side of the above equation is equal to $K^i_{n_T^-} x^*_{n_T^-} + \beta^i_{n_T^-}$. As the relation has to hold true for all $x^*_{n_T^-}$, collecting the coefficients of $x^*_{n_T^-}$ we obtain the equations for $K^i_{n_T^-}$ and $\beta^i_{n_T^-}$. Next, since the matrix $\Lambda^{T-2}_{T-1} = I + \sum_{l\in\mathbb{N}} S^l_{T-2} K^l_{T-1}$ is invertible we repeat this process for $n_t \in \mathbf{n}_t, t = T-2, \cdots, 1$ to obtain the backward equations (16) and (18) on the scenario tree.

We address the uniqueness part as follows. If the set of backward equations (15-18) admits a solution, that is, the matrices $\{\Lambda^{n_t}_{n_t^+},\ n_t \in n_0^{++}\backslash \mathbf{n}_T\}$ are invertible, then the two-point boundary value problem (8 -10) has a unique solution. To see this, let $\bar\lambda^i_{n_t} = \lambda^i_{n_t} - \big(K^i_{n_t} x^*_{n_t} + \beta^i_{n_t}\big)$ be any other solution of (8 -10). Then, substituting this in (8 -10), we write the two-point boundary value problem in $(x^*, \bar\lambda)$ coordinates as follows:

$$
\begin{aligned}
x^*_\nu =& (\Lambda^{n_t}_{n_t^+})^{-1} \Big(A_{n_t} x^*_{n_t} - \sum_{l\in\mathbb{N}} S^l_{n_t} \bar\lambda^l_{\pi,n_t^+} - \sum_{l\in\mathbb{N}} S^l_{n_t} {\beta_\circ}^l_{\pi,n_t^+} \Big), \\
\bar\lambda^i_{n_t} =& A'_t \bar\lambda^i_{\pi,n_t^+} - A'_{n_t} K^i_{\pi,n_t^+} \sum_{l\in\mathbb{N}} S^l_{n_t} \bar\lambda^l_{\pi,n_t^+}.
\end{aligned}
$$

The above system of equations is decoupled. From the terminal conditions we have $\bar\lambda^i_{n_T} = 0$, and as a result, we have $\bar\lambda^i_{n_t} = 0$ for all $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}$, and therefore the solution is unique.

The equilibrium actions (7), in parametric form, are given by

$$
\begin{aligned}
{u^i_{n_t}}^* =& -(R^{ii}_{n_t})^{-1} {B^i_{n_t}}' K^i_{\pi,n_t^+} \big(\Lambda^{n^t}_{n_t^+}\big)^{-1} A_{n_t} x^*_{n_t} \\
&- (R^{ii}_{n_t})^{-1} {B^i_{n_t}}' \Big({\beta_\circ}^i_{\pi,n_t^+} - \big(\Lambda^{n_t}_{n_t^+}\big)^{-1} \sum_{l\in\mathbb{N}} S^l_{n_t} {\beta_\circ}^l_{\pi,n_t^+} \Big), \quad (23) \\
& n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}, \\
x^*_\nu =& \big(\Lambda^{n_t}_{n_t^+}\big)^{-1} A_{n_t} x^*_{n_t} - \big(\Lambda^{n_t}_{n_t^+}\big)^{-1} \sum_{l\in\mathbb{N}} S^l_{n_t} {\beta_\circ}^l_{\pi,n_t^+}, \quad (24) \\
& \nu \in n_t^+,\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{0\},\ x_{n_0} = x_0.
\end{aligned}
$$

3.1 *Procedure for solving the necessary conditions*

Notice that restricting the costate variables to be affine functions in the state variable provides a unique solution to the parametric TPBVP. The resulting backward equations (15-16) evolve independently of the backward equations (18). Further, we obtain parametric linear backward difference equations (18) parametrized by $\{(\mathbf{v}^*_{n_t}, \boldsymbol{\mu}^*_{n_t}),\ n_t \in n_0^{++}\}$. From this observation it is then possible to represent the state variables as functions of the parameters $\{(\mathbf{v}^*_{n_t}, \boldsymbol{\mu}^*_{n_t}),\ n_t \in n_0^{++}\}$. In the following, we seek to derive an explicit relation between the state variables and the parameters.

Letting $\boldsymbol{\beta}_{\circ n_t} := \mathrm{col}(\beta^i_{\circ n_t})_{i=1}^N$, $\mathbf{p}_{n_t} := \mathrm{col}(p^i_{n_t})_{i=1}^N$, $\mathbf{L}_{n_t} := \mathrm{col}(L^i_{n_t})_{i=1}^N$ and $\mathbf{M}_{n_t} := \oplus_{i=1}^N M^i_{n_t}$, the vector representation of (18) is then given by

$$\boldsymbol{\beta}_{\circ n_t} = \mathbf{p}_{n_t} + \begin{bmatrix}\mathbf{L}_{n_t} & -\mathbf{M}'_{n_t}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{n_t} & \boldsymbol{\mu}^{*\prime}_{n_t}\end{bmatrix}' + \mathbf{G}_{\circ\ n_t^+}^{n_t}\,\boldsymbol{\beta}_{\circ\pi,n_t^+}, \tag{25}$$

$$n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\},$$

$$\boldsymbol{\beta}_{\circ n_T} = \mathbf{p}_{n_T} + \begin{bmatrix}\mathbf{L}_{n_T} & -\mathbf{M}'_{n_T}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{n_T} & \boldsymbol{\mu}^{*\prime}_{n_T}\end{bmatrix}',\ n_T \in \mathbf{n}_T, \tag{26}$$

where

$$[\mathbf{G}_{\circ\ n_t^+}^{n_t}]_{ij} = \begin{cases} -A'_{n_t}K^i_{\pi,n_t^+}(\Lambda^{n_t}_{n_t^+})^{-1}S^j_{n_t} & i \neq j, \\ A'_{n_t} - A'_{n_t}K^i_{\pi,n_t^+}(\Lambda^{n_t}_{n_t^+})^{-1}S^i_{n_t} & i = j. \end{cases}$$

Let $n_t^+ := \{\nu_1, \nu_2, \cdots, \nu_{|n_t^+|}\}$ denote the successor nodes, then (25) can be represented as

$$\begin{aligned}\boldsymbol{\beta}_{\circ n_t} &= \mathbf{p}_{n_t} + \begin{bmatrix}\mathbf{L}_{n_t} & -\mathbf{M}'_{n_t}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{n_t} & \boldsymbol{\mu}^{*\prime}_{n_t}\end{bmatrix}' \\ &\quad + \left(\boldsymbol{\pi}^{n_t^+}_{n_t} \otimes \mathbf{G}_{\circ\ n_t^+}^{n_t}\right)\begin{bmatrix}\boldsymbol{\beta}'_{\circ\nu_1} & \boldsymbol{\beta}'_{\circ\nu_2} & \cdots & \boldsymbol{\beta}'_{\circ\nu_{|n_t^+|}}\end{bmatrix}', \\ &\qquad n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}.\end{aligned}$$

Enumerating the nodes at time t as $\mathbf{n}_t := \{n_t^1, n_t^2, \cdots, n_t^{|\mathbf{n}_t|}\}$, we obtain the following backward recursion relating the set of all nodes at time $t+1$ with the set of all nodes at time t, that is,

$$\boldsymbol{\beta}_{\circ\mathbf{n}_t} = \mathbf{p}_{\mathbf{n}_t} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_t} & -\mathbf{M}'_{\mathbf{n}_t}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{\mathbf{n}_t} & \boldsymbol{\mu}^{*\prime}_{\mathbf{n}_t}\end{bmatrix}' + \mathbf{G}_{\circ t+1}^{\ t}\boldsymbol{\beta}_{\circ\mathbf{n}_{t+1}},$$

where $\boldsymbol{\beta}_{\circ\mathbf{n}_t} := \mathrm{col}(\boldsymbol{\beta}_{\circ n_t^i})_{i=1}^{|\mathbf{n}_t|}$, $\mathbf{v}^*_{\mathbf{n}_t} := \mathrm{col}(\mathbf{v}^*_{n_t^i})_{i=1}^{|\mathbf{n}_t|}$, $\boldsymbol{\mu}^*_{\mathbf{n}_t} := \mathrm{col}(\boldsymbol{\mu}^*_{n_t^i})_{i=1}^{|\mathbf{n}_t|}$, $\mathbf{p}_{\mathbf{n}_t} := \mathrm{col}(\mathbf{p}_{n_t^i})_{i=1}^{|\mathbf{n}_t|}$, $\mathbf{L}_{\mathbf{n}_t} := \oplus_{i=1}^{|\mathbf{n}_t|}\mathbf{L}_{n_t^i}$, $\mathbf{M}_{\mathbf{n}_t} := \oplus_{i=1}^{|\mathbf{n}_t|}\mathbf{M}_{n_t^i}$, and $\mathbf{G}_{\circ t+1}^{\ t} := \oplus_{i=1}^{|\mathbf{n}_t|}(\pi^{n_t^{i+}}_{n_t^i} \otimes \mathbf{G}_{\circ\ n_t^{i+}}^{n_t^i})$. Here, (25) represents a backward linear difference equation evolving on the event tree, with terminal condition (26) defined at the leaf nodes. We define the transition matrix associated with the linear backward difference equation (25) as $\psi_\circ(t,\tau) = \mathbf{G}^t_{t+1}\cdots\mathbf{G}^{\tau-2}_{\tau-1}\mathbf{G}^{\tau-1}_{\tau}$ when $\tau > t$, and as $\psi_\circ(t,\tau) = \mathbf{I}$ when $\tau = t$, and (25) is given by:

$$\boldsymbol{\beta}_{\circ\mathbf{n}_t} = \sum_{\tau=t}^{T}\psi_\circ(t,\tau)\left(\mathbf{p}_{\mathbf{n}_\tau} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_\tau} & -\mathbf{M}'_{\mathbf{n}_\tau}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{\mathbf{n}_\tau} & \boldsymbol{\mu}^{*\prime}_{\mathbf{n}_\tau}\end{bmatrix}'\right). \tag{27}$$

From (24) and (25), it is fairly easy to see that the equilibrium state trajectory $\{x^*_{n_t},\ n_t \in n_0^{++}\backslash n_0\}$ is an affine function of the parameters $\{(\mathbf{v}^*_{n_t}, \boldsymbol{\mu}^*_{n_t}),\ n_t \in n_0^{++}\backslash n_0\}$. Next, we derive this explicit relation. Towards this end, the state variables at all the successor nodes can be written in the following vector form:

$$\begin{aligned}\begin{bmatrix}x^{*\prime}_{\nu_1} & x^{*\prime}_{\nu_2} & \cdots & x^{*\prime}_{\nu_{|n_t^+|}}\end{bmatrix}' &= \left(\mathbf{1} \otimes \left((\Lambda^{n_t}_{n_t^+})^{-1}A_{n_t}\right)\right)x^*_{n_t} \\ &\quad + \left(-\boldsymbol{\pi}^{n_t^+}_{n_t} \otimes \left((\Lambda^{n_t}_{n_t^+})^{-1}\begin{bmatrix}S^1_{n_t} & S^2_{n_t} & \cdots & S^N_{n_t}\end{bmatrix}\right)\right) \\ &\qquad \begin{bmatrix}\boldsymbol{\beta}'_{\circ\nu_1}\boldsymbol{\beta}'_{\circ\nu_2} & \cdots & \boldsymbol{\beta}'_{\circ\nu_{|n_t^+|}}\end{bmatrix}'.\end{aligned}$$

