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On the P_2' and P_2-properties in the semidefinite linear complementarity problem

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Motivated by the so-called P_2-property in the semidefinite linear complementarity problems, in this article, we introduce the concept of P_2'-property for a linear transformation on the space of real $n \times n$ symmetric matrices. While these two properties turn out to be different, we show that they are equivalent for the Lyapunov transformation L_A, double-sided multiplicative transformation M_A and a particular class of Stein transformations. We also show that P_2' implies the *SSM* and Q-properties.

1. Introduction

Given a linear transformation $L : S^n \to S^n$ and a matrix $Q \in S^n$, the semidefinite linear complementarity problem, *SDLCP*(L, Q), is the problem of finding a matrix $X \in S^n$ such that

$$X \in S^n_+, \quad Y := L(X) + Q \in S^n_+, \quad \text{and} \quad \langle X, Y\rangle = 0,$$

where S^n denotes the space of all real $n \times n$ symmetric matrices, S^n_+ denotes the set of all positive semidefinite matrices in S^n and $\langle X, Y\rangle$ denotes the trace of the (matrix) product XY. If such an X exists, then we call X to be the solution of *SDLCP*(L, Q). In the last decade significant work done in the area of semidefinite linear complementarity problem starting from Kojima et al. [16] in 1997 and Gowda

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with others, see the articles [9,8,12,13,7]. *SDLCP* is also a special case of semidefinite programming, see Shida et al. [19]. Then in 2000, Gowda and Song [9] regarded it as a natural generalization of linear complementarity problem (*LCP*) [5]. The problem *SDLCP* has been studied extensively for the following transformations. Given $A \in \mathbb{R}^{n\times n}$,

(i) The *Lyapunov transformation* $L_A : X \in S^n \longmapsto AX + XA^T$,
(ii) The *Stein transformation* $S_A : X \in S^n \longmapsto X - AXA^T$,
(iii) The *Multiplicative transformation* $R_A : X \in S^n \longmapsto AXA^T$.

In this article we introduce the P_2'-property for a linear transformation. Let $L : S^n \to S^n$ be a linear transformation. We say L has the P_2'-property, if

$$[0 \preceq X \in S^n, XL(X)X \preceq 0] \Rightarrow X = 0.$$

P_2'-property was motivated by the strict semimonotone property in the linear complementarity problems. In *LCP* if a matrix A has the strict semimonotone property, then it is equivalent to saying that $LCP(A, q)$ has unique solution whenever $q \geqslant 0$. We observe a similar result (Remark 2.3) for the P_2'-property in the *SDLCP* setting. Also we establish the equivalence of the P_2 and P_2'-property for the Lyapunov transformation L_A, the multiplicative transformation M_A and for a particular class of Stein transformations S_A. Further, we study the relationship of P_2'-property, Q-property and P-property.

1.1. Preliminaries

We use S^n_+ (S^n_-) to denote the set of all positive (negative) semidefinite matrices in S^n. The trace of a matrix $A \in \mathbb{R}^{n\times n}$ is the sum of its diagonal entries and it is denoted by $tr(A)$. By $A \succeq 0$ ($A \preceq 0$), we mean A is positive (negative) semidefinite. By the symbol $\|A\|$ we denote the Frobenius norm of A on $\mathbb{R}^{n\times n}$. The following results are well known, see [6,14].

(i) $tr(A) = tr(A^T)$;
(ii) $tr(AB) = tr(BA)$;
(iii) If $A \succeq 0$, then $UAU^T \succeq 0$ for any orthogonal matrix U;
(iv) If A and B are two commuting symmetric matrices, then there exists an orthogonal matrix U such that $A = UDU^T$ and $B = UEU^T$;
(v) If $0 \preceq X$ and $0 \preceq Y$ with $tr(XY) = 0$, then $XY = 0$.

In the following, we recall some definitions in the setting of *SDLCP*.