As before, we collect the state vectors defined at all nodes at time t

$$x^*_{\mathbf{n}_{t+1}} = \bar{A}^\circ_t x^*_{\mathbf{n}_t} + \bar{B}^\circ_{t+1}\boldsymbol{\beta}_{\circ\mathbf{n}_{t+1}}, \tag{28}$$

where $\bar{A}^\circ_t = \oplus_{i=1}^{|\mathbf{n}_t|}\left(\mathbf{1} \otimes \left((\Lambda^{n_t^i}_{n_t^{i+}})^{-1}A_{n_t^i}\right)\right)$ and $\bar{B}^\circ_{t+1} = \oplus_{i=1}^{|\mathbf{n}_t|}\left(-\pi^{n_t^+}_{n_t} \otimes \left((\Lambda^{n_t}_{n_t^+})^{-1}\begin{bmatrix}S^1_{n_t} & S^2_{n_t} & \cdots & S^N_{n_t}\end{bmatrix}\right)\right)$. The above equation represents the forward linear difference equation that evolves on the event tree, and relates the state variables defined at all nodes at time t with the state variables defined at all nodes at time $t+1$. Denoting the associated transition matrix as $\phi_\circ(\rho,t) = \bar{A}^\circ_{t-1}\bar{A}^\circ_{t-2}\cdots\bar{A}^\circ_\rho$ for $\rho < t$ and $\phi_\circ(t,\rho) = \mathbf{I}$ when $\rho = t$, we have:

$$x^*_{\mathbf{n}_t} = \phi_\circ(0,t)x_0 + \sum_{\rho=0}^{t-1}\phi_\circ(\rho+1,t)\,\bar{B}^\circ_{\rho+1}\,\boldsymbol{\beta}_{\circ\mathbf{n}_{\rho+1}},\ t \in \mathbb{T}\backslash\{0\}. \tag{29}$$

Using (27) in (29), we have for $t \in \mathbb{T}\backslash\{0\}$

$$\begin{aligned}x^*_{\mathbf{n}_t} &= \phi_\circ(0,t)x_0 + \sum_{\rho=0}^{t-1}\Bigg[\Bigg(\phi_\circ(\rho+1,t)\ \bar{B}^\circ_{\rho+1} \\ &\qquad \sum_{\tau=\rho+1}^{T}\psi_\circ(t,\tau)\left(\mathbf{p}_{\mathbf{n}_\tau} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_\tau} & -\mathbf{M}'_{\mathbf{n}_\tau}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{\mathbf{n}_\tau} & \boldsymbol{\mu}^{*\prime}_{\mathbf{n}_\tau}\end{bmatrix}'\right)\Bigg)\Bigg] \\ &= \phi_\circ(0,t)x_0 + \sum_{\tau=1}^{T}\Bigg[\Bigg(\Big(\sum_{\rho=1}^{\min(t,\tau)}\phi_\circ(\rho,t)\ \bar{B}^\circ_\rho\ \psi_\circ(\rho,\tau)\Big) \\ &\qquad \left(\mathbf{p}_{\mathbf{n}_\tau} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_\tau} & -\mathbf{M}'_{\mathbf{n}_\tau}\end{bmatrix}\begin{bmatrix}\mathbf{v}^{*\prime}_{\mathbf{n}_\tau} & \boldsymbol{\mu}^{*\prime}_{\mathbf{n}_\tau}\end{bmatrix}'\right)\Bigg]. \end{aligned} \tag{30}$$

Next, we aggregate the variables in (30) as $x^*_{\mathbf{T}} := \mathrm{col}(x^*_{\mathbf{n}_t})_{t=1}^T$, $\mathbf{p}_{\mathbf{T}} := \mathrm{col}(\mathbf{p}_{\mathbf{n}_t})_{t=1}^T$, $\mathbf{v}^*_{\mathbf{T}} := \mathrm{col}(\mathbf{v}^*_{\mathbf{n}_t})_{t=1}^T$ and $\boldsymbol{\mu}^*_{\mathbf{n}_T} := \mathrm{col}(\boldsymbol{\mu}^*_{\mathbf{n}_t})_{t=1}^T$ the above equation can be compactly represented as

$$x^*_{\mathbf{T}} = \boldsymbol{\Phi}^\circ_0 x_0 + \boldsymbol{\Phi}^\circ_1\mathbf{p}_{\mathbf{T}} + \boldsymbol{\Phi}^\circ_2\mathbf{v}^*_{\mathbf{T}} + \boldsymbol{\Phi}^\circ_3\boldsymbol{\mu}^*_{\mathbf{T}}, \tag{31}$$

where $[\boldsymbol{\Phi}^\circ_0]_t = \phi_\circ(0,t)$, $[\boldsymbol{\Phi}^\circ_1]_{t\tau} = \sum_{\rho=1}^{\min(t,\tau)}\phi_\circ(\rho,t)\bar{B}^\circ_\rho\psi_\circ(\rho,\tau)$, $[\boldsymbol{\Phi}^\circ_2]_{t\tau} = \sum_{\rho=1}^{\min(t,\tau)}\phi_\circ(\rho,t)\bar{B}^\circ_\rho\psi_\circ(\rho,\tau)\mathbf{L}_{\mathbf{n}_\tau}$ and $[\boldsymbol{\Phi}^\circ_3]_{t\tau} = -\sum_{\rho=1}^{\min(t,\tau)}\phi_\circ(\rho,t)\bar{B}^\circ_\rho\psi_\circ(\rho,\tau)\mathbf{M}'_{\mathbf{n}_\tau}$ for $t,\tau \in \mathbb{T}\backslash\{0\}$. Similarly, the parametric linear complementarity problems (13) over the nodes $n_t \in \mathbf{n}_t$ at time $t \in \mathbb{T}\backslash\{0\}$ are aggregated as follows:

$$\text{pLCP}(x^*_{\mathbf{T}}) \ : 0 \leq \begin{bmatrix} \mathbf{D_T} & -\mathbf{N}'_{\mathbf{T}} \\ \mathbf{N_T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}^*_{\mathbf{T}} \\ \boldsymbol{\mu}^*_{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{L}}'_{\mathbf{T}} \\ \bar{\mathbf{M}}_{\mathbf{T}} \end{bmatrix} x^*_{\mathbf{T}} + \begin{bmatrix} \mathbf{d_T} \\ \mathbf{r_T} \end{bmatrix} \perp \begin{bmatrix} \mathbf{v}^*_{\mathbf{T}} \\ \boldsymbol{\mu}^*_{\mathbf{T}} \end{bmatrix} \geq 0, \tag{32}$$

where $\mathbf{D_T} = \oplus_{t=1}^{T}\left(\oplus_{i=1}^{|\mathbf{n}_t|}\mathbf{D}_{n_t^i}\right), \mathbf{N_T} = \oplus_{t=1}^{T}\left(\oplus_{i=1}^{|\mathbf{n}_t|}\mathbf{N}_{n_t^i}\right)$, $\bar{\mathbf{L}}_{\mathbf{T}} = \oplus_{t=1}^{T}\left(\oplus_{i=1}^{|\mathbf{n}_t|}\bar{\mathbf{L}}_{n_t^i}\right), \bar{\mathbf{M}}_{\mathbf{T}} = \oplus_{t=1}^{T}\left(\oplus_{i=1}^{|\mathbf{n}_t|}\bar{\mathbf{M}}_{n_t^i}\right), \mathbf{d_T} = \text{col}\left(\text{col}(\mathbf{d}_{n_t^i})_{i=1}^{|\mathbf{n}_t|}\right)_{t=1}^{T}$ and $\mathbf{r_T} = \text{col}\left(\text{col}(\mathbf{r}_{n_t^i})_{i=1}^{|\mathbf{n}_t|}\right)_{t=1}^{T}$. Now, substituting for $x^*_{\mathbf{T}}$ in (32) using (31) we have the following single large-scale linear complementarity problem

$$\text{LCP}_{\circ} : 0 \leq \mathbf{M}^{\circ} \begin{bmatrix} \mathbf{v}^*_{\mathbf{T}} \\ \boldsymbol{\mu}^*_{\mathbf{T}} \end{bmatrix} + \mathbf{q}^{\circ} \perp \begin{bmatrix} \mathbf{v}^*_{\mathbf{T}} \\ \boldsymbol{\mu}^*_{\mathbf{T}} \end{bmatrix} \geq 0, \tag{33}$$

where $\mathbf{M}^{\circ} = \begin{bmatrix} \mathbf{D_T} + \bar{\mathbf{L}}'_{\mathbf{T}}\mathbf{\Phi}_2^o & -\mathbf{N}'_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}}\mathbf{\Phi}_3^o \\ \mathbf{N_T} + \bar{\mathbf{M}}_{\mathbf{T}}\mathbf{\Phi}_2^o & \bar{\mathbf{M}}_{\mathbf{T}}\mathbf{\Phi}_3^o \end{bmatrix}$ and

$$\mathbf{q}^{\circ} = \begin{bmatrix} \mathbf{d_T} + \bar{\mathbf{L}}'_{\mathbf{T}}\mathbf{\Phi}_1^o\mathbf{p_T} + \bar{\mathbf{L}}'_{\mathbf{T}}\mathbf{\Phi}_0^o x_0 \\ \mathbf{r_T} + \bar{\mathbf{M}}_{\mathbf{T}}\mathbf{\Phi}_1^o\mathbf{p_T} + \bar{\mathbf{L}}'_{\mathbf{T}}\mathbf{\Phi}_0^o x_0 \end{bmatrix}.$$

To summarize, using assumptions 2 and 6 we have reformulated the necessary conditions as solvability of a single large-scale linear complementarity problem. The solutions of pLCP(x_{n_0}) and LCP$_{\circ}$ along with (7) constitute candidates for the S-adapted open-loop Nash equilibrium strategies for LQDGET.

3.2 Sufficient conditions

Let $\begin{bmatrix} \mathbf{v}^*_{\mathbf{n}_0} \\ \boldsymbol{\mu}^*_{\mathbf{n}_0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{v}^*_{\mathbf{T}} \\ \boldsymbol{\mu}^*_{\mathbf{T}} \end{bmatrix}$ be the solutions of pLCP(x_{n_0}) and LCP$_{\circ}$, respectively. We can write player i's objective as follows:

$$\begin{aligned} J_i(x_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = & \sum_{t=0}^{T}\sum_{n_t\in\mathbf{n}_t}\pi_{n_t}\Big(\frac{1}{2}x'_{n_t}Q^i_{n_t}x_{n_t} \\ & + \big(p^i_{n_t} + L^i_{n_t}\mathbf{v}^*_{n_t} - M^i_{n_t}{}'\mu^i_{n_t}{}^*\big)'x_{n_t}\Big) \\ & + \sum_{t=0}^{T-1}\sum_{n_t\in\mathbf{n}_t}\pi_{n_t}\frac{1}{2}\Big(u^i_{n_t}{}'R^{ii}_{n_t}u^i_{n_t} + \sum_{j\in i^-}u^j_{n_t}{}'R^{ij}_{n_t}u^j_{n_t}\Big) \\ & + \sum_{t=0}^{T}\sum_{n_t\in\mathbf{n}_t}\pi_{n_t}\Big(\big[d^i_{n_t}\big]'_i v^i_{n_t} + x'_{n_t}\big[L^i_{n_t}\big]_i(v^i_{n_t} - v^i_{n_t}{}^*) \\ & + x'_{n_t}\big[L^i_{n_t}\big]_{i^-}(v^{i^-}_{n_t} - v^{i^-}_{n_t}{}^*) + \big[d^i_{n_t}\big]'_{i^-}v^{i^-}_{n_t} + \mu^i_{n_t}{}^{*\prime}M^i_{n_t}x_{n_t} \\ & + \frac{1}{2}\Big(v^i_{n_t}{}'\big[D^i_{n_t}\big]_{ii}v^i_{n_t} + 2v^i_{n_t}{}'\big[D^i_{n_t}\big]_{ii^-}v^{i^-}_{n_t} \\ & + v^{i^-}_{n_t}{}'\big[D^i_{n_t}\big]_{i^-i^-}v^{i^-}_{n_t}\Big)\Big). \end{aligned} \tag{34}$$

Next, we provide conditions under which the objective function (34) is strictly convex in player i's decision variables for an arbitrary choice of decision variables of players in i^-.