Definition 1.1. For a linear transformation $L : S^n \to S^n$, we say that L has the

(i) Q-property if $SDLCP(L, Q)$ has a solution for all $Q \in S^n$;
(ii) P-property if

$$[X \text{ and } L(X) \text{ commute, and } XL(X) \preceq 0] \Rightarrow X = 0;$$

(iii) Globally Uniquely Solvable (GUS) property, if for all $Q \in S^n$, $SDLCP(L, Q)$ has a unique solution;
(iv) Strong-monotonicity property if $tr(XL(X)) > 0$ for any nonzero $X \in S^n$;
(v) R_0-property if $SDLCP(L, 0)$ has a unique solution;
(vi) P_2-property (also called ultra P-property in [13]) if

$$[X \succeq 0, Y \succeq 0, (X - Y)L(X - Y)(X + Y) \preceq 0] \Rightarrow X = Y.$$

1.1.1. Lipschitzian property

The set of all solutions to the problem $SDLCP(L, Q)$ is denoted by $SOL(L, Q)$. The multivalued map $\phi_L : S^n \to S^n_+$ defined by $\phi_L(Q) := SOL(L, Q)$ is called the solution map.

Definition 1.2. Let $L : S^n \to S^n$. We say that ϕ_L is Lipschitzian, if there exists $C > 0$ such that

$$\phi_L(Q) \subseteq \phi_L(Q') + C\|Q - Q'\|B \tag{1}$$

for all $Q, Q' \in S^n$ satisfying $\phi_L(Q) \neq \emptyset$ and $\phi_L(Q') \neq \emptyset$. Here B is the closed unit ball in S^n. We say, L is a Lipschitzian map if ϕ_L has the Lipschitzian property.

1.1.2. The linear transformation R_A

Apart from the linear transformations mentioned in the beginning, we have one more linear transformation R_A, for which one can find a detailed discussion in the article by Gowda and Song [11]. For a given $X \in S^n$, let $diag(X)$ denote the n dimensional(column) vector formed by the diagonal entries of X. Also, for a vector $q \in \mathbb{R}^n$, we call by $\hat{q}$ or $Diag(q)$ the diagonal matrix whose diagonal vector is q. And for a given $X \in S^n$, with $X = (x_{ij})$, we say $X_0 := (x_{ij}(1 - \delta_{ij}))$, where δ_{ij} is one if $i = j$ and zero otherwise. The problem $SDLCP(R_A, \hat{q})$ is called the semidefinite relaxation of the problem $LCP(A, q)$. Now, the linear transformation $R_A : S^n \to S^n$ is defined by

$$R_A(X) := Diag(Adiag(X)) + X_0.$$

Now we will recall some results, that are required for this paper.

Theorem 1.1 [15]. *Let $L : S^n \to S^n$ be a linear transformation. If the problem $SDLCP(L, 0)$ and $SDLCP(L, I)$ have unique solutions, then L has the Q-property. Here, I denotes the identity matrix of order n.*

Lemma 1.2 [2]. *Let $A \in \mathbb{R}^{n\times n}$. Then A is positive definite if and only if every diagonal entry of UAU^T is positive, for any orthogonal matrix U.*

2. The P_2'-property

2.1. Strict semimonotone property and the P-property in LCP

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are non-positive. We say A has the strict semimonotone property (SSM-property) if $x \geqslant 0$ and $x_i(Ax)_i \leqslant 0$ for all $i \Rightarrow x = 0$. In other words, restricted to $\mathbb{R}^n_+$, A does not reverse the sign of (nonnegative) vector. If A has SSM-property then $A \in Q$. In fact A is completely Q, that is, A and all its principal submatrices are in Q, for details see [5].

We say A has the P-property, if for $x \in \mathbb{R}^n$ and $x_i(Ax)_i \leqslant 0$ for all $i \Rightarrow x = 0$. In case of a Z-matrix, SSM-property and P-property are equivalent.

Theorem 2.1. *Let A be a Z-matrix. Then the following are equivalent.*

(i) *A has the P-property.*
(ii) *A has the SSM-property.*
(iii) *$x \geqslant 0$ and $x_i(Ax)_i x_i \leqslant 0$ for all $i \Rightarrow x = 0$.*

Proof. The equivalence of (ii) and (iii) are obvious. (i) $\Rightarrow$ (ii) is well known, see the monograph by Cottle et al. [5]. For (ii) $\Rightarrow$ (i), we know SSM-property implies the Q-property and Q-property along with Z-property implies the P-property, refer to Berman and Plemons [3]. $\square$

2.2. Strict semimonotone property and the P-property in SDLCP

In case of the *SDLCP* the strict semimonotone property was first introduced by Gowda and Song in [9]. Given a linear transformation $L : S^n \to S^n$, we say that L has the strict semimonotone property (SSM-property), if

$$[X \succeq 0, X \text{ and } L(X) \text{ commute}, XL(X) \preceq 0] \Rightarrow 0.$$

At this point, we define the P_2'-property as follows.

Definition 2.1. Let $L : S^n \to S^n$ be a linear transformation. Then we say that L has the P_2'-property, if

$$[0 \preceq X \in S^n, XL(X)X \preceq 0] \Rightarrow X = 0.$$

Now we prove the following theorem.

Theorem 2.2. *Let $L : S^n \to S^n$ be a linear transformation. Consider the following statements.*

(i) *L has the P_2-property.*
(ii) *L has the P_2'-property.*
(iii) *L has the SSM-property.*
(iv) *L has the Q-property.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Proof. (i) $\Rightarrow$ (ii) is evident from the definition of P_2'-property. Now we prove (ii) $\Rightarrow$ (iii). Assume L has the P_2'-property. Let $X \succeq 0$ with $XL(X) = L(X)X \preceq 0$. We claim that $X = 0$. Since X and $L(X)$ are symmetric commuting matrices, we have $XL(X)X \preceq 0$. Thus by the P_2'-property we have $X = 0$. Now let us see the proof of (iii) $\Rightarrow$ (iv). Assume L has the SSM-property. Suppose there exists a nonzero $X \succeq 0$ and $L(X) \succeq 0$, such that $XL(X) = 0$. Then, by the SSM-property, such X must be 0. This asserts that $SOL(L, 0) = \{0\}$. Let $Q = I$ and suppose there exists a nonzero $X \succeq 0$ and $L(X) + I \succeq 0$, such that $X(L(X) + I) = (L(X) + I)X = 0$. Then $XL(X) = L(X)X = -X \preceq 0$. Now, by the SSM-property, we have $X = 0$, that is $SOL(L, I) = \{0\}$. Thus, L has the Q-property by Theorem 1.1. Hence the proof. $\square$

The following are some remarks which one can observe as a consequence of the above theorem.

Remark 2.1. P_2'-property does not imply the P_2-property. The following is a linear transformation that has the P_2'-property, but not the P_2-property.

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Consider the transformation R_A. Then

$$\begin{aligned} R_A(X) &= \begin{pmatrix} x + 2z & y \\ y & 2x + z \end{pmatrix}, \\ XR_A(X)X &= \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} x + 2z & y \\ y & 2x + z \end{pmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} \\ &= \begin{pmatrix} x(x^2 + 2xz + y^2) + y^2(3x + z) & * \\ * & y^2(x + 3z) + z(z^2 + 2xz + y^2) \end{pmatrix}. \end{aligned}$$

Now, suppose $X \succeq 0, XR_A(X)X \preceq 0$. This clearly implies that $X = 0$. In otherwords R_A has the P_2'-property. But here A is not a P-matrix. We know from Gowda and Song [11, Proposition 5], that saying A is a P-matrix is equivalent to saying R_A has the P-property. This says that R_A does not have the P-property. Hence R_A does not possess P_2-property as well.