Theorem 8 *Let the solutions of $E^i_{n_t}$ of the symmetric difference matrix Riccati equations defined on the event tree*

$$\begin{aligned} E^i_{n_t} = & A'_{n_t}E^i_{\pi,n_t^+}A_{n_t} + Q^i_{n_t} \\ & - A'_{n_t}E^i_{\pi,n_t^+}B^i_{n_t}T^i_{n_t}{}^{-1}B^i_{n_t}E^i_{\pi,n_t^+}A_{n_t}, \\ & \qquad n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\} \\ E^i_{n_T} = & Q^i_{n_T},\ n_T \in \mathbf{n}_T,\ t \in \mathbb{T} \\ T^i_{n_t} = & R^{ii}_{n_t} + B^i_{n_t}{}'E^i_{\pi,n_t^+}B^i_{n_t} \end{aligned} \tag{35}$$

exist for $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}$, (hence we assume $T^i_{n_t} = R^{ii}_{n_t} + B^i_{n_t}{}'E^i_{\pi,n_t^+}B^i_{n_t}$ is invertible for all $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}$). Then, for any set of decision variables $(\tilde{u}_i, \tilde{v}_i),\ i \in \mathbb{N}$ the following recursive difference equations defined on the event tree:

$$\begin{aligned} f^i_{n_t} = & (p^i_{n_t} + L^i_{n_t}\mathbf{v}^*_{n_t} - M^i_{n_t}{}'\mu^i_{n_t}{}^*) + A'_{n_t}E^i_{\pi,n_t^+}\delta^i_{n_t} \\ & + A'_{n_t}f^i_{\pi,n_t^+}{}' - A'_{n_t}E^i_{\pi,n_t^+}B^i_{n_t}b^i_{n_t}, \end{aligned} \tag{36}$$

$$f^i_{n_T} = (p^i_{n_T} + L^i_{n_T}\mathbf{v}^*_{n_T} - M^i_{n_T}{}'\mu^i_{n_T}{}^*), \tag{37}$$

$$b^i_{n_t} = T^i_{n_t}{}^{-1}B^i_{n_t}{}'\left(E^i_{\pi,n_t^+}\delta^i_{n_t} + f^i_{\pi,n_t^+}\right),\ b^i_{n_T} = 0, \tag{38}$$

$$\begin{aligned} g^i_{n_t} = & g^i_{\pi,n_t^+} + \frac{1}{2}\delta^i_{n_t}{}'E^i_{\pi,n_t^+}\delta^i_{n_t} + f^i_{\pi,n_t^+}{}'\delta^i_{n_t} \\ & - \frac{1}{2}b^i_{n_t}{}'T^i_{n_t}b^i_{n_t} + \frac{1}{2}\sum_{j\in i^-}u^j_{n_t}{}'R^{ij}_{n_t}u^j_{n_t}, \end{aligned} \tag{39}$$

$$g^i_{n_T} = 0,$$

are solvable backwards for $t \in \mathbb{T}$, where $\delta^i_{n_t} = \sum_{j\in i^-}B^j_{n_t}u^j_{n_t}$. Define the matrices $\mathbf{T}_i$, $\mathbf{C}_i$ and $\mathbf{D}_i$ for $i \in \mathbb{N}$ as follows: $\mathbf{T}_i = \oplus_{t=1}^{T}\left(\oplus_{k=1}^{|\mathbf{n}_t|}T^i_{n_t^k}\right)$, $\mathbf{D}_i = \oplus_{t=1}^{T}\left(\oplus_{k=1}^{|\mathbf{n}_t|}\left[D^i_{n_t^k}\right]_{ii}\right)$ and $\mathbf{C}_i = \left[\frac{\partial^2 J_i}{\partial u^i_{n_t}\partial v^i_{n_\tau}}\right]_{\{n_t\in\mathbf{n}_t,\ t\in\mathbb{T}\backslash\{T\},\ n_\tau\in\mathbf{n}_\tau,\ \tau\in\mathbb{T}\}}$. If the matrices $\begin{bmatrix} \mathbf{T}_i & \mathbf{C}'_i \\ \mathbf{C}_i & \mathbf{D}_i \end{bmatrix}$ are positive definite for $i \in \mathbb{N}$, then player i's objective function is strictly convex in $(\tilde{u}_i, \tilde{v}_i)$ for an arbitrary choice of $(\tilde{u}_{i^-}, \tilde{v}_{i^-})$.

PROOF. The proof of the theorem follows the same reasoning as in [20, Theorem 2.1] when adapted to the event tree setting. We define $V^i_{n_t} = \frac{1}{2}x'_{n_t}E^i_{n_t}x_{n_t} + f^i_{n_t}{}'x_{n_t} + g'_{n_t}$ for $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}$. We start with the difference $\sum_{\nu\in n_t^+}\pi^{\nu}_{n_t} - V^i_{n_t}$ and take $\xi^i_{n_t} = T^i_{n_t}{}^{-1}B^i_{n_t}{}'E^i_{\pi,n_t^+}A_{n_t}x_{n_t} + b^i_{n_t}$. Then after a few algebraic manipulations using the recursive equations (35)-(39), we can rewrite the objective function (34) as

$$\begin{aligned} &J_i(x_0,((\tilde{u}_i,\tilde{v}_i),(\tilde{u}_{i^-},\tilde{v}_{i^-}))) \\ &= V^i_{n_0} + \sum_{t=0}^{T-1}\sum_{n_t\in\mathbf{n}_t}\pi_{n_t}\left(\frac{1}{2}||u^i_{n_t}+\xi^i_{n_t}||_{T^i_{n_t}}\right) \\ &+ \sum_{t=0}^{T}\sum_{n_t\in\mathbf{n}_t}\pi_{n_t}\Big(x'_{n_t}\left[L^i_{n_t}\right]_i\left(v^i_{n_t}-v^{i\ *}_{n_t}\right) \\ &+ x'_{n_t}\left[L^i_{n_t}\right]_{i^-}\left(v^{i^-}_{n_t}-v^{i^-*}_{n_t}\right)+\left[d^i_{n_t}\right]'_i v^i_{n_t}+\left[d^i_{n_t}\right]'_{i^-} v^{i^-}_{n_t} \\ &+ \frac{1}{2}\Big(||v^i_{n_t}||^2_{\left[D^i_{n_t}\right]_{ii}} + 2v^{i\ \prime}_{n_t}\left[D^i_{n_t}\right]_{ii^-} v^{i^-}_{n_t} \\ &+ v^{i^-\prime}_{n_t}\left[D^i_{n_t}\right]_{i^-i^-} v^{i^-}_{n_t}\Big) + \mu^{i\ *\prime}_{n_t} M^i_{n_t} x_{n_t}\Big). \end{aligned} \tag{40}$$

The matrix $\begin{bmatrix}\mathbf{T}_i & \mathbf{C}'_i \\ \mathbf{C}_i & \mathbf{D}_i\end{bmatrix}$ is the Hessian matrix corresponding the player i's objective computed with the decision variables $(\tilde{u}_i,\tilde{v}_i)$ for an arbitrary choice of $(\tilde{u}_{i^-},\tilde{v}_{i^-})$. Clearly, convexity of (34) follows from the positive definiteness of the Hessian matrix.

Remark 9 *The matrix $\mathbf{C}_i$ captures the coupling between the decision variables $\tilde{u}_i$ and $\tilde{v}_i$. Further, due to the linear state dynamics the elements of $\mathbf{C}_i$ do not depend on the choice of the decision variables $(\tilde{u}_i,\tilde{v}_i)$.*

Using Theorem 8 we have transformed the dynamic game into a static game. If assumption 2 holds true then players' action sets are non-empty, convex and bounded. Further, if conditions in Theorem 8 hold true, then players' objectives are strictly convex. Therefore, the LQDGET is a convex game; see Rosen [21], where player i minimizes the objective using $(\tilde{u}_i,\tilde{v}_i)$ subject to constraints while the remaining players' actions are fixed at $(\tilde{u}_{i^-},\tilde{v}_{i^-})$. We recall the following result for the existence of a Nash equilibrium for convex games.

Theorem 10 ([21, Theorem 1]) *An equilibrium point exists for every convex game.*

The following theorem provides a sufficient condition for the existence of an S-adapted open-loop Nash equilibrium.

Theorem 11 *Let assumptions 2 and 6 hold true. Let the solutions of the symmetric difference matrix Riccati equations (35) exist, and the Hessian matrices $\begin{bmatrix}\mathbf{T}_i & \mathbf{C}'_i \\ \mathbf{C}_i & \mathbf{D}_i\end{bmatrix}$ be positive definite for $i\in\mathbb{N}$. Then, the solutions of pLCP(x_{n_0}) and LCP$_\circ$, denoted by $\{\mathbf{v}^*_{n_t}, n_t\in n_0^{++}\}$, along with (23) and (24) provide a constrained open-loop S-adapted Nash equilibrium for LQDGET.*

PROOF. We want to show that the solutions obtained from solving pLCP(x_{n_0}) and LCP$_\circ$ along with (23) indeed provide an S-adapted open-loop Nash equilibrium. To establish this, we fix the strategies of players in i^- at their Nash equilibrium values. Then, the best reply of player i is obtained by minimizing the objective (40) with respect to $(\tilde{u}_i,\tilde{v}_i)$ $x_\nu = A_{n_t}x_{n_t} + B^i_{n_t}u^i_{n_t} + \sum_{j\in i^-} B^j_{n_t}u^{j\ *}_{n_t}$, $\nu\in n_t^+$, $n_t\in\mathbf{n}_t$, $t\in\mathbb{T}\backslash\{T\}$, $x_{n_0}=x_0$ and $M^i_{n_t}x_{n_t}+N^i_{n_t}v^i_{n_t}+r^i_{n_t}\geq 0$, $v^i_{n_t}\geq 0$, $n_t\in\mathbf{n}_t$, $t\in\mathbb{T}$. Since the objective function is convex in the decision variables, the KKT conditions provide the necessary and sufficient conditions for a minimum. The KKT conditions are as follows:

$$\begin{aligned} &\text{for } n_t\in\mathbf{n}_t, t\in\mathbb{T}\backslash\{T\}, \\ &T^i_{n_t}(u^i_{n_t}+\xi^i_{n_t}) + \sum_{\tau=t+1}^{T}\sum_{\alpha\in\mathbf{n}_\tau\cap n_t^{++}}\pi^\alpha_{n_t}\frac{\partial x_\alpha}{\partial u^i_{n_t}}' \\ &\quad\Big(\left[L^i_\alpha\right]_i(v^i_\alpha - v^{i\ *}_\alpha) - M^{i\ \prime}_\alpha(\mu^i_\alpha-\mu^{i\ *}_\alpha)\Big) = 0, \\ &\text{for } n_t\in\mathbf{n}_t, t\in\mathbb{T}, \\ &0\leq \left[D^i_{n_t}\right]_{ii} v^i_{n_t} + \left[D^i_{n_t}\right]_{ii^-} v^{i^-*}_{n_t} - N^i_{n_t}\mu^i_{n_t} \\ &\quad + \left[L^i_{n_t}\right]'_i x_{n_t} + \left[d^i_{n_t}\right]_i \perp v^i_{n_t}\geq 0, \\ &0\leq M^i_{n_t}x_{n_t} + N^i_{n_t}v^i_{n_t} + r^i_{n_t}\perp \mu^i_{n_t}\geq 0, \\ &\text{where } x_{n_t} = A_{n_t^-}x_{n_t^-} + B^i_{n_t^-}u^i_{n_t^-} + \sum_{j\in i^-} B^j_{n_t^-}u^{j\ *}_{n_t^-}. \end{aligned}$$

Next, we substitute for $v^i_{n_t} = v^{i\ *}_{n_t}$ in the above equations to obtain the following equations:

$$\begin{aligned} &\text{for } n_t\in\mathbf{n}_t, t\in\mathbb{T}\backslash\{T\}, \\ &T^i_{n_t}(u^i_{n_t}+\xi^i_{n_t}) = 0, \\ &\text{for } n_t\in\mathbf{n}_t, t\in\mathbb{T}, \\ &0\leq \left[D^i_{n_t}\right]_{ii} v^{i\ *}_{n_t} + \left[D^i_{n_t}\right]_{ii^-} v^{i^-*}_{n_t} - N^i_{n_t}\mu^{i\ *}_{n_t} \\ &\quad + \left[L^i_{n_t}\right]'_i x_{n_t} + \left[d^i_{n_t}\right]_i \perp v^{i\ *}_{n_t}\geq 0, \\ &0\leq M^i_{n_t}x_{n_t} + N^i_{n_t}v^{i\ *}_{n_t} + r^i_{n_t}\perp \mu^{i\ *}_{n_t}\geq 0, \\ &\text{where } x_{n_t} = A_{n_t^-}x_{n_t^-} + B^i_{n_t^-}u^i_{n_t^-} + \sum_{j\in i^-} B^j_{n_t^-}u^{j\ *}_{n_t^-}. \end{aligned}$$

The decision variables $u^{i\ *}_{n_t}$ are obtained by solving the TPBVP (8-9). The control action of player i given by $u^{i\ *}_{n_t} := -B^{i\ \prime}_{n_t}R^i_i\lambda^i_{\pi,n_t^+}$ solves $T^i_{n_t}(u^i_{n_t}+\xi^i_{n_t})=0$. Then, the state trajectory generated by $u^{i\ *}_{n_t}$ is $x^*_{n_t}$. Therefore, it remains to verify if $v^{i\ *}_{n_t}$ satisfy the remaining KKT conditions, that is,

$$\begin{aligned} &\text{for } n_t\in\mathbf{n}_t, t\in\mathbb{T}, \\ &0\leq \left[D^i_{n_t}\right]_{ii} v^{i\ *}_{n_t} + \left[D^i_{n_t}\right]_{ii^-} v^{i^-*}_{n_t} - N^i_{n_t}\mu^{i\ *}_{n_t} \\ &\quad + \left[L^i_{n_t}\right]'_i x^*_{n_t} + \left[d^i_{n_t}\right]_i \perp v^{i\ *}_{n_t}\geq 0, \\ &0\leq M^i_{n_t}x^*_{n_t} + N^i_{n_t}v^{i\ *}_{n_t} + r^i_{n_t}\perp \mu^{i\ *}_{n_t}\geq 0. \end{aligned}$$

Clearly, the above conditions are satisfied by $\{v_{n_t}^{i\ *}, \mu_{n_t}^{i\ *},\ n_t \in n_0^{++}\}$ as they are precisely the parametric linear complementarity problems evaluated along the equilibrium state trajectory (8).

4 *S*-adapted feedback Nash equilibrium

We introduce the reachable set $\mathcal{X}_{n_t} \in \mathbb{R}^n$, i.e., the set of all state variables $x_{n_t} \in \mathbb{R}^n$ at node n_t that are reachable when the players use some combination of admissible actions along a realized sample path $n_0 \rightsquigarrow n_t$ starting from the root node n_0. Recall that the set of admissible actions taken by the players along a sample path $n_0 \rightsquigarrow n_t$ is given by $\{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_t\}$; see Figure 1.

Definition 12 (*S*-adapted feedback Nash equilibrium) *The set of strategies $(\tilde{\psi}_1^*, \tilde{\psi}_2^*, \cdots, \tilde{\psi}_N^*)$ constitute an S-adapted feedback Nash equilibrium for the LQDGET if it satisfies the following set of backward recursive inequalities:*

for every $n_T \in \mathbf{n}_T$

$$J_i\bigg(x_0, \{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_T^-\}, (\psi_{n_T}^i, \psi_{n_T}^{i-*})\bigg) \leq J_i\bigg(x_0, \{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_T^-\}, (\psi_{n_T}^{i\ *}, \psi_{n_T}^{i-*})\bigg), \quad (41)$$

for every $n_t \in \mathbf{n}_t,\ t = T-1, T-2, \cdots, 0$ (recursively)

$$J_i\bigg(x_0, \{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_t^-\}, (\psi_{n_t}^i, \psi_{n_t}^{i-*}), \{\boldsymbol{\psi}_{n_\tau}^*,\ n_\tau \in n_t^{++}\backslash n_t\}\bigg) \leq J_i\bigg(x_0, \{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_t^-\}, (\psi_{n_t}^{i\ *}, \psi_{n_t}^{i-*}),, \{\boldsymbol{\psi}_{n_\tau}^*,\ n_\tau \in n_t^{++}\backslash n_t\}\bigg). \quad (42)$$

In the above definition, the first inequality holds true for all the admissible actions $\{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_T^-\}$ taken along the sample path $n_0 \rightsquigarrow n_T^-$. Similarly, in (42), the inequalities defined recursively backward at time $t = T-1, \cdots, 0$, hold true for all the admissible strategies $\{\boldsymbol{\psi}_\alpha, \alpha \in n_0 \rightsquigarrow n_t^-\}$ taken along the realized sample path $n_0 \rightsquigarrow n_t^-$. We have the following proposition.

Proposition 13 *Every N-tuple $(\tilde{\psi}_1^*, \tilde{\psi}_2^*, \cdots, \tilde{\psi}_N^*)$ that satisfies the set of inequalities* (42) *also satisfies the set of N inequalities* (5).

The proof of the above proposition follows directly from Proposition 3.9 of [18] when adapted to an event tree setting. The proposition says that every *S*-adapted feedback Nash equilibrium, obtained through solving (41-42), is also an *S*-adapted Nash equilibrium, i.e., they also solve (5).

Remark 14 *The above recursive formulation naturally leads to a static game at node n_t where the downstream decisions at the nodes $n_t^{++}\backslash n_t$ are fixed at their equilibrium values. During this recursive procedure, if there exist at any time period more than one Nash equilibrium, then we will need to solve the upstream static games for each one of these equilibria to obtain the complete set of feedback Nash equilibria.*

For any $x_{n_t} \in \mathcal{X}_{n_t}$ the subgame starting at (n_t, x_{n_t}) associated with the LQDGET is then given by

$$\mathtt{sg}(n_t, x_{n_t}): \min_{\left\{\substack{\boldsymbol{\psi}_{n_\tau} := (\psi_{n_\tau}^i, \psi_{n_\tau}^{i-}),\\ n_\tau \in n_t^{++}}\right\}} J_i(n_t, x_{n_t}, \{(\psi_{n_\tau}^i, \psi_{n_\tau}^{i-}),\ n_\tau \in n_t^{++}\}), \quad (43)$$

subject to

$$\hat{x}_{n_\tau} = A_{n_\tau^-}\hat{x}_{n_\tau^-} + B_{n_\tau^-}^i u_{n_\tau^-}^i + \sum_{j\in i^-} B_{n_\tau^-}^j u_{n_\tau^-}^j, \quad n_\tau \in \mathbf{n}_\tau,\ \tau \in \mathbb{T}\backslash\{0\},\ \hat{x}_{n_t} = x_{n_t},$$

and

$$M_{n_\tau}^i \hat{x}_{n_\tau} + N_{n_\tau}^i v_{n_\tau}^i + r_{n_\tau}^i \geq 0,\ v_{n_\tau}^i \geq 0,\ n_\tau \in n_t^{++},$$

where

$$J_i(n_t, x_{n_t}, (\psi_{n_t}^i, \psi_{n_t}^{i-}), \{\boldsymbol{\psi}_{n_\tau},\ n_\tau \in n_t^{++}\backslash n_t\}) = \sum_{\tau=t}^{T-1} \sum_{n_\tau \in \mathbf{n}_\tau \cap n_t^{++}} \pi_{n_\tau} g_i(n_\tau, x_{n_\tau}, (u_{n_\tau}^i, u_{n_\tau}^{i-}), (v_{n_\tau}^i, v_{n_\tau}^{i-})) + \sum_{n_T \in \mathbf{n}_T \cap n_t^{++}} \pi_{n_T} gv_i(n_T, x_{n_T}, (v_{n_T}^i, v_{n_T}^{i-})). \quad (44)$$

We make the following additional assumption, besides assumption 2, concerning the players' objectives.

Assumption 15 *We define $s \times s$ matrices $\hat{D}_{n_t}$ for $n_t \in n_0^{++}$ as*

$$\hat{D}_{n_t} := \begin{bmatrix} [D_{n_t}^1]_{11} & [D_{n_t}^1]_{12} & \cdots & [D_{n_t}^1]_{1N} \\ [D_{n_t}^2]_{21} & [D_{n_t}^2]_{22} & \cdots & [D_{n_t}^2]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ [D_{n_t}^N]_{N1} & [D_{n_t}^N]_{N2} & \cdots & [D_{n_t}^N]_{NN} \end{bmatrix} \quad (45)$$

and assume that the matrix $\hat{D}_{n_t} + \hat{D}_{n_t}'$ is positive definite for all $n_t \in n_0^{++}$. Here, $[D_{n_t}^i]_{jk}, j, k \in \mathbb{N}$ is the $s_j \times s_k$ submatrix associated with the quadratic term

$\mathbf{v}_{n_t}' D^i_{n_t} \mathbf{v}_{n_t} := \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} v^j_{n_t}{}' \left[D^i_{n_t}\right]_{jk} v^k_{n_t}$ *in the player i's objective* (4).

We have the following theorem characterizing the S-adapted feedback Nash equilibrium of LQDGET.

Theorem 16 *Let assumptions 2 and 15 hold true. For an N-person constrained LQDGET the set of strategies $\{u^i_{n_t}{}^* \equiv \gamma^{i*}_{\mathtt{f}\,n_t}(x_{n_t}),\ n_t \in n_0^{++}\backslash \mathbf{n}_T,\ v^i_{n_t}{}^*,\ n_t \in n_0^{++}\}$ provides a feedback-Nash equilibrium solution if and only if there exist value functions $W^i(n_t, x_{n_t}) : \mathcal{X}_{n_t} \to \mathbb{R}$ such that the following backward recursive equations defined on the event tree are satisfied:*

$$
\begin{aligned}
&W^i(n_t, x_{n_t}) \\
&= \min_{\{u^i_{n_t}, v^i_{n_t}\}} \Big\{ g_i(n_t, x_{n_t}, (u^i_{n_t}, \gamma_{\mathtt{f}\,n_t}^{i^-{}^*}(x_{n_t})), (v^i_{n_t}, v^{i^-{}^*}_{n_t})) \\
&\qquad + \sum_{\nu\in n_t^+} \pi^\nu_{n_t} W^i(\nu, x_\nu) \Big\} \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{subject to } x_\nu = A_{n_t} x_{n_t} + B^i_{n_t} u^i_{n_t} + \sum_{j\in i^-} B^j_{n_t} \gamma_{\mathtt{f}\,n_t}^{j}{}^*, \\
&\qquad \nu \in n_t^+,\ t \in \mathbb{T}\backslash\{T\},\ x_{n_0} = x_0 \\
&\qquad M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t} + r^i_{n_t} \geq 0,\ v^i_{n_t} \geq 0 \\
&\textit{and} \quad W^i(n_T, x_{n_T}) = gv_i(n_T, x_{n_T}, (v^i_{n_T}{}^*, v^{i^-{}^*}_{n_T})) \\
&\qquad - \mu^i_{n_T}{}^*(M^i_{n_T} x_{n_T} + N^i_{n_T} v^i_{n_T}{}^* + r^i_{n_T}),\ \forall n_T \in \mathbf{n}_T \\
&\textit{where} \quad (\mathbf{v}^*_{n_T}, \boldsymbol{\mu}^*_{n_T}) = \mathrm{SOL}(\mathrm{pLCP}(x_{n_T})),\ \forall n_T \in \mathbf{n}_T.
\end{aligned}
$$

PROOF. The proof of the theorem follows from [2, Theorem III.3] when adapted to the event tree setting.