Remark 2.2. The above example also illustrates that in general P_2'-property need not imply P-property.

Remark 2.3. P_2'-property implies that for all $Q \in S^n_+$, $SOL(L, Q) = \{0\}$; in particular it has R_0-property.

P_2'-property does not require the operator commutativity of X and $L(X)$. If we include this commutativity, then P_2'-property reduces to SSM-property.

Theorem 2.3. *Let $L : S^n \to S^n$ be a linear transformation. Then the following two statements are equivalent.*

(i) $[X \succeq 0, XL(X) = L(X)X,$ *and* $XL(X)X \preceq 0] \Rightarrow X = 0$.
(ii) *L has the SSM-property.*

Proof. From Theorem 2.2, it follows that (i) $\Rightarrow$ (ii). We now prove that (ii) $\Rightarrow$ (i). Suppose $X \succeq 0, XL(X) = L(X)X$, and $XL(X)X \preceq 0$. Since X and $L(X)$ are symmetric commuting matrices, there exists an orthogonal matrix U such that $UEU^T = X$ and $UFU^T = L(X)$. From hypothesis we have $E \succeq 0$ and $EFE \preceq 0$. This implies that $EF \preceq 0$ (since E is a nonnegative diagonal matrix and F is a diagonal matrix) and hence $E = 0$ or $X = 0$ by the *SSM*-property of L. This completes the proof of (ii) $\Rightarrow$ (i). $\square$

Remark 2.4. In the standard *LCP* situation *SSM*-property is equivalent to (the matrix version of) P_2'-property, see Theorem 2.1.

In general P_2' and its commutative version are not equivalent. We now give an example to show that P_2' is not equivalent to its commutative version.

Example 2.1. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$.

Note that A is positive stable. Hence L_A has the *SSM*-property, see [9]. Hence from the above result it follows that L_A has the commutative version of P_2'-property. Since A is not positive definite, it follows from Theorems 3.1 and 3.3, L_A does not possess the P_2'-property.

Remark 2.5. The P-property need not imply P_2'-property. The following (from Gowda and Song [9]) is an example of a linear transformation having P-property but not P_2'-property.

Let $A = \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix}$. Now, for $Q = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \succeq 0$, apart from $X = 0, X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, is also a solution to $SDLCP(L_A, Q)$. Thus, by Remark 2.3, L_A fails to have P_2'-property, but it has P-property.

Definition 2.2. Let $L : S^n \to S^n$ be a linear transformation. For $X_k \in S^k$, define $L_k : S^k \to S^k$ by $L_k(X_k) = (L(X))_k$, where $X = \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix}$ and $(L(X))_k$ is the $k \times k$ leading principal submatrix of $L(X)$. That is $\begin{pmatrix} L_k(X_k) & B \\ C & D \end{pmatrix}_k = L_k(X_k)$. We call L_k, a principal subtransformation of L.

Definition 2.3. Let $L : S^n \to S^n$ be a linear transformation. Then L has the completely Q-property (completely R_0-property) if every principal subtransformation L_k of L have the Q-property (R_0-property).

Theorem 2.4. *Let $L : S^n \to S^n$ be a linear transformation. If L has the P_2'-property then it is inherited by all its principal subtransformations.*

Proof. Let $X_k \in S^k$. Suppose $X_k \succeq 0$ with $X_k L_k(X_k) X_k \preceq 0$ then we need to show that $X_k = 0$. Let $X = \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix}$. Then $L_k(X_k) = (L(X))_k = \begin{pmatrix} L_k(X_k) & B \\ C & D \end{pmatrix}_k$, where B, C and D are matrices of appropriate order.

$$\begin{aligned} XL(X)X &= \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L_k(X_k) & B \\ C & D \end{pmatrix} \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} X_k L_k(X_k) & X_k B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} X_k L_k(X_k) X_k & 0 \\ 0 & 0 \end{pmatrix} \preceq 0 \\ &\Rightarrow X = 0 \text{ or } X_k = 0 \text{ for } L \text{ has the } P_2'\text{-property.} \end{aligned}$$

Thus, if L has the P_2'-property then every principal subtransformation, also has the P_2'-property. $\square$

Corollary 2.1. *If $L = L_A$ or S_A, has the P_2'-property, then L has the P-property.*

Proof. L_A has the P-property from a result due to Gowda and Song [9, Theorem 5]. Similarly S_A has the P-property by a result from Gowda and Parthasarathy [8, Theorem 11]. $\square$

We know from [9], that P_2-property in *SDLCP*'s is equivalent to the P-property in *LCP*'s. Now one can ask when the P_2-property is equivalent to P_2'-property. The answer is yes in the case of the Lyapunov transformations, the multiplicative transformations and a particular class of Stein transformations. We will see those equivalence results in the following sections.

3. The Lyapunov transformation L_A

In [17], Parthasarathy et al. have shown the following result.

Theorem 3.1 [17]. *Let $A \in \mathbb{R}^{n\times n}$. Then the following are equivalent for the Lyapunov transformation L_A.*

(i) *A is positive definite.*
(ii) *L_A has the strong-monotonicity property.*
(iii) *L_A has the P_2-property.*

Now, we will establish the equivalence of P_2-property and P_2'-property for the Lyapunov transformation.

Lemma 3.2. *Let $A \in \mathbb{R}^{n\times n}$. If L_A has the P_2'-property then L_{UAU^T} also has the P_2'-property for any orthogonal matrix U.*

Proof. Let us assume that L_A has the P_2'-property. We now claim that L_{UAU^T} also has the P_2'-property. Let $0 \preceq X \in S^n$, with

$$XL_{UAU^T}(X)X = X(UAU^TX + XUA^TU^T)X \preceq 0.$$

Then

$$\begin{aligned}
&XUU^T(UAU^TX + XUA^TU^T)UU^TX \preceq 0\\
&\Rightarrow XU(AU^TXU + U^TXUA^T)U^TX \preceq 0\\
&\Rightarrow U^TXU(AU^TXU + U^TXUA^T)U^TXU \preceq 0.
\end{aligned}$$

Taking $Y = U^TXU$, the above equation becomes,

$$Y(AY + YA^T)Y \preceq 0$$

with $0 \preceq Y \in S^n$. Now, by the P_2'-property of L_A we must have $Y = 0$, which implies that $X = 0$. $\square$

Remark 3.1. The inheritance of the P_2'-property for S_A and M_A to S_{UAU^T} and M_{UAU^T} for any orthogonal matrix U can be proved similarly.

Theorem 3.3. *Let $A \in \mathbb{R}^{n\times n}$. Then the following are equivalent.*

(i) *L_A has the P_2-property.*
(ii) *L_A has the P_2'-property.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Now, let us assume that L_A has the P_2'-property and claim that A is positive definite which is equivalent to saying L_A has the P_2-property (Theorem 3.1). To prove A is positive definite, because of Lemmas 1.2 and 3.2 it is enough to show that every diagonal entry of A is positive. Suppose $a_{11} \leqslant 0$.

Then take $X = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$. Now,

$$XAX = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = XAXX = XXA^TX$$

$$\Rightarrow XL_A(X)X = X(AX + XA^T)X = \begin{pmatrix} 2a_{11} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \preceq 0.$$

Now, by P_2'-property X must be zero. Thus, a_{11} cannot be non-positive. So, A is positive definite. □

4. The multiplicative transformation M_A

Now, we establish the equivalence of P_2'-property and the P_2-property for the multiplicative transformation M_A. The following result in the form given below is available in the thesis of Sampangi Raman [18]. But originally the equivalence of (i) and (ii) was proved by Gowda et al. [13, Corollary 6]. The equivalence of (i), (iii), (iv), and (v) is proved in Bhimasankaram et al. [4, Theorem 17].