Although Theorem 16 provides a characterization of S-adapted feedback-Nash equilibrium, through the value function (46), it does not give additional information on the solvability of (46). Notice that the minimization problem (46) involves one constrained minimization problem for each player $i \in \mathbb{N}$. Due to separability of the objective function and due to the unique solution $(\mathbf{v}^*_{n_t}, \boldsymbol{\mu}^*_{n_t}) = \mathtt{Sol}(\mathrm{pLCP}(x_{n_t}))$ the value function at node n_t is obtained by substituting the feedback-Nash equilibrium solution in the associated Lagrangian as

$$
\begin{aligned}
&W^i(n_t, x_{n_t}) = \\
&\quad g_i(n_t, x_{n_t}, (\gamma_{\mathtt{f}\,n_t}^{i}{}^*(x_{n_t}), \gamma_{\mathtt{f}\,n_t}^{i^-{}^*}(x_{n_t})), (v^i_{n_t}{}^*, v^{i^-{}^*}_{n_t})) \\
&-\mu^i_{n_t}{}^{*\prime}(M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t}{}^* + r^i_{n_t}) + \sum_{\nu\in n_t^+} \pi^\nu_{n_t} W^i(\nu, x_\nu).
\end{aligned}
$$

The second term $\mu^i_{n_t}{}^{*\prime}(M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t}{}^* + r^i_{n_t}) = 0$ due to complementarity condition associated with $\mathrm{pLCP}(x_{n_t})$. In the remaining section, we show that this observation enables to relate the value function (46) with the value function associated with a parametric unconstrained dynamic game defined on the event tree; thereby providing a way for solving (46) and computing the S-adapted feedback Nash equilibrium. To this end, we define the following parametric dynamic game defined on the event tree. Introduce the parametric linear-quadratic game defined on the event tree as follows:

$$
\textbf{pLQDGET}: \quad \min_{\left\{u^i_{n_\tau},\ n_\tau \in n_0^{++}\backslash \mathbf{n}_T\right\}} J_i^p(x_0, \tilde{\mathbf{u}}; (\tilde{\mathbf{w}}, \tilde{\boldsymbol{\theta}})) \qquad (47)
$$

$$
\text{subject to (1)},
$$

where

$$
\begin{aligned}
&J_i^p(x_0, \tilde{\mathbf{u}}; (\tilde{\mathbf{w}}, \tilde{\boldsymbol{\theta}})) = J_i(x_0, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \\
&\qquad - \sum_{t=0}^{T} \sum_{n_t\in\mathbf{n}_t} \pi_{n_t} \theta^i_{n_t}{}' \left(M^i_{n_t} x_{n_t} + N^i_{n_t} w^i_{n_t} + r^i_{n_t}\right).
\end{aligned}
$$

Theorem 17 *For an N-person unconstrained pLQDGET the set of strategies $\{u^i_{n_t}{}^* := \xi^i_{n_t}(x_{n_t}),\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}\}$ provides a feedback-Nash equilibrium solution if and only if there exist value functions $W^i_p(n_t, \cdot\,; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\}) : \mathbb{R}^n \to \mathbb{R}$ such that the following backward recursive equations defined on the event tree are satisfied:*

$$
\begin{aligned}
&W^i_p(n_t, x_{n_t}; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\}) \\
&= \min_{u^i_{n_t}} \Big\{ g_i\Big(n_t, x_{n_t}, (u^i_{n_t}, \xi^i_{n_t}(x_{n_t}; (\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++})) \\
&\quad , (w^i_{n_t}, w^{i^-}_{n_t})\Big) - \theta^i_{n_t}{}' \left(M^i_{n_t} x_{n_t} + N^i_{n_t} w^i_{n_t} + r^i_{n_t}\right) \\
&\quad + \sum_{\nu\in n_t^+} \pi^\nu_{n_t} W^i_p(\nu, x_\nu; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in \nu^{++}\}) \Big\}, \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
&W^i_p(n_T, x_{n_T}; \{(\mathbf{w}_{n_T}, \boldsymbol{\theta}_{n_T}),\ n_T \in \mathbf{n}_T\}) \\
&\qquad = gv_i(n_T, x_{n_T}, (w^i_{n_T}, w^{i^-}_{n_T})) \\
&\qquad - \theta^i_{n_T}{}' \left(M^i_{n_T} x_{n_T} + N^i_{n_T} w^i_{n_T} + r^i_{n_T}\right), \qquad (49)
\end{aligned}
$$

where $x_\nu = A_{n_t} x_{n_t} + B^i_{n_t} u^i_{n_t} + \sum_{j\in i^-} B^j_{n_t} \xi^j_{n_t}(x_{n_t}; (\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}),\ \nu \in n_t^+,\ n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\},\ x_{n_0} = x_0$. Every such parametric Nash equilibrium solution admits a representation given by

$$
u^i_{n_t} := \xi^i_{n_t}(x_{n_t}; (\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}). \qquad (50)
$$

PROOF. The existence of a feedback-Nash equilibrium for pLQDGET follows directly from the proof of [2, Lemma III.4] when adapted to event tree. Since there are no cross terms between the decision variables $u^i_{n_t}$ and the parameters $(\mathbf{w}_{n_t}, \boldsymbol{\theta}_{n_t})$ the decision variable $u^i_{n_t}$ in the KKT conditions is a function the current state and the parameters $\{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha), \alpha \in n_t^{++}\}$. Therefore, the feedback Nash equilibrium strategy $\{u^i_{n_t},\ n_t \in n_0^{++} \backslash \mathbf{n}_T\}$ admits the representation (50).

The following theorem relates the Nash equilibrium strategy associated with LQDGET and the parametric feedback Nash equilibrium strategy associated with pLQDGET.

Theorem 18 *Let assumptions 2 and 15 hold true. The constrained S-adapted feedback Nash equilibrium is given by the set of strategies where*

$$u^{i\ *}_{n_t} := \gamma^{i\ *}_{\mathtt{f}n_t}(x_{n_t}) = \xi^i_{n_t}(x_{n_t}; \{(\mathbf{v}^*_\alpha, \boldsymbol{\mu}^*_\alpha),\ \alpha \in n_t^{++}\}), \tag{51}$$

with $(\mathbf{v}^*_\alpha, \boldsymbol{\mu}^*_\alpha) = \mathtt{sol}(\mathtt{pLCP}(x_\alpha))$ *and the state trajectory evolves on the scenario tree as follows:*

$$x_\nu = A_{n_t} x_{n_t} + \sum_{j \in \mathbb{N}} B^j_{n_t} \xi^j_{n_t}(x_{n_t}; \{(\mathbf{v}^*_\alpha, \boldsymbol{\mu}^*_\alpha),\ \alpha \in n_t^{++}\})$$

Further, we have

$$W^i(n_t, x_{n_t}) = W^i_p(n_t, x_{n_t}; \{(\mathbf{v}^*_\alpha, \boldsymbol{\mu}^*_\alpha),\ \alpha \in n_t^{++}\}). \tag{52}$$

PROOF. The proof of the theorem follows from [2, Theorem III.5] when adapted to the event tree setting.

To summarize, besides obtaining the S-adapted Nash equilibrium of LQDGET through a parametric Nash equilibrium strategy (51), Theorem 18 also provides a way to obtain the value function (46), from the parametric value function (48), through the relation (52).

4.1 Procedure for solving the dynamic programming equation

In this subsection, we show that under a few additional assumptions on strategy spaces, and a functional form of the parametric value function it is possible to solve the dynamic programming equation (48). We have the following assumption.

Assumption 19 *The equilibrium controls that enter the dynamics* (1) *are restricted to be affine in the state variable, i.e.,*

$$u^{i\ *}_{n_t} \equiv E^i_{n_t} x_{n_t} + F^i_{n_t},\ E^i_{n_t} \in \mathbb{R}^{m_i \times n},\ F^i_{n_t} \in \mathbb{R}^{m_i},\ \forall i \in \mathbb{N}. \tag{53}$$

Further, the value function for player $i \in \mathbb{N}$ at time t takes the following parametric form:

$$\begin{aligned} &W^i_p(n_t, x_{n_t}; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\}) \\ &\qquad = \frac{1}{2} x'_{n_t} P^i_{n_t} x_{n_t} + \beta^{i\,\prime}_{\mathtt{f}n_t} x_{n_t} + \eta^i_{n_t} \\ &+ \sum_{\alpha \in n_t^{++}} \pi^\alpha_{n_t} \Big(\frac{1}{2} \begin{bmatrix} \mathbf{w}_\alpha \\ \boldsymbol{\theta}_\alpha \end{bmatrix}' \begin{bmatrix} D^i_\alpha & -\mathbf{N}'_\alpha \\ -\mathbf{N}_\alpha & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w}_\alpha \\ \boldsymbol{\theta}_\alpha \end{bmatrix} + d^{i\ \prime}_\alpha \mathbf{w}_\alpha \Big). \end{aligned}$$

We write down the Lagrangian associated with player i's minimization problem (48) as follows:

$$\begin{aligned} \mathcal{L}^i_{n_t} &:= g_i\Big(n_t, x_{n_t}, u^i_{n_t}, \xi^{i-*}_{n_t}(x_{n_t}; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\}), \\ &\quad (v^i_{n_t}, v^{i-*}_{n_t}) \Big) - \mu^{i\prime}_{n_t}\big(M^i_{n_t} x_{n_t} + N^i_{n_t} v^i_{n_t} + r^i_{n_t} \big) \\ &\quad + \sum_{\nu \in n_t^+} \pi^\nu_{n_t}\ W^i_p\big(\nu, x_\nu; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in \nu^{++}\}\big), \end{aligned} \tag{54}$$

where $x_\nu = A_{n_t} x_{n_t} + B^i_{n_t} u^i_{n_t} + \sum_{j \in i^-} B^j_{n_t} \xi^{j\ *}_{n_t}(x_{n_t}; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\})$. The Karush-Kuhn-Tucker conditions associated with the minimization problem (48) are given by

$$\begin{aligned} &\big(R^{ii}_{n_t} + B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} B^i_{n_t} \big)\big(E^i_{n_t} x_{n_t} + F^i_{n_t} \big) \\ &\qquad + B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} \big[A_{n_t} + \sum_{j \in i^-} B^j_{n_t} E^j_{n_t} \big] x_{n_t} \\ &\qquad + B^{i\,\prime}_{n_t} P^i_{\pi, n_t} \sum_{j \in i^-} B^j_{n_t} F^j_{n_t} + B^{i\,\prime}_{n_t} \beta^i_{\mathtt{f}\pi, n_t^+} = 0. \end{aligned} \tag{55}$$

Since the above equalities have to hold true for all $x_{n_t} \in \mathcal{X}_{n_t}$, equating the coefficients of x_{n_t}, we get for every $i \in \mathbb{N}$ the following equations:

$$\begin{aligned} &\big(R^{ii}_{n_t} + B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} B^i_{n_t} \big) E^i_{n_t} \\ &\qquad + \sum_{j \in i^-} B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} B^j_{n_t} E^j_{n_t} = -B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} A_{n_t}, \end{aligned} \tag{56}$$

$$\begin{aligned} &\big(R^{ii}_{n_t} + B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} B^i_{n_t} \big) F^i_{n_t} \\ &\qquad + \sum_{j \in i^-} B^{i\,\prime}_{n_t} P^i_{\pi, n_t^+} B^j_{n_t} F^j_{n_t} = -B^{i\,\prime}_{n_t} \beta^i_{\mathtt{f}\pi, n_t^+}. \end{aligned} \tag{57}$$

The following theorem characterizes the feedback-Nash equilibrium strategy associated with pLQDGET.