Theorem 4.1 [18]. *Let $A \in \mathbb{R}^{n\times n}$. Then, for the double-sided multiplicative transformation M_A the following are equivalent*:

(i) *A is positive definite or negative definite*.
(ii) *M_A has the P_2-property*.
(iii) *M_A has the GUS-property*.
(iv) *M_A has the P-property*.
(v) *M_A has the R_0-property*.

Theorem 4.2. *Let $A \in \mathbb{R}^{n\times n}$. Then, for the double-sided multiplicative transformation M_A the following are equivalent*:

(i) *M_A has the P_2-property*.
(ii) *M_A has the P_2'-property*.

Proof. (i) $\Rightarrow$ (ii) Let M_A have the P_2-property. By Theorem 4.1, A is positive definite(or $-A$ is positive definite). Assume that $X \succeq 0$ with $XM_A(X)X = XAXA^TX \preceq 0$. But $XAXA^TX \succeq 0$. This implies that $tr(XAXA^TX) = tr(AXA^TXX) \leqslant 0$. That is, $AXA^TXX = 0 \Rightarrow XA^TXX = 0$. Now, premultiplying by X we get $XXA^TXX = 0$. This yields $XX = 0$ and $X = 0$. Thus, M_A has the P_2'-property.

(ii) $\Rightarrow$ (i) Let M_A have the P_2'-property. Then by Remark 2.3, M_A has the R_0-property. Now, the result follows from Theorem 4.1. □

Remark 4.1. In [1], Balaji and Parthasarathy prove that Q and P-properties are equivalent for the multiplicative transformations M_A, provided $A \in \mathbb{R}^{n\times n}$ is normal. So, if $A \in \mathbb{R}^{n\times n}$ is normal and M_A has the P_2'-property then M_A has the P-property. Further, Sampangi Raman [18] proves that Q and P properties are equivalent when $A \in \mathbb{R}^{2\times 2}$, that is, whenever $A \in \mathbb{R}^{2\times 2}$ with P_2'-property then M_A has the P-property. Recently for M_A with $A \in \mathbb{R}^{n\times n}$, Balaji (oral communication) has proved that P and Q-properties are equivalent.

Remark 4.2. We know from Gowda and Song [10], that $P_2 \Rightarrow GUS$ for any linear transformations. Hence we can deduce that for L_A and M_A, $P_2' \Rightarrow GUS$. But in general $P_2' \Rightarrow GUS$, need not hold, see Remark 2.1.

In the following theorem we will prove that the Lipschitzian property implies the P_2'-property for the transformation M_A.

Theorem 4.3. *Let $A \in \mathbb{R}^{n\times n}$ and the corresponding M_A be a Lipschitzian map. Then*

(i) *M_A has the R_0-property.*
(ii) *A is positive definite or negative definite.*
(iii) *M_A has the P_2-property.*
(iv) *M_A has the P_2'-property.*

Proof. We show (i), assuming that M_A has the Lipschitzian property. Note that $SDLCP(M_A, Q)$ has a solution namely the zero solution for all $Q \in S^n_+$. If M_A does not have the R_0-property then there exists a nonzero $X_0 \succeq 0$ such that $AX_0A^T \succeq 0$ and $X_0AX_0A^T = 0$. In fact $\lambda X_0 \in SOL(M_A, 0)$ for all $\lambda \geqslant 0$. Now, if $X \succeq 0, M_A(X) + I \succeq 0$ and $X(M_A(X) + I) = (M_A(X) + I)X = 0$, we have $XM_A(X) = M_A(X)X = -X^2 \preceq 0$. This yields that $X = 0$. Thus $SOL(M_A, I) = \{0\}$. Whereas $SOL(M_A, 0)$ is an unbounded set. This will contradict the Lipschitzian property of M_A. Thus, M_A has the R_0-property. Now, (ii) and (iii) follows from Theorem 4.1. □

Remark 4.3. The above result brings us to the following converse question. If A is positive definite or negative definite, does it follow that M_A has the Lipschitzian property? The answer to the above question is yes if $A \in S^n$. In this case M_A is strongly monotone and consequently M_A has the Lipschitzian property. Now, one can also ask whether there is an example where M_A has the Lipschitzian property, that does not have the strong-monotonicity property?