Theorem 20 *Consider the N-person pLQDGET with assumption 19 holding true. Then, $\{x_{n_t},\ n_t \in n_0^{++}\}$ constitutes a feedback Nash equilibrium state trajectory, if the following set of backward recursive equations are satisfied:*

$$P^i_{n_t} = Q^i_{n_t} + \big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)' P^i_{\pi,n_t^+} \big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big) + \sum_{l\in\mathbb{N}} E^{l\,\prime}_{n_t} R^{il}_{n_t} E^l_t, \quad (58)$$

$$\beta^i_{\mathtt{f}\,n_t} = p^i_{n_t} + L^i_{n_t}\mathbf{v}^*_{n_t} + \sum_{l\in\mathbb{N}} E^{l\,\prime}_{n_t} R^{il}_{n_t} F^l_{n_t} + \big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)' \big(\beta^i_{\mathtt{f}\,\pi,n_t^+} + P^i_{\pi,n_t^+}\sum_{l\in\mathbb{N}} B^l_{n_t}F^l_{n_t}\big), \quad (59)$$

$$\eta^i_{n_t} = \eta^i_{\pi,n_t^+} + \frac{1}{2}\sum_{l\in\mathbb{N}} F^{l\,\prime}_{n_t} R^{il}_{n_t} F^l_{n_t} + \frac{1}{2}\Big(\sum_{l\in\mathbb{N}} B^l_{n_t}F^l_{n_t}\Big)' P^i_{\pi,n_t^+}\Big(\sum_{l\in\mathbb{N}} B^l_{n_t}F^l_{n_t}\Big) + \Big(\sum_{l\in\mathbb{N}} B^l_{n_t}F^l_{n_t}\Big)'\beta^i_{\mathtt{f}\,\pi,n_t^+}, \quad (60)$$

with terminal conditions given by

$$P^i_{n_T} = Q^i_{n_T},\ \beta^i_{\mathtt{f}\,n_T} = p^i_{n_T} + L^i_{n_T}\mathbf{v}^*_{n_T},\ \text{and } m^i_{n_T} = 0.$$

Further, if $(R^{ii}_{n_t} + B^{i\,\prime}_{n_t} P^i_{\pi,n_t^+} B^i_{n_t})$ is positive definite, then the equilibrium control $u^{i*}_{n_t}$ is unique and is given by

$$u^{i\ *}_{n_t} = \xi^{i\ *}_{n_t}(x_{n_t}; \{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha),\ \alpha \in n_t^{++}\}) = E^i_{n_t}x_{n_t} + F^i_{n_t},$$

where the matrices $E^i_{n_t}$ and $F^i_{n_t}$ are obtained by solving (56) *and* (57) *for $i \in \mathbb{N},\ t \in \mathbb{T}\backslash\{T\}$.*

PROOF. To prove the result, it is sufficient to apply the verification result (48) from Theorem 17. The positive definiteness of the matrices guarantee a unique solution to the dynamic programming equation (48). The parametric feedback-Nash equilibrium is obtained by solving (56) and (57).

Next, we derive the relation between parametric feedback Nash equilibrium strategy and the parameters $\{(\mathbf{w}_\alpha, \boldsymbol{\theta}_\alpha), \alpha \in n_0^{++}\}$. We represent $\mathbf{R}_{n_t} := \oplus_{i=1}^N R^{ii}_{n_t}$, $\mathbf{P}_{n_t} := \mathrm{col}(P^i_{n_t})_{i=1}^N$, $\boldsymbol{\beta}_{\mathtt{f}\,n_t} : \mathrm{col}(\beta^i_{\mathtt{f}\,n_t})_{i=1}^N$, $\bar{\mathbf{B}}_{n_t} := \oplus_{i=1}^N B^i_{n_t}$, $\mathbf{B}_{n_t} := \mathrm{row}(B^i_{n_t})_{i=1}^N$, $\mathbf{E}_{n_t} := \mathrm{col}(E^i_{n_t})_{i=1}^N$ and $\mathbf{F}_{n_t} := \mathrm{col}(F^i_{n_t})_{i=1}^N$. Then, (56) and (57) can be written in vector form as follows:

$$\big(\mathbf{R}_{n_t} + \bar{\mathbf{B}}'_{n_t}\mathbf{P}_{\pi,n_t^+}\mathbf{B}_{n_t}\big)\mathbf{E}_{n_t} = -\bar{\mathbf{B}}'_{n_t}\mathbf{P}_{\pi,n_t^+}A_{n_t}, \quad (61)$$

$$\big(\mathbf{R}_{n_t} + \bar{\mathbf{B}}'_{n_t}\mathbf{P}_{\pi,n_t^+}\mathbf{B}_{n_t}\big)\mathbf{F}_{n_t} = -\bar{\mathbf{B}}'_{n_t}\boldsymbol{\beta}_{\mathtt{f}\,\pi,n_t^+}. \quad (62)$$

In order to solve the above equations, we make the following assumption:

Assumption 21 *The matrix $\big(\mathbf{R}_{n_t} + \bar{\mathbf{B}}'_{n_t}\mathbf{P}_{\pi,n_t^+}\mathbf{B}_{n_t}\big)$ is invertible for $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}$.*

Let $\boldsymbol{\Gamma}_{n_t} = -\big(\mathbf{R}_{n_t} + \bar{\mathbf{B}}'_{n_t}\mathbf{P}_{\pi,n_t^+}\mathbf{B}_{n_t}\big)^{-1}$ for $n_t \in \mathbf{n}_t,\ t \in \mathbb{T}\backslash\{T\}$, then equation (59) can be represented in vector form as follows:

$$\begin{bmatrix}\beta^1_{\mathtt{f}\,n_t}\\ \beta^2_{\mathtt{f}\,n_t}\\ \vdots\\ \beta^N_{\mathtt{f}\,n_t}\end{bmatrix} = \begin{bmatrix}p^1_{n_t} + L^1_{n_t}\mathbf{w}_{n_t} - M^{1\,\prime}_{n_t}\theta^1_{n_t}\\ p^2_{n_t} + L^2_{n_t}\mathbf{w}_{n_t} - M^{2\,\prime}_{n_t}\theta^2_{n_t}\\ \vdots\\ p^N_{n_t} + L^N_{n_t}\mathbf{w}_{n_t} - M^{N\prime}_{n_t}\theta^N_{n_t}\end{bmatrix} + \begin{bmatrix}E^{1\prime}_{n_t}R^{11}_{n_t} & E^{2\prime}_{n_t}R^{12}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{1N}_{n_t}\\ E^{1\prime}_{n_t}R^{21}_{n_t} & E^{2\prime}_{n_t}R^{22}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{2N}_{n_t}\\ \vdots & \vdots & \ddots & \vdots\\ E^{1\prime}_{n_t}R^{N1}_{n_t} & E^{2\prime}_{n_t}R^{N2}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{NN}_{n_t}\end{bmatrix}\begin{bmatrix}F^1_{n_t}\\ F^2_{n_t}\\ \vdots\\ F^N_{n_t}\end{bmatrix}$$
$$+ \begin{bmatrix}\big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)'P^1_{\pi,n_t^+}\\ \big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)'P^2_{\pi,n_t^+}\\ \vdots\\ \big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)'P^N_{\pi,n_t^+}\end{bmatrix}\begin{bmatrix}B^{1\,\prime}_{n_t}\\ B^{2\,\prime}_{n_t}\\ \vdots\\ B^{N\prime}_{n_t}\end{bmatrix}'\begin{bmatrix}F^1_{n_t}\\ F^2_{n_t}\\ \vdots\\ F^N_{n_t}\end{bmatrix} + \Big(\mathbf{I}\otimes\big(A_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)'\Big)\begin{bmatrix}\beta^1_{\mathtt{f}\,\pi,n_t^+}\\ \beta^2_{\mathtt{f}\,\pi,n_t^+}\\ \vdots\\ \beta^N_{\mathtt{f}\,\pi,n_t^+}\end{bmatrix},$$

$$\boldsymbol{\beta}_{\mathtt{f}\,n_t} = \mathbf{p}_{n_t} + \big[\mathbf{L}_{n_t} \quad -\mathbf{M}'_{n_t}\big]\big[\mathbf{w}'_{n_t} \quad \boldsymbol{\theta}'_{n_t}\big]' + \mathbf{G}_{\mathtt{f}\,n_t^+}^{\ n_t}\boldsymbol{\beta}_{\mathtt{f}\,\pi,n_t^+}, \quad (63)$$

where

$$\mathbf{G_f}^{n_t}_{n_t^+} := \Big(\mathbf{I}\otimes \big(A_{n_t}+\sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)'\Big) - \begin{bmatrix} E^{1\prime}_{n_t}R^{11}_{n_t} & E^{2\prime}_{n_t}R^{12}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{1N}_{n_t} \\ E^{1\prime}_{n_t}R^{21}_{n_t} & E^{2\prime}_{n_t}R^{22}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{2N}_{n_t} \\ \vdots & \vdots & \ddots & \vdots \\ E^{1\prime}_{n_t}R^{N1}_{n_t} & E^{2\prime}_{n_t}R^{N2}_{n_t} & \cdots & E^{N\prime}_{n_t}R^{NN}_{n_t} \end{bmatrix} \mathbf{\Gamma}_{n_t}\bar{\mathbf{B}}'_{n_t} - \begin{bmatrix} \big(A_{n_t}+\sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)' P^1_{\pi,n_t^+} \\ \big(A_{n_t}+\sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)' P^2_{\pi,n_t^+} \\ \vdots \\ \big(A_{n_t}+\sum_{l\in\mathbb{N}} B^l_{n_t}E^l_{n_t}\big)' P^N_{\pi,n_t^+} \end{bmatrix} \mathbf{B}_{n_t}\mathbf{\Gamma}_{n_t}\bar{\mathbf{B}}'_{n_t},$$

and the terminal conditions at the leaf nodes are given by

$$\boldsymbol{\beta}_{\mathbf{f}\,n_T} = \mathbf{p}_{n_T} + \begin{bmatrix}\mathbf{L}_{n_T} & -\mathbf{M}'_{n_T}\end{bmatrix}\begin{bmatrix}\mathbf{w}'_{n_T} & \boldsymbol{\theta}'_{n_T}\end{bmatrix}'. \qquad (64)$$

Like before, enumerating the nodes at time t as $\mathbf{n}_t := \{n_t^1, n_t^2, \cdots, n_t^{|\mathbf{n}_t|}\}$ we obtain the following backward recursion relating the set of all nodes at time $t+1$ with the set of all nodes at time t

$$\boldsymbol{\beta}_{\mathbf{f}\,\mathbf{n}_t} = \mathbf{p}_{\mathbf{n}_t} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_t} & -\mathbf{M}'_{\mathbf{n}_t}\end{bmatrix}\begin{bmatrix}\mathbf{w}'_{\mathbf{n}_t} & \boldsymbol{\theta}'_{\mathbf{n}_t}\end{bmatrix}' + \mathbf{G_f}^{t}_{t+1}\boldsymbol{\beta}_{\mathbf{f}\,\mathbf{n}_{t+1}}$$

where $\boldsymbol{\beta}_{\mathbf{f}\,\mathbf{n}_t} := \mathrm{col}(\boldsymbol{\beta}_{\mathbf{f}\,n_t^i})_{i=1}^{|n_t|}$, $\mathbf{w}_{\mathbf{n}_t} := \mathrm{col}(\mathbf{w}_{n_t^i})_{i=1}^{|n_t|}$, $\boldsymbol{\theta}_{\mathbf{n}_t} := \mathrm{col}(\boldsymbol{\theta}_{n_t^i})_{i=1}^{|n_t|}$ and $\mathbf{G_f}^{t}_{t+1} := \oplus_{i=1}^{|\mathbf{n}_t|}(\pi^{n_t^{i+}}_{n_t^i} \otimes \mathbf{G_f}^{n_t^i}_{n_t^{i+}})$. Here, (63) represents a linear difference equation evolving backwards on the event tree, with terminal condition (64) defined at the leaf nodes. We define the transition matrix associated with the linear backward difference equation (63) as $\psi_{\mathbf{f}}(t,\tau) = \mathbf{G_f}^{t}_{t+1}\cdots\mathbf{G_f}^{\tau-2}_{\tau-1}\mathbf{G_f}^{\tau-1}_{\tau}$ when $\tau > t$ and $\psi_{\mathbf{f}}(t,\tau) = \mathbf{I}$ when $\tau = t$, and (63) is given by:

$$\boldsymbol{\beta}_{\mathbf{f}\,n_t} = \sum_{\tau=t}^{T}\psi_{\mathbf{f}}(t,\tau)\big(\mathbf{p}_{\mathbf{n}_\tau} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_\tau} & -\mathbf{M}'_{\mathbf{n}_\tau}\end{bmatrix}\begin{bmatrix}\mathbf{w}'_{\mathbf{n}_\tau} & \boldsymbol{\theta}'_{\mathbf{n}_\tau}\end{bmatrix}'\big)$$