One can see the following sharper result for the transformation M_A.

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$. Then the following conditions are equivalent.*

(i) *A is positive definite or negative definite.*
(ii) $X \succeq 0, XM_A(X) \preceq 0 \Rightarrow X = 0.$
(iii) $X \succeq 0, tr(XM_A(X)) \leq 0 \Rightarrow X = 0.$

Proof. The implication (i) $\Rightarrow$ (ii) is known, for A positive definite implies that M_A has the P-property; see [4]. Let us see the proof of (ii) $\Rightarrow$ (iii). Since X and $M_A(X)$ are both symmetric positive semidefinite matrices, $tr(XM_A(X))$ cannot be negative. That says $tr(XM_A(X)) = 0$ and hence $XM_A(X) = 0$. This yields $X = 0$ from (ii). Now, we will prove that (iii) $\Rightarrow$ (i). Assume (iii). Suppose A is neither positive definite or negative definite, then there exists a nonzero x such that $x^tAx = 0$ or $X_0AX_0 = 0$ or $X_0AX_0A^T = 0$, where $X_0 = xx^t$. This X_0 is nonzero and positive semidefinite. But $tr(X_0M_A(X_0)) = tr(X_0AX_0A^T) = 0$, which is a contradiction. Hence the result. □

In general, statement (ii) need not imply (iii). The following example illustrates the above statement.

Example 4.1. Let $A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$. Let $X_t = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$. Then $R_A(X_t) = \begin{pmatrix} 1-3t & 0 \\ 0 & t \end{pmatrix}$ and $tr(X_tR_A(X_t)) = 1 + t^2 - 3t$. Then $tr(X_tR_A(X_t)) < 0$ at $t = \frac{1}{2}$. In other words R_A has P-property but statement (iii) of Theorem 4.4 does not hold good.

5. The Stein transformation S_A

Theorem 5.1. *Let A be normal. Then for S_A, P_2'-property is equivalent to P_2-property.*

Proof. Note P_2'-property implies Q-property for S_A, refer Theorem 2.3. Since A is normal, S_A has strong monotonicity property by a result of Gowda et al. [9]. This implies S_A has P_2-property, from a result due to Parthasarathy et al. [17]. $\square$

Remark 5.1. If A is not normal, for S_A we do not know whether $P_2' \Rightarrow P_2$. It is an open problem.

6. Concluding remarks

We have shown in this paper the new property P_2' is equivalent to P_2-property for Lyapunov, Multiplicative and in some special cases for Stein transformations. We have also given the relationship between P_2'-property, Q-property and P-property. Examples are given to show the sharpness of the results that are proved. We end the paper with the following conjecture:

Gowda et al. in [7], introduces Z-transformations in the *SDLCP* setting, based on the Z-matrices in *LCP*. They extend many properties of the Z-matrices to Z-transformations. A linear transformation $L : S^n \to S^n$ is said to have the Z-property (or called a Z-transformation) if

$$\left[X \in S^n_+, Y \in S^n_+, \text{ and } \langle X, Y\rangle = 0\right] \Rightarrow \langle L(X), Y\rangle \leqslant 0.$$

The Lyapunov and Stein transformations are examples of Z-transformations.

If L is a linear Z-transformation from $S^n \to S^n$ with P_2'-property then L has the P_2-property. If not, then L must have at least P-property.

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