The state variable evolves according to the following forward difference equation:

$$\begin{aligned} x_\nu &= A_{n_t}x_{n_t} + \sum_{l\in\mathbb{N}} B^l_{n_t}u^l_{n_t} \\ &= (A_{n_t} + \mathbf{B}_{n_t}\mathbf{E}_{n_t})x_{n_t} - \mathbf{B}_{n_t}\mathbf{\Gamma}_{n_t}\bar{\mathbf{B}}'_{n_t}\boldsymbol{\beta}_{\mathbf{f}\,\pi,n_t^+}, \end{aligned}$$

that is,

$$\begin{bmatrix}x'_{\nu_1} & x'_{\nu_2} & \cdots & x'_{\nu_{|n_t^+|}}\end{bmatrix}' = \Big(\mathbf{1}\otimes(A_{n_t}+\mathbf{B}_{n_t}\mathbf{E}_{n_t})\Big)x_{n_t} + \Big(-\boldsymbol{\pi}^{n_t^+}_{n_t}\otimes\big(\mathbf{B}_{n_t}\mathbf{\Gamma}_{n_t}\bar{\mathbf{B}}'_{n_t}\big)\Big)\begin{bmatrix}\boldsymbol{\beta}'_{\mathbf{f}\,\nu_1} & \boldsymbol{\beta}'_{\mathbf{f}\,\nu_2} & \cdots & \boldsymbol{\beta}'_{\mathbf{f}\,\nu_{|n_t^+|}}\end{bmatrix}'$$

As before, we collect the state vectors $x_{\mathbf{n}_t} := \mathrm{col}(x_{n_t^i})_{i=1}^{|n_t|}$ defined at all the nodes at time t, and write the above equation as:

$$x_{\mathbf{n}_{t+1}} = \bar{A}^{\mathbf{f}}_t x_{\mathbf{n}_t} + \bar{B}^{\mathbf{f}}_{t+1}\boldsymbol{\beta}_{\mathbf{f}\,\mathbf{n}_{t+1}},$$

where $\bar{A}^{\mathbf{f}}_t = \oplus_{i=1}^{|\mathbf{n}_t|}\Big(\mathbf{1}\otimes(A_{n_t}+\mathbf{B}_{n_t}\mathbf{E}_{n_t})\Big)$ and $\bar{B}^{\mathbf{f}}_{t+1} = \oplus_{i=1}^{|\mathbf{n}_t|}\Big(-\boldsymbol{\pi}^{n_t^+}_{n_t}\otimes\big(\mathbf{B}_{n_t}\mathbf{\Gamma}_{n_t}\bar{\mathbf{B}}'_{n_t}\big)\Big)$. Denoting the associated state transition matrix as $\phi_{\mathbf{f}}(\rho,t) = \bar{A}^{\mathbf{f}}_{t-1}\bar{A}^{\mathbf{f}}_{t-2}\cdots\bar{A}^{\mathbf{f}}_{\rho}$ for $\rho < t$ and $\phi_{\mathbf{f}}(t,\rho) = \mathbf{I}$ when $\rho = t$, then the solution of the above difference equation is given by

$$x_{\mathbf{n}_t} = \phi_{\mathbf{f}}(0,t)x_0 + \sum_{\rho=0}^{t-1}\phi_{\mathbf{f}}(\rho+1,t)\bar{B}^{\mathbf{f}}_{\rho+1}\boldsymbol{\beta}_{\mathbf{f}\,\mathbf{n}_{\rho+1}},\ t\in\mathbb{T}\backslash\{0\}. \qquad (65)$$

Using (63) in (65), we have for $t\in\mathbb{T}\backslash\{0\}$

$$x_{\mathbf{n}_t} = \phi_{\mathbf{f}}(0,t)x_0 + \sum_{\tau=1}^{T}\Big(\Big(\sum_{\rho=1}^{\min(t,\tau)}\phi_{\mathbf{f}}(\rho,t)\bar{B}^{\mathbf{f}}_{\rho}\psi_{\mathbf{f}}(\rho,\tau)\Big) \big(\mathbf{p}_{\mathbf{n}_\tau} + \begin{bmatrix}\mathbf{L}_{\mathbf{n}_\tau} & -\mathbf{M}'_{\mathbf{n}_\tau}\end{bmatrix}\begin{bmatrix}\mathbf{w}'_{\mathbf{n}_\tau} & \boldsymbol{\theta}'_{\mathbf{n}_\tau}\end{bmatrix}'\big). \qquad (66)$$

Next, we aggregate the variables in (66) as $x_{\mathbf{T}} := \mathrm{col}(x_{\mathbf{n}_t})_{t=1}^{T}$, $\mathbf{w}_{\mathbf{T}} := \mathrm{col}(\mathbf{w}_{\mathbf{n}_t})_{t=1}^{T}$, and $\boldsymbol{\theta}_{\mathbf{T}} := \mathrm{col}(\boldsymbol{\theta}_{\mathbf{n}_t})_{t=1}^{T}$ and represent the above equations for all $t\in\mathbb{T}\backslash\{0\}$ compactly as follows:

$$x_{\mathbf{T}} = \mathbf{\Phi}^{\mathbf{f}}_0 x_0 + \mathbf{\Phi}^{\mathbf{f}}_1\mathbf{p}_{\mathbf{T}} + \mathbf{\Phi}^{\mathbf{f}}_2\mathbf{w}_{\mathbf{T}} + \mathbf{\Phi}^{\mathbf{f}}_3\boldsymbol{\theta}_{\mathbf{T}}, \qquad (67)$$

where $\big[\mathbf{\Phi}^{\mathbf{f}}_0\big]_t = \phi_{\mathbf{f}}(0,t)$, $\big[\mathbf{\Phi}^{\mathbf{f}}_1\big]_{t\tau} = \sum_{\rho=1}^{\min(t,\tau)}\phi_{\mathbf{f}}(\rho,t)\bar{B}^{\mathbf{f}}_{\rho}\psi_{\mathbf{f}}(\rho,\tau)$ and $\big[\mathbf{\Phi}^{\mathbf{f}}_2\big]_{t\tau} = \sum_{\rho=1}^{\min(t,\tau)}\phi_{\mathbf{f}}(\rho,t)\bar{B}^{\mathbf{f}}_{\rho}\psi_{\mathbf{f}}(\rho,\tau)\mathbf{L}_{\mathbf{n}_\tau}$ $\big[\mathbf{\Phi}^{\mathbf{f}}_3\big]_{t\tau} = -\sum_{\rho=1}^{\min(t,\tau)}\phi_{\mathbf{f}}(\rho,t)\bar{B}^{\mathbf{f}}_{\rho}\psi_{\mathbf{f}}(\rho,\tau)\mathbf{M}'_{\mathbf{n}_\tau}$ for $t,\tau\in\mathbb{T}\backslash\{0\}$.

Here, (67) represents the equilibrium state trajectory associated with pLQDGET, and is an affine function of the parameters $\{(\mathbf{w}_\alpha,\boldsymbol{\theta}_\alpha),\ \alpha\in n_0^{++}\backslash n_0\}$.

Recall from Theorem 18 that the feedback-Nash equilibrium associated with LQDGET is obtained by setting the parameters $\{(\mathbf{w}_\alpha,\boldsymbol{\theta}_\alpha),\ \alpha\in n_0^{++}\}$ as solutions of $\{(\mathbf{v}^*_{n_t},\boldsymbol{\mu}^*_{n_t}) := \mathrm{sol}(\mathrm{pLCP}(x_{n_t})), n_t\in n_0^{++}\}$ where $\{x_{n_t}, n_t\in n_0^{++}\}$ represents the equilibrium state trajectory given by (67). The vector representation of the parametric linear complementarity problems is given by

(32) where $x_{\mathbf{T}}$ is given by (see also (31)):

$$x_{\mathbf{T}} = \mathbf{\Phi}_0^{\mathbf{f}} x_0 + \mathbf{\Phi}_1^{\mathbf{f}} \mathbf{p}_{\mathbf{T}} + \mathbf{\Phi}_2^{\mathbf{f}} \mathbf{v}_{\mathbf{T}}^* + \mathbf{\Phi}_3^{\mathbf{f}} \boldsymbol{\mu}_{\mathbf{T}}^*. \qquad (68)$$

Notice, that (68) and (32) represent a fixed-point problem as intersection of two parametric maps. Substituting for $x_{\mathbf{T}}$ in (32) using (68) the associated fixed points are obtained as solutions of the following single large scale linear complementarity problem

$$\text{LCP}_{\mathbf{f}} : 0 \leq \mathbf{M}^{\mathbf{f}} \begin{bmatrix} \mathbf{v}_{\mathbf{T}}^* \\ \boldsymbol{\mu}_{\mathbf{T}}^* \end{bmatrix} + \mathbf{q}^{\mathbf{f}} \perp \begin{bmatrix} \mathbf{v}_{\mathbf{T}}^* \\ \boldsymbol{\mu}_{\mathbf{T}}^* \end{bmatrix} \geq 0, \qquad (69)$$

where $\mathbf{M}^{\mathbf{f}} = \begin{bmatrix} \mathbf{D}_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}} \mathbf{\Phi}_2^{\mathbf{f}} & -\mathbf{N}'_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}} \mathbf{\Phi}_3^{\mathbf{f}} \\ \mathbf{N}_{\mathbf{T}} + \bar{\mathbf{M}}_{\mathbf{T}} \mathbf{\Phi}_2^{\mathbf{f}} & \bar{\mathbf{M}}_{\mathbf{T}} \mathbf{\Phi}_3^{\mathbf{f}} \end{bmatrix}$ and $\mathbf{q}^{\mathbf{f}} = \begin{bmatrix} \mathbf{d}_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}} \mathbf{\Phi}_1^{\mathbf{f}} \mathbf{p}_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}} \mathbf{\Phi}_0^{\mathbf{f}} x_0 \\ \mathbf{r}_{\mathbf{T}} + \bar{\mathbf{M}}_{\mathbf{T}} \mathbf{\Phi}_1^{\mathbf{f}} \mathbf{p}_{\mathbf{T}} + \bar{\mathbf{L}}'_{\mathbf{T}} \mathbf{\Phi}_0^{\mathbf{f}} x_0 \end{bmatrix}$.

The main result of this section is provided in the following theorem.

Theorem 22 *Let assumptions 2, 15 and 21 hold true. Let $(\mathbf{v}_0^*, \boldsymbol{\mu}_0^*)$ and $(\mathbf{v}_{\mathbf{T}}^*, \boldsymbol{\mu}_{\mathbf{T}}^*)$ represent the solutions of* pLCP(x_{n_0}) *and* LCP$_{\mathbf{f}}$ *respectively. Define for every player $i \in \mathbb{N}$, the strategies $\tilde{\psi}_i^* := \left\{ \{ {\gamma_{n_t}^i}^*(x_{n_t}), n_t \in \mathbf{n}_t,\ t \in \mathbb{T} \backslash \{T\} \}, \{ {v_{n_T}^i}^*,\ n_T \in \mathbf{n}_T \} \right\}$, with ${\gamma_{n_t}^i}^* = E_{n_t}^i x_{n_t} + F_{n_t}^i$, where $E_{n_t}^i$ and $F_{n_t}^i$ are given by* (56) *and* (57)*, and the state trajectory $\{x_{n_t},\ n_t \in n_0^{++}\}$ is given by* (68)*. Then, $\{\tilde{\psi}_1^*, \tilde{\psi}_2^*, \cdots, \tilde{\psi}_N^*\}$ provides a feedback Nash equilibrium strategy for LQDGET.*

PROOF. From assumption 2, pLCP(x_{n_0}) and LCP$_{\mathbf{f}}$ are feasible, and from assumption 15 SOL(pLCP$(x_{n_0})) \neq \emptyset$. Let $(\mathbf{v}_{\mathbf{T}}^*, \boldsymbol{\mu}_{\mathbf{T}}^*)$ be a solution of LCP$_{\mathbf{f}}$. Again, from assumption 15 we have that $(\mathbf{v}_{\mathbf{T}}^*, \boldsymbol{\mu}_{\mathbf{T}}^*)$ is the unique solution of pLCP$(x_{\mathbf{T}})$. Here, $x_{\mathbf{T}}$ given by (68) represents the equilibrium state trajectory associated with pLQDGET where the parameters are set to $(\mathbf{v}_{\mathbf{T}}^*, \boldsymbol{\mu}_{\mathbf{T}}^*)$. The remaining part of the proof follows immediately from Theorems 17 and 18.

5 Computation of S-adapted Nash equilibrium

In this section, we provide some geometric insights related to the computation of S-adapted Nash equilibrium. Notice, as it is evident from (33) and (69) that both the open-loop and feedback S-adapted Nash equilibria are obtained as solutions of a single large-scale linear complementarity problems. In the open-loop (respectively, feedback) case, (33) ((69)) is obtained by substituting for the state variables in (32) using (31) (respectively (68)). So, the S-adapted Nash equilibria with both information structures are obtained as intersection points of two parametric maps. The first mapping is given by $x_{\mathbf{T}}(.,.) : \mathbb{R}^{s(|n_0^{++}|-1)} \times \mathbb{R}^{c(|n_0^{++}|-1)} \to \mathbb{R}^{n(|n_0^{++}|-1)}$, from (31) and (68) for a given x_{n_0}. The second mapping is given by $\texttt{sol}(\text{pLCP}(.)) : \mathbb{R}^{n(|n_0^{++}|-1)} \to \mathbb{R}^{s(|n_0^{++}|-1)} \times \mathbb{R}^{c(|n_0^{++}|-1)}$, from (32). Clearly, this geometric interpretation hints about the number of S-adapted equilibrium solutions in LQDGET. There may not exist any equilibrium solution, and there could be one, more than one or a continuum of solutions.

6 Numerical illustration

To illustrate our results in the most simplest way, we consider a duopoly selling a homogeneous good over a finite horizon. Let $Q_{n_t}^i$ be the quantity produced by firm i at node n_t, and let the price be given by the following affine stochastic inverse demand function:

$$P\left(Q_{n_t}^1, Q_{n_t}^2\right) = A_{n_t} - B_{n_t}\left(Q_{n_t}^1 + Q_{n_t}^2\right),\ n_t \in n_0^{++}.$$

Here, we assume fixed slope ($B_{n_t} = 10$) and intercept parameter A_{n_t} varies as the game evolves over the event tree. In each node of the event tree, A_{n_t} goes up (high) or down (low) by 30% with equal probability and $A_{n_0} = 25$.

Firms invest in R&D to decrease their unit-production cost, which depends on accumulated knowledge. Let $R_{n_t}^i$ be the investment in R&D, and denote by $X_{n_t}^i$ the stock of knowledge of firm i, whose evolution is described by the difference equation

$$X_\nu^i = \mu_i X_{n_t}^i + R_{n_t}^i + \lambda_i R_{n_t}^j,\ j \neq i,\ i, j \in \{1, 2\},\ \nu \in n_t^+$$

where μ_i is a positive parameter, and $\lambda_i \in (0, 1)$ is the spillover parameter. This means that each firm cannot fully appropriate its investment in R&D, as part of it spills over to its rival. Let $K_{n_t}^i$ be the production capacity, and denote by $I_{n_t}^i$ the investment in this capacity. The evolution of this capacity is given by

$$K_\nu^i = \delta_i K_{n_t}^i + I_{n_t}^i,\ i \in \{1, 2\}$$

where $(1 - \delta_i)$ is the depreciation rate. The quantity produced is subject to available capacity, i.e.,

$$0 \leq Q_{n_t}^i \leq K_{n_t}^i,\ i \in \{1, 2\}.$$

The production cost is given by $h_i(X_{n_t}^i, Q_{n_t}^i) = (c^i - \gamma^i X_{n_t}^i) Q_{n_t}^i$. The investment costs in R&D and production capacity are given by $g^i(R_{n_t}^i) = (\frac{a^i}{2})(R_{n_t}^i)^2$ and $f^i(I_{n_t}^i) = (\frac{b^i}{2})(I_{n_t}^i)^2$ respectively. Here, c^i is the initial cost, γ^i is the positive cost learning parameter, and a^i and b^i are positive parameters.

The salvage value at a terminal node n_T is given by $S_i(X^i_{n_T}, K^i_{n_T}) = \omega_i(X^i_{n_T})^2 + \theta_i(K^i_{n_T})^2$, where ω_i and θ_i are positive parameters. Assuming cost-minimization behavior, player i's objective is given by

$$\Pi_i = \sum_{n_T \in \mathbf{n}_T} \pi_{n_T} \beta^T S_i\left(X^i_{n_T}, K^i_{n_T}\right) + \sum_{t=0}^{T-1} \sum_{n_t \in \mathbf{n}_t} \beta^t \left(g^i\left(R^i_{n_t}\right) + f^i\left(I^i_{n_t}\right)\right) + \sum_{t=0}^{T} \sum_{n_t \in \mathbf{n}_t} \beta^t \left(h_i\left(X^i_{n_t}, Q^i_{n_t}\right) - P\left(Q^1_{n_t}, Q^2_{n_t}\right) Q^i_{n_t}\right).$$

Clearly, this game belongs to the LQGET class. Indeed, we have three control variables, namely, production, investment in capacity and investment in R&D. The two investment variables enter the dynamics, that is, the state equations of capacity and knowledge accumulation, and are not constrained. The output decision is constrained at each period by the available production capacity and does not play any role in the dynamics. To attribute any difference in the results to only the information structure, we assume that the game is symmetric and adopt the following parameter values:

$$\begin{aligned}
\lambda^i &= 0.25,\ \mu^i = 0.8,\ \delta^i = 0.8,\ \gamma^i = 0.2,\ c^i = 0.5 \\
a^i &= 1,\ b^i = 0.75,\ \omega^i = -0.1,\ \theta^i = -0.25,\ i = 1,2 \\
T &= 9,\ A_{n_0} = 25,\ B_{n_0} = 10, \\
\beta &= 0.9,\ X^i_{n_0} = 3,\ K^i_{n_0} = 3,\ i = 1,2.
\end{aligned}$$

Table 1 illustrates the comparison of open-loop and constrained feedback-Nash equilibrium strategies for two realizations of the uncertainty. In both scenarios, we observe that the investments in the knowledge stock and the production capacity are consistently higher when feedback strategies are used, which is somewhat expected, as feedback intensifies competition with respect to precommitment (open-loop). In the sample path - 1, the demand in the periods $t = 1, 2, 3, 4$ is high, hence the players produce more, and this explains why the production constraints are binding during the periods $t = 2, 3, 4$. The demand in the subsequent periods $t = 5, 6, 7, 8, 9$ is low and as a result the constraints are not binding. Notice also that investments in the knowledge stock and production capacity increase during the periods $t = 1, 2, 3, 4$ due to high demand and decrease in subsequent periods.

In sample path -2, the demand is high for the periods $t = 1, 2$ and as a result production constraints are binding during the period $t = 2$. The demand is low during period $t = 3$, so production constraints are not binding during this period. The demand in the periods $t = 4, 5, 6$ is high hence the production constraints are binding during the periods $t = 5, 6$. The decrease in investment behavior follows the demand trend, i.e., investments decrease during the periods $t = 3$ and $t = 7$. We have used the freely available software, the PATH solver (see http://pages.cs.wisc.edu/ferris/path/), for solving linear-complementarity problems.

7 Concluding remarks

In this paper, we extended the results in [1,2] obtained for deterministic games to dynamic games played over event trees. The characterization of both open-loop and feedback-Nash equilibria, as well as the proposed computational approaches, rely heavily on the fact that the control variables that enter the dynamics are not constrained, while those part of joint state-control constraints do not enter the dynamics. In some applications, e.g., electricity markets, the investment in production capacity is irreversible, that is, it cannot assume negative values. Extending this work to the case where the control variables entering the dynamics are constrained is methodologically interesting and empirically relevant. Finally, allowing for coupling constraints, that is, making the players' action sets interdependent, would provide the tools to deal with many applications in environmental economics and power markets, see, e.g., Schiro et al. [22].

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Table 1
Comparison of open-Loop and feedback Nash equilibrium strategies for two sample paths. 'H' refers to an increase in demand by 30% and 'L' refers to a decrease in demand by 30%. The bold faced items in the table illustrate periods where production constraints are binding along the sample paths.

Period (t)	0	1	2	3	4	5	6	7	8	9
Sample path - 1 (demand)		H	H	H	H	L	L	L	L	L
X_{o}^{i}	3.0000	3.7571	4.7633	6.0711	7.7441	9.8296	10.4319	10.2194	9.6575	9.0658
K_{o}^{i}	3.0000	2.7093	**2.8526**	**3.6749**	**4.7987**	6.2717	5.9680	5.8856	6.2518	7.1449
Q_{o}^{i}	1.6733	2.1890	**2.8526**	**3.6749**	**4.7987**	3.5207	2.5609	1.8843	1.4087	1.0787
R_{o}^{i}	1.0857	1.4061	1.8084	2.3097	2.9075	2.0546	1.4991	1.1856	1.0718	0.0000
I_{o}^{i}	0.3093	0.6852	1.3928	1.8588	2.4327	0.9507	1.1112	1.5433	2.1435	0.0000
X_{f}^{i}	3.0000	3.7607	4.7724	6.0884	7.7736	9.8765	10.4892	10.2816	9.7198	9.1224
K_{f}^{i}	3.0000	2.7093	**2.8528**	**3.6752**	**4.7993**	6.2727	5.9691	5.8867	6.2529	7.1462
Q_{f}^{i}	1.6733	2.1890	**2.8528**	**3.6752**	**4.7993**	3.5218	2.5623	1.8860	1.4106	1.0807
R_{f}^{i}	1.0886	1.4111	1.8164	2.3223	2.9261	2.0704	1.5122	1.1956	1.0773	0.0000
I_{f}^{i}	0.3093	0.6853	1.3930	1.8591	2.4333	0.9509	1.1114	1.5436	2.1439	0.0000
Sample path - 2 (demand)		H	H	L	H	H	H	L	H	H
X_{o}^{i}	3.0000	3.7571	4.7633	6.0711	6.4600	7.1785	8.2054	9.4745	9.6320	9.7584
K_{o}^{i}	3.0000	2.7093	**2.8526**	3.6749	3.3381	**3.4606**	**4.4414**	5.7909	6.1512	7.0299
Q_{o}^{i}	1.6733	2.1890	**2.8526**	2.0494	2.6611	**3.4606**	**4.4414**	3.2630	4.2069	5.4270
R_{o}^{i}	1.0857	1.4061	1.8084	1.2825	1.6084	1.9701	2.3282	1.6419	1.6423	0.0000
I_{o}^{i}	0.3093	0.6852	1.3928	0.3982	0.7902	1.6729	2.2378	1.5185	2.1090	0.0000
X_{f}^{i}	3.0000	3.7607	4.7724	6.0884	6.4830	7.2103	8.2488	9.5301	9.6904	9.8115
K_{f}^{i}	3.0000	2.7093	**2.8528**	3.6752	3.3385	**3.4613**	**4.4424**	5.7924	6.1528	7.0317
Q_{f}^{i}	1.6733	2.1890	**2.8528**	2.0497	2.6616	**3.4613**	**4.4424**	3.2646	4.2087	5.4288
R_{f}^{i}	1.0886	1.4111	1.8164	1.2898	1.6191	1.9845	2.3448	1.6531	1.6474	0.0000
I_{f}^{i}	0.3093	0.6853	1.3930	0.3983	0.7905	1.6734	2.2385	1.5189	2.1095	0.0000

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